DEFECT GROUPS IN THE THEORY OF REPRESENTATIONS OF FINITE GROUPS

BY

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Dedicated to Oscar Zariski

1. Introduction[1]

Let G be a finite group. Let Ξ be an algebraically closed field. As is well known, the study of the characters of G is closely related to that of the *group algebra* $\Xi[G]$ and of its center $Z = Z(\Xi[G])$. We call Z the *class algebra* of G. We are concerned here with a further investigation of Z continuing the work in [1].

The dimension of Z as a Ξ-space is the class number $k(G)$ of G. Since we are interested in characters and related functions, we also consider the dual space $\hat{Z}$ consisting of all linear functions defined on Z with values in Ξ.

Write Z as a direct sum

$$Z = \oplus \sum B \tag{1.1}$$

of *block ideals* of Z, i.e. of indecomposable ideals of Z. This decomposition (1.1) corresponds to the decomposition

$$\hat{Z} = \oplus \sum F_B \tag{1.2}$$

where F_B is the subspace of $\hat{Z}$ consisting of those $f \in \hat{Z}$ which vanish on all block ideals $B_1 \neq B$ in (1.1). Then B and F_B are themselves dual vector spaces and they have the same dimension k_B.

Each B is a commutative ring with a unit element η_B. If 1 is the unit element of Z, we have

$$1 = \sum_B \eta_B \tag{1.3}$$

and (1.3) is the decomposition of 1 into primitive orthogonal idempotents. It follows that

$$\Xi[G] = \oplus \sum_B \eta_B \, \Xi[G]$$

is the decomposition of the group algebra into (two-sided) block ideals.

Since B is indecomposable, the residue class ring $\bar{B}$ of B modulo its radical is simple and hence an extension field of finite degree of Ξ. Since Ξ was algebraically closed, $\bar{B}$ is isomorphic to Ξ. We then have an algebra homomorphism ω of B onto Ξ. Clearly, ω can be extended to an algebra homomorphism ω_B of Z onto Ξ such that ω_B vanishes for all block ideals $B_1 \neq B$ in (1.1). Thus $\omega_B \in F_B$. Conversely, it is seen at once that each non-zero algebra homomorphism of Z into Ξ coincides with ω_B for some B.

[1] This work has been supported in part by a National Science Foundation grant.

The case that Ξ has characteristic 0 is well known and fairly trivial. Let $\chi_1, \chi_2, \cdots, \chi_{k(G)}$ denote the irreducible characters of G. Each χ_j defines an algebra-homomorphism ω_j onto Ξ given by Frobenius' formula

$$\omega_j(\mathcal{S}K) = |K| \chi_j(\sigma_K)/\chi_j(1) \tag{1.4}$$

where K is a class of conjugate elements of G, where $\mathcal{S}K \in \Xi[G]$ is the sum of the $|K|$ elements of K, and where $\sigma_K \in K$. Since the $k(G)$ homomorphism ω_j are distinct, we have $k(G)$ block ideals $B \cong \Xi$ in (1.1) and Z is semi-simple.

We now turn to fields of prime characteristic. Throughout this paper, p will be a fixed prime number and we shall reserve the letter Ω for an algebraically closed field of characteristic p. Take then $\Xi = \Omega$ above and set

$$Z = Z(G) = Z(\Omega[G]).$$

It is clear in principle that if we know the irreducible characters $\chi_1, \chi_2, \cdots, \chi_{k(G)}$, we can construct the block ideals B, or as we shall simply say, the *blocks* B of G. Actually, this can be done in an explicit fashion (§2, 2). In particular, the dimension k_B turns out to be the number of irreducible characters χ_i in B in the sense of [1].

In a way, our aim lies in the opposite direction. This is part of our effort to find new links between characters of G and group theoretical properties of G. The main result of [1, I] is already of this type. With each block B of G, we associate a p-subgroup D of G, the *defect group* of B. If we know[2] the normalizer $N_G(D)$ of D, we can construct the algebra homomorphism ω_B for the blocks B of G with the defect group D. This gives us the values (1.4) for the characters $\chi_j \in B$ modulo a prime ideal divisor of p in an appropriate algebraic number field.

The defect group D of B is determined up to conjugacy. We shall associate with B a system of p-subgroups of G which we shall call the lower defect groups of B. Again, they are really only determined up to conjugacy. In order to fix ideas, it will be convenient to choose a set $\mathcal{P}(G)$ of representatives for the classes of conjugate p-subgroups of G. We then take defect groups and lower defect groups in $\mathcal{P}(G)$.

Let K be a conjugate class of G. There is a unique element $P \in \mathcal{P}(G)$ such that P is a p-Sylow subgroup of the centralizer $C_G(\sigma)$ for suitable $\sigma \in K$. We then call P the *defect group D_K of the class K.*

Let B now be a block. A member P of $\mathcal{P}(G)$ will be called a *lower defect group* of B, if there exist elements f of the space F_B in (1.2) with the following properties:

(i) There exist conjugate classes K with the defect group P such that $f(\mathcal{S}K) \neq 0$ with $\mathcal{S}K$ defined as in (1.4).

[2] When we say that a subgroup H of G is known, we usually assume that we know H not only as an abstract group but also the imbedding of H in G, i.e. the manner in which the conjugate classes of H lie in the conjugate classes of G.

(ii) We have $f(\mathcal{S}K) = 0$ for all conjugate classes K for which the order $|D_K|$ of the defect group D_K is smaller than the order $|P|$ of P.

More generally, we consider subspaces V of F_B such that all $f \neq 0$ in V have properties (i) and (ii). Let $m_B(P)$ denote the maximal dimension of such a space V. We count P exactly $m_B(P)$ times as lower defect group of B. Let $\mathfrak{D}_B$ denote the system consisting of the groups $P \in \mathcal{P}(G)$, each P taken with the multiplicity $m_B(P) \geqq 0$. This is the *system* $\mathfrak{D}_B$ *of lower defect groups* of B. We shall show (§4) that $\mathfrak{D}_B$ consists of exactly k_B groups. In other words,

$$(1.5) \qquad k_B = \sum_P m_B(P); \qquad (P \in \mathcal{P}(G)).$$

If P is a lower defect group of B, i.e. if $m_B(P) > 0$, then P is conjugate to a subgroup of the defect group D of B, and D itself is a lower defect group of B. If we know the normalizer $N_G(P)$ of $P \in \mathcal{P}(G)$, we are able to construct a subspace V_P of dimension $m_B(P)$ of F_B with the properties (i), (ii) above such that F_B is the direct sum of the V_P for the various $P \in \mathcal{P}(G)$. If $P \neq 1$, $N_G(P)$ is a 'local subgroup' of G. However, since $P = 1$ occurs in $\mathcal{P}(G)$, our construction falls short of a full construction of F_B based on a knowledge of the local subgroups of G. In particular, in (1.5) the term $m_B(1)$ cannot be determined, and we can only give a lower estimate for k_B.

By a *p-section* $\mathfrak{S}(\tau)$ of an element τ of G, we mean the set of all elements $\xi \in G$ such that the p-factor ξ_p of ξ is conjugate to the p-factor τ_p of τ, cf. [1, II, §3]. Each p-section is a union of conjugate classes. We shall denote by Π a set of representatives for the conjugate classes of p-elements of G. Each p-section has the form $\mathfrak{S}(\pi)$ with $\pi \in \Pi$ and G is the disjoint union of these $\mathfrak{S}(\pi)$. In §6, we shall associate each lower defect group of B with one of the sections. Let $m_B^{(\pi)}(P)$ of the $m_B(P)$ members P of $\mathfrak{D}_B$ be associated with $\mathfrak{S}(\pi)$ so that

$$(1.6) \qquad \sum_\pi m_B^{(\pi)}(P) = m_B(P); \qquad (\pi \in \Pi).$$

We shall show that $m_B^{(\pi)}(P)$ can be determined when we know the centralizer $C_G(\pi)$ of π and the blocks b of $C_G(\pi)$ with $b^G = B$ (in the sense of [1, II, §2]. It suffices to know the lower defect groups of b associated with the section of the unit element in $C_G(\pi)$.

The numbers $m_B^{(1)}(P)$ have some remarkable properties. If l_B is the number of modular irreducible characters in B, then

$$(1.7) \qquad l_B = \sum_P m_B^{(1)}(P); \qquad P \in \mathcal{P}(G).$$

This is a kind of analogue of (1.5). If in (1.7) we sum only over the $P \in \mathcal{P}(G)$ of a fixed order p^r, the partial sum represents the multiplicity of p^r as elementary divisor of the Cartan matrix C_B of B. This refines a result announced without proof in [2].

Notation. Most of the notation used has been explained above. The letter G will always stand for a finite group and p will be a fixed prime number.

We shall denote by Ω an algebraically closed field of characteristic p. The class algebra $Z(\Omega[G])$ of G over Ω will be denoted by Z or $Z(G)$. Occasionally in §2, a particular field Ω will be used, but it is clear that the results concerning Z will not depend on the choice of Ω. If M is a subset of G we denote by $\mathcal{S}M$ the sum of the elements of M in the group algebra of G.

