PRINCIPAL FACTORS, MAXIMAL SUBGROUPS AND CONDITIONAL IDENTITIES OF FINITE GROUPS

BY
REINHOLD BAER

The three principles for the classification of finite groups that we are going to discuss here may be loosely described as follows:

A principal factor of a finite group G is just a minimal normal subgroup M of an epimorphic image H of G. [Jordan-Hölder's Theorem determines the multiplicity of a principal factor which will not concern us here.] Two structural invariants may be derived from this principal factor: the structure of M as an abstract group and the group $\Gamma = \Gamma_H M$ of automorphisms, induced in M by H.

Typical examples. The group G is nilpotent if, and only if, $\Gamma_H M = 1$ for every principal factor. The group G is supersoluble if, and only if, every principal factor M is cyclic of order a prime.

It should be noted as a feature of particular interest that such a class may be described in essentially different fashions.

Example. G is supersoluble if, and only if, $\Gamma_H M$ is cyclic of exponent $p - 1$ whenever the order of the principal factor M is a multiple of the prime p.

The point of view indicated here is closely related with Gaschütz' locally defined formations.

It should be noted that the structure of a principal factor and of the group of automorphisms, induced in it, are not at all independent. Example: Such a group of automorphisms is cyclic if it is abelian.

Maximal subgroups. We just quote two typical examples: Wielandt's Theorem that a group is nilpotent if, and only if, its maximal subgroups are normal; and Huppert's Theorem that a group is supersoluble if, and only if, its maximal subgroups are of index a prime.

Conditional identities. Noting that the only variety [= class of groups, defined by identical relations (B. H. Neumann)] which contains so important a class as the class of all finite p-groups is the variety of all groups, we have to look for something less restrictive. The following theorems may indicate the direction in which to look: The group G is nilpotent if, and only if, $xy = yx$ whenever the elements x and y in G have relatively prime order. The group G is supersoluble if, and only if, $x^{p-1}y = yx^{p-1}$ whenever y is an element in G' of order a power of p and x is an element in G of order prime to p.

Immersion. If a class Λ of finite groups has been defined by defining some requirements on the principal factors, then this leads to the concept of Λ-immersion of a normal subgroup N of G by imposing these requirements only on those principal factors of G which are contained [covered by] N.

Examples. The normal subgroups which are part of the hypercenter are "nilpotently immersed"; and supersoluble immersion has been investigated under just this name.

This generalization appears to be justified by the following well known result: G is nilpotent if, and only if, every primary subgroup of G is "nilpotently immersed" in its normalizer. Likewise we should quote the theorem that a group is supersoluble if, and only if, its primary subgroups are supersolubly immersed in their normalizers.

Every immersion principle Δ leads us to a class of groups: the groups that are Δ-immersed in themselves. It is worth noting that essentially different immersion principles may lead in this fashion to the same class of groups; see Remark 7.6, **D**.

In this preliminary and somewhat cursory discussion we have restricted ourselves to finite groups only and by and large our results will be most striking when restricted to finite groups. But for the more general parts of our discussion such a restriction will not be needed at all [§§1, 2]; and in almost all our discussion of immersion it will suffice to assume the finiteness of the immersed normal subgroup.

In §3 we prove and discuss a general theorem, reducing immersion properties of a normal subgroup N in G to the corresponding immersion properties of the primary subgroups of N in their normalizers [Theorem 3.4].

In §4 a quite simplified type of immersion is discussed: no requirement is imposed upon the structure of the principal factors and the induced groups of automorphisms are all required to belong to one and the same class α of groups. This paves the way for §5 where this class α is supposed to be the class of finite abelian groups. The type of immersion obtained is subdivided into subtypes; and the general theory is then executed in some detail for these subtypes. One of them is supersoluble immersion. Here again more can be said and this is done in §6.

In §8 quite a different type of immersion is discussed. It stems from the concept of dispersed groups and shows some new phenomena. The results of §7, discussing dispersion, prepare the way for this part of our discussion.

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Notations. $\{\cdots\}$ = subgroup, generated by the enclosed subset.
$x \circ y = x^{-1}y^{-1}xy = x^{-1}x^{y}$.
$X \circ Y$ = *set* of all $x \circ y$ for x in X and y in Y.
$X' = \{X \circ X\}$.
S_G = core of the subgroup S in $G = \bigcap_{x \in G} S^{x}$
= product of all normal subgroups X of G with $X \subseteq S$.
$\mathfrak{c}_X Y$ = centralizer of Y in X.
$\mathfrak{z} X$ = center of X.
$\mathfrak{n}_X Y$ = normalizer of Y in X.

$\mathfrak{h}X$ = hypercenter of X = intersection of all normal subgroups Y of X with $\mathfrak{z}(X/Y) = 1$.

$\mathfrak{e}X$ = intersection of all normal subgroups Y of X such that X/Y is an $\mathfrak{e}$-group.

ΦG = Frattini subgroup of G = intersection of all maximal subgroups of G.

factor of a group G = epimorphic image of a subgroup of G.

$o(x)$ = order of the element x.

$o(G)$ = order of the finite group G.

p-element = element of order a power of p.

p-group = group all of whose elements are p-elements.

p'-element = element whose order is prime to p.

p'-group = group all of whose elements are p'-elements.

$\mathfrak{T}$ = trivial class, consisting of 1 only.

$\mathfrak{U}$ = universal class, containing all groups.

$X \subset Y$ signifies that X is a proper subgroup of Y.

1. We denote throughout by θ a class of ordered pairs $(\mathfrak{A}, \mathfrak{B})$ of group-theoretical properties $\mathfrak{A}$ and $\mathfrak{B}$. This class may consist of one pair only and it will usually be a countably infinite set.

We shall say that N is a *θ-immersed normal subgroup of* G, in short $N \; \theta \; G$, if N is a normal subgroup of the group G, meeting the following requirement:

If σ is an epimorphism of G upon H with $N^\sigma \neq 1$, then there exists a pair $(\mathfrak{A}, \mathfrak{B})$ in θ and a normal subgroup M of H with $1 \subset M \subseteq N^\sigma$ such that M is an $\mathfrak{A}$-group and $H/\mathfrak{c}_H M$ is a $\mathfrak{B}$-group.

It is clear that $1 \; \theta \; G$ is always true. We recall that $H/\mathfrak{c}_H M$ is essentially the same as the group of automorphisms, induced by H in M. If θ consists of the one pair $(\mathfrak{A}, \mathfrak{B})$ only, then we say $(\mathfrak{A}, \mathfrak{B})$-immersed and $N \; (\mathfrak{A}, \mathfrak{B}) \; G$ instead of θ-immersed and $N \; \theta \; G$.

Finally, we shall term G a *θ-group whenever $G