The set of conjugate classes of G will be denoted by $\mathcal{C}\ell(G)$. For $K \in \mathcal{C}\ell(G)$, we shall denote by σ_K a representative element in K. If f is a function defined on Z, we shall usually write $f(K)$ instead of $f(\mathcal{S}K)$. The set of blocks of G (for given p) will be denoted by $\mathcal{B}\ell(G)$.

We choose a set $\mathcal{P}(G)$ of representatives for the classes of conjugate p-subgroups of G. If P, $Q \in \mathcal{P}(G)$, we write $P \preceq Q$ when P is conjugate in G to a subgroup of Q. Then $\mathcal{P}(G)$ is partially ordered. A set of representatives for the conjugate classes of p-elements of G will be denoted by Π.

If M is a subset of G, the centralizer of M in G is denoted by $C_G(M)$ and the normalizer of M is denoted by $N_G(M)$. We write $|M|$ for the cardinality of M.

In summations, the range of the summation is often indicated in parentheses at the end of the line, e.g. see (1.5). We frequently have to use determinants Δ of the following kind. We have a set F of n functions f and a set X of n arguments. Each row of Δ correspond to one $f \in F$ and each column of Δ corresponds to one $x \in X$. We then write[3]

$$\Delta = \det(f(x)); \qquad (f \in F, \quad x \in X).$$

2. Preliminaries

1. In the following, a simple method developed in [1, I, §7] will play an important role. We discuss it briefly. We shall say that a pair of subgroups (T, H) of G is an *admissible pair*, if there exists a p-subgroup Q of G such that

$$T = C_G(Q), \quad QT \subseteq H \subseteq N_G(Q). \tag{2.1}$$

(Actually, these conditions could be replaced by weaker ones.)

As shown in [1, I, §7], there exists a unique algebra homomorphism μ of $Z(G) = Z(\Omega[G])$ into $Z(H) = Z(\Omega[H])$ such that

$$\mu : \mathcal{S}K \longrightarrow \mathcal{S}(K \cap T) \quad \text{for} \quad K \in \mathcal{C}\ell(G). \tag{2.2}$$

The dual mapping λ then maps the dual space $\hat{Z}(H)$ of $Z(H)$ into $\hat{Z}(G)$. For $\varphi \in \hat{Z}(H)$, we have

$$\lambda : \varphi \longrightarrow \varphi^{\lambda} = \varphi \circ \mu. \tag{2.3}$$

In particular, if b is a block of H and if φ is the corresponding algebra-homomorphism ω_b of $Z(H)$ onto Ω then ω_b^{λ} is an algebra homomorphism of $Z(G)$ onto Ω. Hence $\omega_b^{\lambda} = \omega_B$ for some block B. We then write $B = b^G$;

[3] The order in which the elements of G and of X are taken will always be immaterial.

cf. [1, II, §2]. We show:

(2A) *Let (T, H) be an admissible pair of subgroups of G. Let b_0 be a block of H and let F_{b_0} denote the subspace of $\hat{Z}(H)$ corresponding to b_0. If $\varphi \in F_{b_0}$ and if λ is the mapping (2.3), then $\varphi^\lambda \in F_{B_0}$ with $B_0 = b_0^G$.*

Proof. Since μ is an algebra homomorphism, it maps the idempotent η_B of $B \in \mathcal{B}\ell(G)$ on an idempotent of $Z(H)$ or on 0. Hence we can set

$$\eta_B^\mu = \sum\nolimits_b \eta_b \tag{2.4}$$

where b ranges over a set Γ_B of blocks of H. If $b_0 \in \mathcal{B}\ell(H)$ and if $B_0 = b_0^G$, by (2.3) and (2.4),

$$\omega_{B_0}(\eta_B) = \omega_{b_0}(\eta_B^\mu) = \sum\nolimits_b \omega_{b_0}(\eta_b); \qquad (b \in \Gamma_B).$$

This shows that $\omega_{B_0}(\eta_B) = 1$, if and only if $b_0 \in \Gamma_B$. Hence Γ_B consists of exactly those $b \in \mathcal{B}\ell(H)$ for which $b^G = B$.

Suppose now that $\varphi \in F_{b_0}$. Then, for $\zeta \in Z(G)$,

$$\varphi^\lambda(\eta_B \zeta) = \varphi(\eta_B^\mu \zeta^\mu) = \sum\nolimits_b \varphi(\eta_b \zeta^\mu); \qquad (b \in \Gamma_B).$$

If $B \neq b_0^G$, then $b_0 \notin \Gamma_B$ and it follows that our expression vanishes. This shows that $\varphi^\lambda \in F_{B_0}$ with $B_0 = b_0^G$.

(2B) *Let (T, H) form an admissible pair of subgroups of G with $T = C_G(Q)$, $Q \in \mathcal{P}(G)$. Let $\varphi \in \hat{Z}(H)$ and $f = \varphi^\lambda$, cf. (2.3). If $f(K) \neq 0$ for some conjugate class, then the defect group D_K of K satisfies $D_K \geq Q$ in the partial ordering of $\mathcal{P}(G)$.*

Indeed, by (2.2) and (2.3)

$$f(K) = \varphi(\mathcal{S}(K \cap C_G(Q))).$$

If $f(K) \neq 0$, the class K meets $C_G(Q)$ and this implies $D_K \geq Q$.

2. We next discuss the connection between the algebras $Z(\Xi[G])$ and $Z(\Omega[G])$ where Ξ is an algebraically closed field of characteristic 0 and Ω (as always) an algebraically closed field of characteristip p. As we have seen in §1, the class algebra $Z(\Xi[G])$ is semi-simple and, if $k(G)$ is the class number of G, we have exactly $k(G)$ distinct algebra homomorphisms ω_i of $Z(\Xi[G])$ onto Ξ, cf. (1.4). These formulas show that this result remains valid, if Ξ is replaced by the field Ξ_0 of the $|G|$-th roots of unity over the field $\mathbf{Q}$ of rational numbers. Indeed, all $\chi_i(\sigma_K)$ in (1.4) lie in Ξ_0.

Let p be a fixed rational prime. Let ν denote a fixed extension of the p-adic (exponential) valuation of $\mathbf{Q}$ to a valuation of Ξ_0. If $\mathfrak{o}$ is the ring of local integers for ν in Ξ_0 and $\mathfrak{p}$ the corresponding prime ideal, we set

$$\mathfrak{o}/\mathfrak{p} = \Omega_0 \tag{2.5}$$

and form the subring

$$J = \sum_K \mathfrak{o}(\mathcal{S}K); \qquad (K \in \mathcal{C}\ell(G)) \tag{2.6}$$

of "integral" elements of $Z(\Xi_0[G])$. If θ_0 is the natural homomorphism of $\mathfrak{o}$ onto Ω_0 in (2.5), clearly θ_0 can be extended to a homomorphism θ of J onto the class algebra $Z(\Omega_0[G])$. If φ is a linear function defined on $Z(\Xi_0[G])$ with values in Ξ_0, and if $\varphi(\alpha) \in \mathfrak{o}$ for all $\alpha \in J$, then the map θ defines a linear function φ^θ, defined on $Z(\Omega_0[G])$ with values in Ω_0. Let Ω denote the algebraic closure of Ω_0. By linearity, φ^θ can be considered as a linear function on the class algebra $Z = Z(\Omega[G])$ with values in Ω, i.e. φ^θ can be considered as an element of the dual space $\hat{Z}$.

Since as is well known the right sides in (1.4) are algebraic integers in Ξ_0, we can apply this to the function $\varphi = \omega_j$. It is clear that ω_j^θ is an algebra homomorphism of Z onto Ω. Hence ω_j^θ must be an ω_B for some block B of G. In [1], the irreducible character χ_j of G was said to *belong* to the block B of G, if $\omega_j^\theta = \omega_B$. We shall also say now that then ω_j is *associated with* B. If this is so for k_B^* values of j, clearly

$$k(G) = \sum_B k_B^*; \qquad (B \in \mathcal{B}\ell(G)). \tag{2.7}$$

Consider the Ξ_0-space W spanned by the ω_j associated with B,

$$W = \sum_j \Xi_0\, \omega_j; \qquad (\chi_j \in B), \tag{2.8}$$

and take the subset M_B consisting of those $\varphi \in W$ for which $\varphi(\alpha) \in \mathfrak{o}$ for all $\alpha \in J$. Then M_B is an $\mathfrak{o}$-module of rank k_B^*. Since $\mathfrak{o}$ is a principal ideal domain, M_B has an $\mathfrak{o}$-basis. It follows that the module $(M_B)^\theta$ of all φ^θ with $\varphi \in M_B$ has again rank k_B^*. On the other hand, the method in [1, II, §4] shows that $(M_B)^\theta \subseteq F_B$. Hence

$$\dim_\Omega B = \dim_\Omega F_B \geqq k_B^*. \tag{2.9}$$

If we add over all B, both sides have the same sum $k(G)$, cf. (1.1) and (2.7). Hence we must have equality in (2.9). Thus

(2C) *Let Ω be an algebraically closed field of characters p. Let B be a block of G. Then* $\dim_\Omega B$ *is equal to the number of ordinary irreducible characters of G in B in the sense of* [1].

With the notation introduced above, we also have

(2D) *If φ ranges over the elements of the $\mathfrak{o}$-module M_B, then φ^θ ranges over F_B.*

3. We add some remarks which will only be used in §6 and §7.

(2E) *Let B be a block of G. Suppose we have coefficients $a_K \in \Xi$ such that*

$$\sum_K a_K\, \omega_j(K) = 0; \qquad (K \in \mathcal{C}\ell(G)) \tag{2.10}$$

for every ω_j associated with B. Then (2.10) *remains valid if we let K range only over the conjugate classes which belong to a fixed p-section.*

Proof. Expressing ω_j by χ_j by means of (1.4), we have

$$\sum | K | \, a_K \, \chi_j(\sigma_K) = 0.$$

We may assume that the p-factor of σ_K is an element $\pi_K \in \Pi$. If $\sigma_K = \pi_K \rho_K$, we can express $\chi_j(\sigma_K)$ by the decomposition numbers belonging to B and the section $\mathfrak{S}(\pi_K)$ and the values of modular irreducible characters of $C_G(\pi_K)$ for the element ρ_K, cf. [1, II (3.2), (6A)]. Since [I1, II (7B)] implies that the matrix of decomposition numbers belonging to B is non-singular the statement is immediate.

(2F) *Let B be a block; $k_B = \dim_\Omega B$. Suppose we have a set F of k_B elements of F_B and a set $\mathfrak{K}$ of k_B conjugate classes such that*

$$\det f(K) \neq 0; \qquad (f \in F, \quad K \in \mathfrak{K}).$$

Let K_0 be a fixed conjugate class. There exist coefficients $c_K \in \mathfrak{o}$ such that

$$\omega_j(K_0) = \textstyle\sum_K c_K \, \omega_j(K); \qquad (K \in \mathfrak{K}) \tag{2.11}$$

for each ω_j associated with B. Here, c_K vanishes when K and K_0 belong to different p-sections. For each $f \in F_B$, then

$$f(K_0) = \textstyle\sum_K c_K^{\theta} f(K); \qquad (K \in \mathfrak{K}). \tag{2.12}$$

Proof. For each $f \in F_B$, there exists a $\varphi \in M_B$ with $\varphi^\theta = f$. If Φ is the system of k_B functions φ obtained from F in this manner,

$$\det(\varphi(K)) \not\equiv 0 \pmod{\mathfrak{p}}; \qquad (\varphi \in \Phi, \quad K \in \mathfrak{K}).$$

It follows that we can find coefficients $c_K \in \mathfrak{o}$ such that

$$\varphi(K_0) = \textstyle\sum_K c_K \, \varphi(K); \qquad (K \in \mathfrak{K})$$

for each $\varphi \in \Phi$. Since the k_B functions φ are certainly linearly independent and belong to W in (2.8), they form a Ξ_0-basis of W and hence

$$\omega_j(K_0) = \textstyle\sum_K c_K \, \omega_j(K); \qquad (K \in \mathfrak{K})$$

for each ω_j associated with B. Now (2E) shows that this result remains valid, if we replace c_K by 0 for all $K \in \mathfrak{K}$ which do not belong to the section of K_0.

The relation (2.11) remains valid if ω_j is replaced by an arbitrary element φ of W. In particular, we may take $\varphi \in M_B$. Now (2.12) is immediate from (2D).

The following result has been observed by M. Osima and K. Iizuka

(2G) *Let B be a block of G. There exists a unique idempotent $\varepsilon_B \in Z(\mathfrak{o}[G])$ such that $\omega_j(\varepsilon_B) = 1$ or 0 according as to whether or not ω_j is associated with B. If K_0 is a fixed conjugate class, we have formulas*

$$(\mathrm{S}K_0)\varepsilon_B = \textstyle\sum_K a_K(\mathrm{S}K); \qquad (K \in \mathcal{C}\ell(G)) \tag{2.13}$$

with $a_K \in \mathfrak{o}$. If K_0 belongs to the p-section $\mathfrak{S}(\pi)$, here $a_K = 0$ for all K not contained in $\mathfrak{S}(\pi)$.

Proof. As shown in [1, II, §4] there exists an idempotent $\varepsilon_B \in Z(\mathfrak{o}[G])$ for which $\omega_j(\varepsilon_B)$ has the values 1 or 0 as indicated. It is clear that ε_B is unique. Then for each $K_0 \in \mathcal{C}\ell(G)$, we have an equation (2.13) with $a_K \in \mathfrak{o}$. This implies that

$$\textstyle\sum_K a_K\, \omega_j(K) = \omega_j(K_0)$$

if ω_j is associated with B while in the other case the sum is 0. In either case, (2E) shows that

$$\textstyle\sum_K a_K\, \omega_j(K) = 0; \qquad (K \in \mathcal{C}\ell(G), \quad K \not\subseteq \mathfrak{S}(\pi)).$$

Since this holds for $j = 1, 2, \cdots, k(G)$, we have $a_K = 0$ for all K not contained in $\mathfrak{S}(\pi)$, Q.E.D.

The map θ of $Z(\mathfrak{o}[G])$ onto $Z(\Omega_0[G])$ clearly maps ε_B onto the idempotent $\eta_B \in B$. Hence

(2H) *Let B be a block of G. Let K_0 be a fixed conjugate class. There exist elements $c_K \in \Omega$ such that*

$$(\mathsf{S}K_0)\eta_B = \textstyle\sum_K c_K\, (\mathsf{S}K).$$

where K ranges over those conjugate classes which are contained in the section of K_0.

3. Selection of sets of conjugate classes for the blocks

(3A) *For each block B of G, we can select a set $\mathfrak{K}_B$ of k_B conjugate classes of G and a set X_B of k_B elements of F_B, denoted by h_K with $K \in \mathfrak{K}_B$, such that:*

(i) *The set $\mathcal{C}\ell(G)$ is the disjoint union of the sets $\mathfrak{K}_B$ with $B \in \mathcal{B}\ell(G)$.*

(ii) *The set X_B is a basis of F_B.*

(iii) *If $Q \in \mathcal{P}(G)$ and if $\mathfrak{K}_B(Q)$ is the subset of $\mathfrak{K}_B$ consisting of those classes with defect group Q, each h_K with $K \in \mathfrak{K}_B(Q)$ has the form $h_K = \varphi^\lambda$ where $\varphi \in \hat{Z}(N_G(Q))$ and where λ is the operator in (2.3) with $T = C_G(Q)$, $H = N_G(Q)$.*

(iv) $h_K(K) = 1$; $h_K(K') = 0$ *for $K, K' \in \mathfrak{K}_B(Q)$ and $K \neq K'$.*

Proof. Consider a fixed $Q \in \mathcal{P}(G)$ and set $H = N_G(Q)$. For each $b \in \mathcal{B}\ell(H)$, let F_b be the subspace of $\hat{Z}(H)$ defined in a manner analogous to the definition of F_B in $\hat{Z}(G)$. Let Y_b denote a basis of F_b.

If $B \in \mathcal{B}\ell(G)$, denote by B_H the set of blocks b of H with $b^G = B$ and let Y_B be the union of the Y_b for these b. Since

$$\hat{Z}(H) = \oplus \textstyle\sum_b F_b\,; \qquad (b \in \mathcal{B}\ell(H)),$$

the union Y of the sets Y_B for all $B \in \mathcal{B}\ell(G)$ is a basis of $\hat{Z}(H)$. Hence

$$\det(\varphi(L)) \neq 0; \qquad (\varphi \in Y, \quad L \in \mathcal{C}\ell(H)). \tag{3.1}$$

It follows from (3.1) that, for each $B \in \mathcal{B}\ell(G)$, we can select a subset $\mathfrak{L}_B$ of $\mathcal{C}\ell(H)$ such that

$$\mathcal{C}\ell(H) = \textstyle\bigcup_B \mathfrak{L}_B \text{ (disjoint)}; \qquad (B \in \mathcal{B}\ell(G)) \tag{3.2}$$

and that $|\mathfrak{L}_B| = |Y_B|$ and

(3.3) $$\det(\varphi(L)) \neq 0; \qquad (\varphi \in Y_B, \quad L \in \mathfrak{L}_B).$$

For $|Y_B| = |\mathfrak{L}_B| = 0$, the determinant in (3.3) is 1 by definition and (3.3) is always satisfied.

Let $\mathfrak{L}_B(Q)$ denote the subset of $\mathfrak{L}_B$ consisting of the classes in $\mathfrak{L}_B$ with the defect group Q in H. If follows from (3.3) that we can find a subset $Y_B(Q)$ of Y_B with $|Y_B(Q)| = |\mathfrak{L}_B(Q)|$ such that

(3.4) $$\det(\varphi(L)) \neq 0; \qquad (\varphi \in Y_B(Q), \quad L \in \mathfrak{L}_B(Q)).$$

It is an immediate consequence of Sylow's theorems that if L is a conjugate class of $H = N_G(Q)$ with the defect group Q in H, then the conjugate class L^G of G which contains L has defect group Q in G. Conversely, every conjugate class K of G with defect group Q is obtained in this fashion; the corresponding class L of H is uniquely determined; $L = K \cap C_G(Q)$. Let $Y_B(Q)^\lambda$ denote the set of functions φ^λ with $\varphi \in Y_B(L)$ and with λ defined in (2.3), with $T = C_G(Q), H = N_G(Q)$. On account of (2A), $Y_B(Q)^\lambda$ is a subset of F_B. Let $\mathfrak{K}_B(Q)$ denote the set of classes L^G with $L \in \mathfrak{L}_B(Q)$. Then each class in $\mathfrak{K}_B(Q)$ has defect group Q. Moreover, for $\varphi \in Y_B(Q)$ and $K = L^G$ with L $\in \mathfrak{L}_B(Q)$, by (2.3)

$$\varphi^\lambda(K) = \varphi(\mathcal{S}(K \cap C_G(Q))) = \varphi(L).$$

Hence (3.4) implies

$$\det(f(K)) \neq 0; \qquad (f \in Y_B(Q)^\lambda, \quad K \in \mathfrak{K}_B(Q)).$$

It is now clear that we can find linear combinations h_K of the elements of $Y_B(Q)^\lambda$ which satisfy the conditions (iv) in (3A). If $\mathfrak{K}_B$ is the union of the sets $\mathfrak{K}_B(Q)$ for all $Q \in \mathcal{P}(G)$, then condition (iii) is likewise satisfied. For each $K \in \mathfrak{K}_B$, the function h_K belongs to F_B.

If K is any class of G and if Q is the defect group, then by (3.2), $L = K \cap C_K(Q)$ belongs to $\mathfrak{L}_B$ for a unique block B. It follows that K belongs to $\mathfrak{K}_B$ for a unique B. Hence condition (i) of (3A) holds.

We show that the set X_B of functions h_K with $K \in \mathfrak{K}_B$ is linearly independent. Suppose we have a non-trivial relation

(3.5) $$\textstyle\sum_K c_K h_K = 0; \qquad (K \in \mathfrak{K}_B)$$

with coefficients $c_K \in \Omega$. Since not all c_K vanish, we can choose a group $P \in \mathcal{P}(G)$ such that $c_K \neq 0$ for some $K \in \mathfrak{K}_B(P)$ while we have $c_K = 0$ for all $K \in \mathfrak{K}_B$ whose defect group D_K has smaller order than $|P|$.

Take $K' \in \mathfrak{K}_B(P)$. Then K' has defect group P. Consider a term $c_K h_K$ in (3.5). If here $K \in \mathfrak{K}_B(Q)$ with $Q \in \mathcal{P}(G)$, by (iii) and (2B), we have $h_K(K') = 0$ except when $P \geq Q$. If $P > Q$, by construction $c_K = 0$. It follows from (3.5) that, for $K' \in K_B(P)$, we have

$$\textstyle\sum_K c_K h_K(K') = 0$$

where K ranges over the classes in $\mathfrak{K}_B$ with the defect group $Q = P$. These are the $K \in \mathfrak{K}_B(P)$. It now follows from (iv) that $c_K = 0$ for all $K \in \mathfrak{K}_B(P)$, a contradiction.

Hence the set $X_B = \{h_K\}$ is linearly independent. This implies

$$| \mathfrak{K}_B | = | X_B | \leqq \dim_\Omega F_B = k_B .$$

If we add here over all $B \in \mathcal{B}\ell(G)$, the sum on the left is $k(G)$ by (i). Since the sum on the right is also $k(G)$ by (1.2), we must have equality for each B. Hence X_B is a basis of F_B. This proves (ii) and the proof of (3A) is complete.

(3B) *Let $\mathfrak{K}_B$ be chosen as in* (3A). *There exists a basis $\{f_K\}$ of F_B with K ranging over $\mathfrak{K}_B$ with the following properties*

$$f_K(K) = 1; f_K(K') = 0 \quad \text{for} \quad K, K' \in \mathfrak{K}_B , K \neq K'.$$

Moreover, if $f_K(K^*) \neq 0$ for some $K^* \in \mathcal{C}\ell(G)$, then $D_{K^*} \geq D_K$.

Proof. Let $Q \in \mathcal{P}(G)$. Suppose that f_K has already been obtained for all $K \in \mathfrak{K}_B(P)$ with $P \in \mathcal{P}(G)$ and $P \succ Q$. Suppose now that $K \in \mathfrak{K}_B(Q)$ and set

$$(3.6) \qquad f_K = h_K - \textstyle\sum_{K_1} h_K(K_1) f_{K_1} ; \qquad (K_1 \in \mathfrak{K}_B , D_{K_1} \succ Q).$$

Here, f_{K_1} is assumed to be defined. If $f_K(K^*) \neq 0$ for $K^* \in \mathcal{C}\ell(G)$, then $h_K(K^*) \neq 0$ or $f_{K_1}(K^*) \neq 0$ for some $K_1 \in \mathfrak{K}_B$ with $D_{K_1} \succ Q$. In the latter case, by assumption $D_{K^*} \geq D_{K_1}$ and hence $D_{K^*} \geq Q$. In the former case, by (3A) (iii) and (2B), $D_{K^*} \geq Q$. This shows that f_K has the last property in (3B).

Suppose now that $K' \in \mathfrak{K}_B$. If $D_{K'} \succ Q$ then K' is one of the K_1 in (3.6) and we see that $f_K(K') = 0$. If $D_{K'} = Q$, then K' is not one of the K_1 and (3.6) yields

$$f_K(K') = h_K(K').$$

Now (3A) (iv) shows that $f_K(K') = 0$ for $K' \neq K$ and that $f_K(K) = 1$. Finally, for the remaining $K' \in \mathfrak{K}_B$, we have $f_K(K') = 0$ since otherwise as shown above $D_{K'} \geq Q$.

Applying this successively for all $Q \in \mathcal{P}(G)$ we obtain the required system $\{f_K\}$. Since $\{h_K\}$ was a basis of F_B, so is $\{f_K\}$.

If for the local subgroups $H = N_G(P)$ with $P \in \mathcal{P}(G)$, $P \neq 1$, we know a basis of F_b with $b \in \mathcal{B}\ell(H)$, we can construct the functions f_K except for the $K \in \mathfrak{K}_B$ with $D_K = 1$.

(3C) *Let B be a block ideal of $Z(G)$ and set*

$$B^* = \oplus \textstyle\sum_{B_1} B_1 , \qquad (B_1 \in \mathcal{B}\ell(G), \quad B_1 \neq B).$$

For each $K^* \in \mathcal{C}\ell(G)$, $K^* \notin \mathfrak{K}_B$ form the element

$$\zeta_{K^*} = \mathcal{S}K^* - \textstyle\sum_K f_K(K^*) \mathcal{S}K; \qquad (K \in \mathfrak{K}_B).$$

These elements form a basis of B^*.

Proof. It is clear that all f_K with $K \in \mathfrak{K}_B$ vanish for the elements ζ_{K^*} and this implies $\zeta_{K^*} \in B^*$. It is clear that the elements ζ_{K^*} are linearly independent and since the number of these elements is equal to $\dim_\Omega B^*$, they form an Ω-basis of B^*.

Remark. The construction in (3A), (3B) can be performed in the case when we have a partition

$$\mathcal{B}\ell(G) = \bigcup B \quad \text{(disjoint)}$$

where each B is a union of blocks. In particular, if we take

$$\mathcal{B}\ell(G) = B \cup B^*$$

with B and B^* as in (3C) and interchange the roles of B and B^*, we obtain an Ω-basis of B.

It should be mentioned that the selection $\mathfrak{K}_B$ of sets of classes for the blocks in (3A) is not uniquely determined.

4. The lower defect groups of a block

The system $\mathfrak{D}_B$ of lower defect groups of a block has been defined in the introduction. We show

(4A) *If $\mathfrak{K}_B$ is as in* (3A), *the system $\mathfrak{D}_B$ of lower defect groups of the block B coincides exactly with the system of defect groups of the k_B classes $K \in \mathfrak{K}_B$.*

Proof. We have to show that for $P \in \mathcal{P}(G)$, the multiplicity $m_B(P)$ of P in $\mathfrak{D}_B$ (cf. §1) is equal to $|\mathfrak{K}_B(P)| = k_B(P)$. Let V_0 denote the subspace of F_B spanned by the $k_B(P)$ functions f_K with $K \in \mathfrak{K}_B(P)$. It is clear from (3B) that V_0 has dimension $k_B(P)$ and that for $v \neq 0$ in V_0, there exist classes K with $D_K = P$ such that $v(K) \neq 0$. We may even choose $K \in \mathfrak{K}_B(P)$. Moreover, if $K^* \in \mathcal{C}\ell(G)$ and if $v(K^*) \neq 0$, then $f_K(K^*) \neq 0$ for some $K \in \mathfrak{K}_B(P)$ and then, by (3B), $D_{K^*} \succeq P$. In particular, $|D_{K^*}| \geqq |P|$. This shows that V_0 has the properties (i) and (ii) required in the definition of $m_B(P)$ in §1 of subspaces V of F_B and hence $k_B(P) \leqq m_B(P)$.

Conversely, let V be any subspace of F_B with these properties (i), (ii), §1. Express $v \in V$ by the basis $\{f_K\}$ of F_B in (3B),

$$v = \textstyle\sum_K a_K f_K\,; \qquad (K \in \mathfrak{K}_B), \quad a_K \in \Omega.$$

Here $a_K = v(K)$ for $K \in \mathfrak{K}_B$. For any $K^* \in \mathcal{C}\ell(G)$, then

$$v(K^*) = \textstyle\sum_K v(K) f_K(K^*); \qquad (K \in \mathfrak{K}_B). \tag{4.1}$$

Because of the property §1, (ii) of V, it suffices to let K range over the classes for which $|D_K| \geqq |P|$.

If $v \neq 0$, then by §1, (i), we can choose K^* with the defect group P such that $v(K^*) \neq 0$. By (3B), $f_K(K^*) = 0$ in (4.1) except when $P \succeq D_K$. It follows that there exist $K \in \mathfrak{K}_B$ with the defect group P for which $v(K) \neq 0$. Since $K \in \mathfrak{K}_B(P)$ and $|\mathfrak{K}_B(P)| = k_B(P)$, this implies that the dimension of

V is at most equal to $k_B(P)$. Hence $m_B(P) \leqq k_B(P)$. We then have equality and the proof is complete.

In particular, the numbers $| \mathfrak{K}_B(P) |$ in (3A) do not depend on the choice of $\mathfrak{K}_B$. As a corollary of (4A), we mention

(4B) *The number k_B of irreducible characters χ_i of G in the block B is given by*

$$k_B = \sum_P m_B(P); \qquad P \in \mathcal{P}(G). \tag{4.2}$$

For each P, the sum

$$\sum_B m_B(P); \qquad (B \in \mathcal{B}\ell(G)) \tag{4.3}$$

represents the number of conjugate classes of G with defect group P.

A re-examination of the proof of (3A) yields

(4C) *For any $B \in \mathcal{B}\ell(G)$ and any $Q \in \mathcal{P}(G)$*

$$m_B(Q) = \sum_b m_b(Q) \tag{4.4}$$

where b ranges over the blocks of $H = N_G(Q)$ with $b^G = B$.

Proof. It follows from (3.3) that, for each $B \in \mathcal{B}\ell(G)$ and each $b \in B_H$, we can find subsets Y_b of Y_B and $\mathfrak{L}_b$ of $\mathfrak{L}_B$ with $| Y_b | = | \mathfrak{L}_b |$ such that Y_B is the disjoint union of the Y_b, that $\mathfrak{L}_B$ is the disjoint union of the $\mathfrak{L}_b$ with b ranging over B_H and that for each b

$$\det (\varphi(L)) \neq 0; \qquad (\varphi \in Y_b, \quad L \in \mathfrak{L}_b).$$

We apply (3A) to the group $H = N_G(Q)$ instead of G. Let $\mathfrak{L}_b(Q)$ denote the set of those $L \in \mathfrak{L}_b$ which have defect group Q in H. Since $L^H = L$, we see that $\mathfrak{L}_b(Q)$ has the same significance for H and b as $\mathfrak{K}_B(Q)$ has for G and B. Hence by (3A)

$$| \mathfrak{L}_b(Q) | = m_b(Q).$$

Since $\mathfrak{L}_B(Q)$ in §3 is the disjoint union of the sets $\mathfrak{L}_b(Q)$ with $b \in B_H$ and since

$$| \mathfrak{K}_B(Q) | = | \mathfrak{L}_B(Q) |$$

cf. §3, (4.4) now is evident.

(4D) *The defect group D of B (in the sense of* [1]*) occurs in $\mathfrak{D}_B$. It is the unique maximal element of $\mathfrak{D}_B$ in the partial ordering of $\mathcal{P}(G)$.*

Proof. The algebra homomorphism ω_B in F_B (cf. §1) vanishes for all $K \in \mathcal{C}\ell(G)$ with $| D_K | < | D |$, but not for all K with $D_K = D$, [1, I, §8]. Hence $D \in \mathfrak{D}_B$.

On the other hand, if $P \in \mathfrak{D}_B$, there exist blocks b of $H = N_G(P)$ with $b^G = B$. Let d be a defect group of b in the sense of [1]. Since $P \lhd H$, then $P \subseteq d$, [1, I, (9F)] and d is conjugate in G to a subgroup of D, [1, II (2B)]. Hence $D \succeq P$ as stated.

If $d = P$, then $D = P$ by [1]. If $P \subset d$, there exist blocks b_0 of $N_H(d)$ with

$b_0^H = b$ and then $b_0^G = B$. Hence we have

(4E) *If B and D are as in* (4D) *and if P is a lower defect group of B with $P \neq D$, there exists a p-subgroup d of G with*

$$P \subset d \subseteq N_G(P)$$

and a block b_0 of $N_G(P) \cap N_G(d)$ with $b_0^G = B$.

We finally prove an extension of (4A).

(4F) *Suppose that for each block B of G we have a subset $\mathfrak{K}_B^*$ of $\mathcal{C}\ell(G)$ such that*

(i) *each $K \in \mathcal{C}\ell(G)$ belongs to at least one $\mathfrak{K}_B^*$.*

(ii)) *If $| \mathfrak{K}_B^* | = k_B^*$, there exists a subset U_B of F_B with $| U_B | = | k_B^* |$* and

$$\det (h(K)) \neq 0; \qquad (h \in U_B, \quad K \in \mathfrak{K}_B^*). \tag{4.5}$$

Then $k_B^* = k_B$ and exactly $m_B(Q)$ classes of $\mathfrak{K}_B^*$ have defect group Q; ($Q \in \mathcal{P}(G)$).

Proof. It follows from (i) that

$$\textstyle\sum_B k_B^* \geqq k(G) = \sum_B k_B ; \qquad (B \in \mathcal{B}\ell(G)).$$

On the other hand, (ii) implies that

$$k_B^* \leqq \dim F_B = k_B .$$

If we add over B, we conclude that $k_B^* = k_B$. Each $K \in \mathcal{C}\ell(G)$ belongs to exactly one $\mathfrak{K}_B^*$.

For any $Q \in \mathcal{P}(G)$, let $r_B(Q)$ denote the number of $K \in \mathfrak{K}_B^*$ with the defect group Q. Then

$$\textstyle\sum_B r_B(Q) = \sum_B m_B(Q); \qquad B \in \mathcal{B}\ell(G), \tag{4.6}$$

since on both sides, we have the number of conjugate classes of G with defect group Q.

If $r_B(Q) \neq m_B(Q)$ for some B and Q, choose a Q of maximal order for which this happens. On account of (4.6), we can then choose B such that

$$r_B(Q) < m_B(Q). \tag{4.7}$$

If $\{f_K\}$ has the same significance as in (3B), it follows from the assumption (ii) and $k_B^* = k_B$ that

$$\det (f_K(K^*)) \neq 0; \qquad (K \in \mathfrak{K}_B , K^* \in \mathfrak{K}_B^*). \tag{4.8}$$

Consider here the rows for which $D_K \geq Q$. By (3B) then $f_K(K^*) = 0$ except when

$$D_{K^*} \geq D_K \geq Q. \tag{4.9}$$

The number of rows in question is

$$R = \textstyle\sum_P m_B(P); \qquad (P \in \mathcal{P}(G), \quad P \geq Q).$$

As shown by (4.9), the non-zero coefficients in the rows occur in

$$C = \sum_P r_B(P); \qquad (P \in \mathcal{P}(G), \quad P \geq Q)$$

columns. Our choice of Q implies that $m_B(P) = r_B(P)$ for $|P| > |Q|$. By (4.7), $R > C$. But this is inconsistent with (4.8) and (4F) is proved.

5. The ideals I_Q of $Z(G)$

We shall give another characterization of the multiplicity $m_B(Q)$ of $Q \in \mathcal{P}(G)$ as lower defect groups of the block B. We first note

(5A) *Let $Q \in \mathcal{P}(G)$. Let K range over the conjugate classes of G which do not meet $T = C(Q)$. The corresponding class sums $\mathcal{S}K$ form the basis of an ideal I_Q of $Z(G)$.*

This is immediate since I_Q is the kernel of the homomorphism μ in (2.2) of $Z(G)$ into $Z(H)$; $H = N_G(Q)$, $T = C_G(Q)$.

(5B) *Let $B \in \mathcal{B}\ell(G)$. If I_Q is as in (5A),*

(5.1) $$\dim_\Omega (B \cap I_Q) = \sum' m_B(P)$$

where P in the sum ranges over the members of $\mathcal{P}(G)$ which do not contain a conjugate of Q.

Proof. Consider B as an algebra over Ω. Then $I = B \cap I_Q$ is an ideal of B. Let R denote the representation of B belonging to the B-module B/I. We then have $R(\zeta) = 0$ for $\zeta \in I$. Conversely, if $\zeta \in B$ and $R(\zeta) = 0$, then $B\zeta \subseteq I$. Since B has a unit element η_B, this implies $\zeta \in I$. Hence R has the kernel I.

Choose an Ω-basis of B/I and write R in matrix form. Each coefficient of R considered as a function of a variable element of B can be viewed as an element of the dual space $\hat{B}$ of B. Let W denote the subspace of $\hat{B}$ spanned by the different coefficients of R. Since R has the kernel I, we have

(5.2) $$\dim_\Omega W = \dim_\Omega (B/I) = k_B - \dim_\Omega I.$$

If $w \in W \subseteq \hat{B}$, we can consider w as an element of F_B. Then

$$w(\zeta) = w(\eta_B \zeta)$$

for $\zeta \in Z$. Express w by the basis $\{f_K\}$ in (3B),

(5.3) $$w = \sum_K w(K) f_K; \qquad (K \in \mathfrak{K}_B).$$

If here $\mathcal{S}K \in I_Q$, then $\eta_B(\mathcal{S}K) \in I$ and

$$w(K) = w(\eta_B(\mathcal{S}K)) = 0.$$

Therefore, it suffices to let K in (5.3) range over those elements K of $\mathfrak{K}_B$ which meet T. These are the K for which $D_K \geq Q$. Then

$$\dim_\Omega W \leqq \sum_P m_B(P); \qquad (P \in \mathcal{P}(G), \quad P \geq Q)$$

since the sum of the right represents the number of K in (5.3). By (5.2) and (5.3),

$$k_B - \sum_P m_B(P) \leqq \dim_\Omega I; \qquad (P \in \mathcal{P}(G), \quad P \geq Q).$$

Here the left side is equal to the sum in (5.1), cf. (4.2). Thus

$$\sum\nolimits_P' m_B(P) \leqq \dim_\Omega I = \dim_\Omega(I_Q \cap B). \tag{5.4}$$

Add here over all $B \in \mathcal{B}\ell(G)$. On the left, we obtain the number of $K \in \mathcal{C}\ell(G)$ whose defect group does not contain a conjugate of Q, cf. (4.3). By (5A), this is the dimension of I_Q. Since I_Q is the direct sum of the $I_Q \cap B$ for the different blocks, we have equality after adding (5.4) and hence equality in (5.4), Q.E.D.

It is clear from (5.1) that, for each $P \in \mathcal{P}(G)$, $m_B(P)$ can be expressed by the dimensions of the ideals $B \cap I_Q$ for suitable $Q \in \mathcal{P}(G)$ if k_B is known.

6. The p-sections of G

The p-sections of a group have been defined in §1. Each $\zeta \in Z = Z(G)$ has a unique representation

$$\zeta = \sum_K a_K(\mathcal{S}K); \qquad (K \in \mathcal{C}\ell(G)). \tag{6.1}$$

If π is a p-element of G, let $\zeta^{(\pi)}$ denote the sum of the terms in (6.1) for which K belongs to the section $\mathfrak{S}(\pi)$ of π. Then

$$\zeta = \sum_\pi \zeta^{(\pi)}; \qquad (\pi \in \Pi). \tag{6.2}$$

We note

(6A) *If ζ belongs to the block B of G, each $\zeta^{(\pi)}$ in* (6.2) *does.*

Indeed, since $\zeta = \eta_B \zeta$, we have

$$\zeta = \sum_\pi \eta_B \zeta^{(\pi)}; \qquad (\pi \in \Pi).$$

On account of (2H), each $\eta_B \zeta^{(\pi)}$ is a linear combination of class sums $\mathcal{S}K$ with $K \subseteq \mathfrak{S}(\pi)$. On comparing this with (6.2), we find

$$\zeta^{(\pi)} = \eta_B \zeta^{(\pi)} \in B.$$

We shall say that an element f of $\hat{Z}$ is *sectional* and is *associated with the section* $\mathfrak{S}(\pi)$, if $f(K) = 0$ for all $K \subseteq \mathcal{C}\ell(G)$ which are not contained in $\mathfrak{S}(\pi)$.

(6B) *The functions f_K in* (3B) *are sectional.*

Proof. Let K_0 be a fixed conjugate class, say $K_0 \subseteq \mathfrak{S}(\pi_0)$. Apply (2F) with $\mathfrak{K} = \mathfrak{K}_B$ and with F consisting of the f_K with $K \in \mathfrak{K}_B$. It follows that for $K \in \mathfrak{K}_B$ there exists elements $c_K^\theta \in \Omega$ such that, for every $f \in F_B$,

$$f(K_0) = \sum_K c_K^\theta f(K); \qquad (K \in \mathfrak{K}_B)$$

and that $c_K^\theta = 0$ if K is contained in a section $\mathfrak{S}(\pi) \neq \mathfrak{S}(\pi_0)$. Taking $f = f_K$ with $K \subseteq \mathfrak{S}(\pi)$, we see that $f_K(K_0) = 0$ as we wished to show.

We shall denote by $F_B^{(\pi)}$ the subset of F_B consisting of the functions $f \in F_B$, which are sectional and are associated with $\mathfrak{S}(\pi)$. As a corollary of (6B), we have

(6C) *The space F_B is the direct sum of the subspaces $F_B^{(\pi)}$ with π ranging over* Π.

Indeed, the functions f_K wiht $K \in \mathfrak{K}_B$ form a basis of F_B. Here f_K is sectional and, if $K \subseteq \mathfrak{S}(\pi)$, then as $f_K(K) = 1, f_K$ is associated with $\mathfrak{S}(\pi)$ and hence $f_K \in F_B^{(\pi)}$.

We now replace F_B by $F_B^{(\pi)}$ in the definition of $m_B(P)$ in §1. For given $B \in \mathcal{B}\ell(G)$, $P \in \mathcal{P}(G)$ and $\pi \in \Pi$, we consider subspaces V of $F_B^{(\pi)}$ such that every $f \neq 0$ in V has the following properties:

(i) There exist conjugate classes K with the defect group P such that $f(K) \neq 0$.

(ii) We have $f(K) = 0$ for every $K \in \mathcal{C}\ell(G)$ for which $|D_K| < |P|$.

Of course, it will suffice here to consider only classes $K \subseteq \mathfrak{S}(\pi)$.

We denote by $m_B^{(\pi)}(P)$ the maximal dimension of a space V with the properties (i), (ii). Let $\mathfrak{D}_B^{(\pi)}$ be the system of groups consisting of the groups $P \in \mathcal{P}(G)$ with each P taken with the multiplicity $m_B^{(\pi)}(P)$. Now a proof quite analogous to that of (4A) yields

(6D) *Let $\mathfrak{K}_B$ be as in* (3A) *and let $\mathfrak{K}_B^{(\pi)}$ be the subset consisting of the $K \in \mathfrak{K}_B$ which are contained in the section $\mathfrak{S}(\pi)$, $\pi \in \Pi$. Then $\mathfrak{D}_B^{(\pi)}$ consists exactly of the defect group D_K of the $K \in \mathfrak{K}_B^{(\pi)}$.*

We shall say that $\mathfrak{D}_B^{(\pi)}$ is the system of lower defect groups of B *associated with the section* $\mathfrak{S}(\pi)$. Now (6D) yields the following:

(6E) *The system $\mathfrak{D}_B$ is the union of the systems $\mathfrak{D}_B^{(\pi)}$ for all $\pi \in \Pi$. In other words*

$$m_B(P) = \textstyle\sum_\pi m_B^{(\pi)}(P); \qquad (\pi \in \Pi) \tag{6.3}$$

for each $P \in \mathcal{P}(G)$. If we set $|\mathfrak{K}_B^{(\pi)}| = l_B^{(\pi)}$, we have

$$l_B^{(\pi)} = \textstyle\sum_P m_B^{(\pi)}(P); \qquad (P \in \mathcal{P}(G)). \tag{6.4}$$

The number of conjugate classes $K \subseteq \mathfrak{S}(\pi)$ with a given defect group P is given by

$$\textstyle\sum_B m_B^{(\pi)}(P); \qquad (B \in \mathcal{B}\ell(G)). \tag{6.5}$$

We can also extend (4F).

(6F) *Suppose that for each $B \in \mathcal{B}\ell(G)$ and each $\pi \in \Pi$, we have a set $\mathfrak{K}_B^*$ of conjugate classes $K \subseteq \mathfrak{S}(\pi)$ such that each $K \in \mathcal{C}\ell(G)$ contained in the section $\mathfrak{S}(\pi)$ belongs to $\mathfrak{K}_B^*$ for some B. Suppose further that for each B, we have* *a subset* U_B *of* $F_B^{(\pi)}$ with $|U_B| = |\mathfrak{K}_B^*|$ such that

$$\det(h(K)) \neq 0; \qquad (h \in U_B, \quad K \in \mathfrak{K}_B^*).$$

Then $|\mathfrak{K}_B^*| = l_B^{(\pi)}$ and exactly $m_B^{(\pi)}(Q)$ classes $K \in \mathfrak{K}_B^*$ have defect group Q; $(Q \in \mathcal{P}(G))$.

This is shown by the same method as (4F) considering only classes contained in the section $\mathfrak{S}(\pi)$ and replacing $m_B(Q)$ by $m_B^{(\pi)}(Q)$ and F_B by $F_B^{(\pi)}$.

Our next result shows that the numbers $m_B^{(\pi)}(P)$ can be expressed by the analogous numbers with G replaced by $C_G(\pi)$ and B replaced by blocks of $C_G(\pi)$.

(6G) *Let* $\pi \in \Pi$ *and set* $C = C_G(\pi)$. *For each* $B \in \mathcal{B}\ell(G)$ *and each* $P \in \mathcal{P}(G)$, *we have*

$$m_B^{(\pi)}(P) = \textstyle\sum_Q \sum_b m_b^{(\pi)}(Q) \tag{6.6}$$

where Q *ranges over those groups in* $\mathcal{P}(C)$ *which are conjugate to* P *in* G *and where* b *ranges over the set* B_C *of blocks* b *of* C *with* $b^G = B$.

Proof. We apply here the method of §2, **1** with $Q = \langle \pi \rangle$, $T = H = C$. Let $\mathfrak{S}_C(\pi)$ denote the p-section of π in C. If $K \in \mathcal{C}\ell(G)$ and $K \subseteq \mathfrak{S}(\pi)$, set

$$K \cap \mathfrak{S}_C(\pi) = L. \tag{6.7}$$

Then L is not empty and L consists of elements of the form $\pi\rho$ with p-regular $\rho \in C$. Any two elements of L are conjugate in G. It follows from their form that they are conjugate in C. It is now evident that L is a conjugate class of C; $L \subseteq \mathfrak{S}_C(\pi)$. Conversely, if $L \in \mathcal{C}\ell(C)$ and $L \subseteq \mathfrak{S}_C(\pi)$, the conjugate class $K = L^G$ of G containing L belongs to $\mathfrak{S}(\pi)$ and satisfies (6.7). Hence we have a one-to-one correspondence between the set of $K \in \mathcal{C}\ell(G)$ with $K \subseteq \mathfrak{S}(\pi)$ and the set of $L \in \mathcal{C}\ell(C)$ with $L \subseteq \mathfrak{S}_C(\pi)$. Moreover, if $Q \in \mathcal{P}(C)$ is the defect group of L in C, the defect group P of $K = L^G$ in G is conjugate to Q.

We now use a method similar to that used in the proof of (3A). If $b \in \mathcal{B}\ell(C)$, let F_b be the subspace of the dual space $\hat{Z}(C)$ of $Z(C)$ defined in a manner analogous to the definition of F_B in $\hat{Z}(G)$. Let Y_b denote a basis of F_b consisting of sectional functions, (cf. (6C)). Let Y_B be the union of all Y_b with $b^G = B$ and let Y be the union of Y_B for all $B \in \mathcal{B}\ell(G)$. Then Y is a basis of $\hat{Z}(C)$ and hence

$$\det(\varphi(L)) \neq 0; \qquad (\varphi \in Y, \quad L \in \mathcal{C}\ell(C)).$$

It follows that, for each B, we can select a subset $\mathfrak{L}_B$ of $\mathcal{C}\ell(C)$, such that $\mathcal{C}\ell(C)$ is the disjoint union of the sets $\mathfrak{L}_B$, that $|Y_B| = |\mathfrak{L}_B|$, and that

$$\det(\varphi(L)) \neq 0; \qquad (\varphi \in Y_B, \quad L \in \mathfrak{L}_B).$$

Let $\mathfrak{L}_B^{(\pi)}$ denote the set of all $L \in \mathfrak{L}_B$ such that $L \subseteq \mathfrak{S}_C(\pi)$. We can then find a subset $Y_B^{(\pi)}$ with $| Y_B^{(\pi)} | = | \mathfrak{L}_B^{(\pi)} |$ such that

$$(6.8) \qquad \det(\varphi(L)) \neq 0; \qquad (\varphi \in Y_B^{(\pi)}, \quad L \in \mathfrak{L}_B^{(\pi)}).$$

Since all $\varphi \in Y_B^{(\pi)}$ are sectional, it follows from (6.8) that they are associated with the section $\mathfrak{S}_C(\pi)$. Let $X_B^{(\pi)}$ denote the system of functions φ^λ with $\varphi \in Y_B^{(\pi)}$ and λ defined by (2.3) with $T = H = C$. Each $f = \varphi^\lambda$ belongs to F_B, cf. (2A). If $K \in \mathcal{C}\ell(G)$, by (2.2) and (2.3)

$$f(K) = \varphi(\mathcal{S}(K \cap C)).$$

If K is not contained in $\mathfrak{S}(\pi)$, then K does not meet $\mathfrak{S}_C(\pi)$ and hence $f(K) = 0$. If $K \subseteq \mathfrak{S}(\pi)$, $K \cap C$ is the union of L in (6.7) and of conjugate classes $L^* \in \mathcal{C}\ell(C)$ contained in sections of C different from $\mathfrak{S}_C(\pi)$. Since $\varphi(L^*) = 0$ for these L^*, we find $f(K) = \varphi(L)$.

If $\mathfrak{K}_B^*$ is the set of classes L^G with $L \in \mathfrak{L}_B^{(\pi)}$, we then have

$$\det(f(K)) \neq 0; \qquad (f \in X_B^{(\pi)}, \quad K \in \mathfrak{K}_B^*).$$

Since every $L \in \mathcal{C}\ell(C)$ with $L \subseteq \mathfrak{S}_C(\pi)$ belongs to $\mathfrak{L}_B^{(\pi)}$ for some $B \in \mathcal{B}\ell(G)$, every $K \in \mathcal{C}\ell(G)$ with $K \subseteq \mathfrak{S}(\pi)$ belongs to some $\mathfrak{K}_B^*$. We can now apply (6F) and see that exactly $m_B^{(\pi)}(P)$ classes $K \in \mathfrak{K}_B^*$ have defect group P. It follows that exactly $m_B^{(\pi)}(P)$ of the classes $L \in \mathfrak{L}_B^{(\pi)}$ have defect groups Q in C with Q conjugate to P. On the other hand, if we set

$$Y_b^{(\pi)} = Y_B^{(\pi)} \cap Y_b$$

for $b \in B_C$, it follows from (6.8) that we can break up $\mathfrak{L}_B^{(\pi)}$ into subsets $\mathfrak{L}_b^{(\pi)}$ with $b \in B_C$ such that $| Y_b^{(\pi)} | = | L_b^{(\pi)} |$ and

$$\det(\varphi(L)) \neq 0; \qquad (\varphi \in Y_b^{(\pi)}, \quad L \in \mathfrak{L}_b^{(\pi)}).$$

Applying (6F) to C and b, we see that $m_b^{(\pi)}(Q)$ of the classes $L \in \mathfrak{L}_b^{(\pi)}$ have defect group $Q \in \mathcal{P}(C)$. It is now clear that (6.6) holds.

7. The section of the unit element

(7A) *Assume that π is a p-element in the center $Z(G)$ of G. Let B be a block of G. Let $\mathfrak{K}_B$ have the same significance as in* (3A). *As before, let $\mathfrak{K}_B^{(\pi)}$ denote the set of classes $K \in \mathfrak{K}_B$ with $K \subseteq \mathfrak{S}(\pi)$ and set $| \mathfrak{K}_B^{(\pi)} | = l_B^{(\pi)}$. We can find a set X_B of $l_B^{(\pi)}$ irreducible characters χ_i in B such that*

$$(7.1) \qquad \det(\chi_i(\sigma_K)) \not\equiv 0 \pmod{\mathfrak{p}}; \qquad (\chi_i \in X_B, \quad K \in \mathfrak{K}_B^{(\pi)}).$$

Here, $\mathfrak{p}$ has the same meaning as in §2, 2. Moreover, $l_B^{(\pi)}$ coincides with the number l_B of modular irreducible characters in B. Finally, $\mathfrak{D}_B^{(\pi)} = \mathfrak{D}_B^{(1)}$.

Proof. As shown in [1, I (5A)] there exists a set X_B of l_B irreducible characters $\chi_i \in B$ and a set $\mathfrak{L}_B^{(1)}$ of l_B conjugate classes $K \subseteq \mathfrak{S}(1)$ such that each conjugate class in $\mathfrak{S}(1)$ belongs to $\mathfrak{L}_B^{(1)}$ for exactly one B, and that for every B

$$\det(\chi(\sigma_K)) \not\equiv 0 \pmod{\mathfrak{p}}; \qquad (\chi \in X_B, \quad K \in \mathfrak{L}_B^{(1)}).$$

The section $\mathfrak{S}(1)$ consists of the p-regular elements of G. If K ranges over the classes in $\mathfrak{S}(1)$, πK ranges over the classes in $\mathfrak{S}(\pi)$. Let $\mathfrak{L}_B^{(\pi)}$ denote the set of classes πK with $K \in \mathfrak{L}_B^{(1)}$. Since

$$\chi(\sigma_K) \equiv \chi(\sigma_{\pi K}) \pmod{\mathfrak{p}}$$

we have

$$(7.2) \qquad \det(\chi(\sigma_K)) \not\equiv 0 \pmod{\mathfrak{p}}; \qquad (\chi \in X_B, \quad K \in \mathfrak{L}_B^{(\pi)})$$

Let $r_B^{(\pi)}(P)$ denote the number of classes in $\mathfrak{L}_B^{(\pi)}$ with the given defect group $P \in \mathcal{P}(G)$. Then

$$(7.3) \qquad \textstyle\sum_B r_B^{(\pi)}(P) = \sum_B m_B^{(\pi)}(B); \qquad (B \in \mathcal{B}\ell(G))$$

since both sides represent the number of classes $K \subseteq \mathfrak{S}(\pi)$ with defect group P, cf. (6.5).

If some class $K \in \mathfrak{L}_B^{(\pi)}$ does not belong to $\mathfrak{K}_B^{(\pi)}$, we try to replace it by a class in $\mathfrak{K}_B^{(\pi)}$ with the same defect group such that the condition (7.2) is preserved after the replacement. We continue in this manner as long as possible.

Assume first that, for every $\pi \in Z(G)$ and for every $B \in \mathcal{B}\ell(G)$, this process only comes to an end when all classes in $\mathfrak{L}_B^{(\pi)}$ have been replaced by classes in $\mathfrak{K}_B^{(\pi)}$. Then, obviously,

$$r_B^{(\pi)}(P) \leqq m_B^{(\pi)}(P)$$

and (7.3) implies that we have equality. This means that we have replaced $\mathfrak{L}_B^{(\pi)}$ by $\mathfrak{K}_B^{(\pi)}$ and hence (7.1) holds. Also,

$$l_B^{(\pi)} = |\,\mathfrak{K}_B^{(\pi)}\,| = |\,\mathfrak{L}_B^{(\pi)}\,| = l_B\,.$$

Since the classes K and πK have the same defect group, we have $r_B^{(\pi)}(P) = r_B^{(1)}(P)$. Since our result above can be applied for $\pi = 1$, we find

$$m_B^{(\pi)}(P) = r_B^{(\pi)}(P) = r_B^{(1)}(P) = m_B^{(1)}(P)$$

Hence $\mathfrak{D}_B^{(\pi)} = \mathfrak{D}_B^{(1)}$, and (7A) holds in the case under discussion.

Assume then that for some $\pi \in Z(G)$ our exchange comes to an end before all classes in $\mathfrak{L}_B^{(\pi)}$ have been replaced. Let H_B denote the set obtained from $\mathfrak{L}_B^{(\pi)}$ when the process terminates. All classes in H_B lie in $\mathfrak{S}(\pi)$. Exactly $r_B^{(\pi)}(P)$ classes in H_B have defect group P, we have $|\,H_B\,| = l_B$ and

$$(7.4) \qquad \Delta = \det(\chi(\sigma_K)) \not\equiv 0 \pmod{\mathfrak{p}}; \qquad (\chi \in X_B, \quad K \in H_B).$$

Finally, for some B, there exist classes $K_0 \in H_B$ which do not belong to $\mathfrak{K}_B^{(\pi)}$ and which cannot be exchanged with a class in $\mathfrak{K}_B^{(\pi)}$ with the same defect group such that (7.4) is preserved. Choose here B and K_0 such that the defect group Q of K_0 has maximal order.

If $P \in \mathcal{P}(G)$ and $|\,P\,| > |\,Q\,|$, our choice implies that, for every block B_1 and every $K \in H_{B_1}$ with $D_K = P$, we have $K \in \mathfrak{K}_{B_1}^{(\pi)}$. This implies that $r_{B_1}^{(\pi)}(P) \leqq m_{B_1}^{(\pi)}(P)$. On account of (7.3), we have equality. This shows that, for $|P| > |Q|$, the same classes of defect group P occur in H_{B_1} and in $\mathfrak{K}_{B_1}^{(\pi)}$.

We shall now derive a contradiction. It follows from (2F) that there exist elements $c_K \in \mathfrak{o}$ for $K \in \mathfrak{K}_B^{(\pi)}$ such that

$$(7.5) \qquad \omega_j(K_0) = \sum_K c_K\, \omega_j(K); \qquad (K \in \mathfrak{K}_B^{(\pi)})$$

for each ω_j associated with B. Then, by (1.4)

$$(7.6) \quad \chi_j(\sigma_{K_0}) = \sum_K c_K(|K|/|K_0|)\chi_j(\sigma_K); \qquad (K \in \mathfrak{K}_B^{(\pi)}).$$

Let Δ_{K_1} denote the determinant obtained form Δ in (7.4) by replacing the column $\chi(\sigma_{K_0})$ by $\chi(\sigma_{K_1})$ with $K_1 \in \mathfrak{K}_B^{(\pi)}$. On account of (7.6), then

$$(7.7) \qquad \Delta = \sum_K c_K(|K|/|K_0|)\Delta_K; \qquad (K \in \mathfrak{K}_B^{(\pi)}).$$

If here K has a defect group D_K with $|D_K| > |Q|$, then as remarked, $K \in H_B$ and hence $\Delta_K = 0$ since two columns are equal. It will therefore suffice to let K range over the classes with $|D_K| \leqq |Q|$. Since $D_{K_0} = Q$, then $|K|/|K_0| \in \mathfrak{o}$. It then follows from (7.7) that there exist classes $K_1 \in \mathfrak{K}_B^{(\pi)}$ for which

$$(7.8) \qquad c_{K_1} \not\equiv 0, \quad |\Delta_{K_1}| \not\equiv 0 \pmod{\mathfrak{p}}, \quad |D_{K_1}| = |Q|.$$

In the notation of §2, **2**, the equation (7.5) remains valid if we replace ω_j by an element of M_B. Then (2F) shows that

$$f(K_0) = \sum_K \overset{\theta}{c}_K f(K); \qquad (K \in \mathfrak{K}_B^{(\pi)})$$

for any $f \in F_B$. For $f = f_{K_1}$, this yields

$$\overset{\theta}{c}_{K_1} = f_{K_1}(K_0)$$

By (7.8), then $f_{K_1}(K_0) \neq 0$ and by (3B) $Q \succeq D_{K_1}$. Now (7.8) shows that $D_{K_1} = Q$ and that we could have exchanged K_0 with $K_1 \in \mathfrak{K}_B^{(\pi)}$ since both have the same defect group and since (7.4) would be preserved. This is a contradiction and the proof is complete.

If π is an arbitrary p-element of the group G, we can apply (7A) to the group $C = C_G(\pi)$. If $b \in \mathcal{B}\ell(C)$ and if $Q \in \mathcal{P}(C)$, then $\mathfrak{D}_b^{(1)} = \mathfrak{D}_b^{(\pi)}$ and hence $m_b^{(1)}(Q) = m_b^{(\pi)}(Q)$. Now (6G) becomes

(7B) *Let π be a p-element of G and set $C = C_G(\pi)$. For each $B \in \mathcal{B}\ell(G)$ and each $P \in \mathcal{P}(G)$,*

$$(7.9) \qquad m_B^{(\pi)}(P) = \sum_Q \sum_b m_b^{(1)}(Q)$$

where Q ranges over the groups in $\mathcal{P}(C)$ which are conjugate to P and where b ranges over the blocks of C with $b^G = B$.

By (7A), $l_b^{(1)} = l_b$ is the number of modular irreducible characters in b. If we add (7.9) over all $P \in \mathcal{P}(G)$, (6E) yields the corollary:

(7C) *If the notation is as in* (7B), *we have*

(7.10) $$l_B^{(\pi)} = \sum_b l_b$$

where b ranges over the blocks of C with $b^G = B$.

On comparing (7.10) with [1, II, §7] we have

(7D) *The number $l_B^{(\pi)}$ of lower defect groups of B associated with the section $\mathfrak{S}(\pi)$ is equal to the number of modular irreducible characters of $C_G(\pi)$ which belong to B in the sense of* [1, II, §7].

In particular, there are l_B lower defect groups of B associated with the section $S(1)$. On account of (6E), we have

(7E) *The number l_B of modular irreducible characters in the block B is given by*

(7.11) $$l_B = l_B^{(1)} = \sum_P m_B^{(1)}(P); \qquad (P \in \mathcal{P}(G)).$$

The proposition (7A) can be applied for $\pi = 1$ for any G. Hence

(7F) *For every $B \in \mathcal{B}\ell(G)$, we have*

$$\det(\chi(\sigma_K)) \not\equiv 0 \pmod{\mathfrak{p}}; \qquad (\chi \in X_B\ ; \quad K \in \mathfrak{K}_B^{(1)}).$$

Since every p-regular class of G belongs to $\mathfrak{K}_B^{(1)}$ for exactly one B, we can apply the method in (1, I, §5) and obtain

(7G) *Let B be a block. Let $r \geqq 0$ be a rational integer. The multiplicity of p^r as an elementary divisor of the Cartan matrix of B is given by*

$$\sum\nolimits_P' m_B^{(1)}(P)$$

where P ranges over all groups in $\mathcal{P}(G)$ of order p^r.

This is a refinement of a result stated without proof in [2].

As a consequence of (7G), we have

(7H) *If B has defect group D (in the sense of* [1]*) with D chosen in $\mathcal{P}(G)$), then D occurs exactly once in $\mathfrak{D}_B^{(1)}$.*

The results of [3] show that D occurs in $\mathfrak{D}_B^{(\pi)}$ for $\pi \in \Pi$, if and only if π is conjugate to an element of $Z(D)$; they also allow us to characterize the multiplicity $m_B^{(\pi)}(D)$. For arbitrary $\pi \in \Pi$, we can determine the maximal elements of $\mathfrak{D}_B^{(\pi)}$.

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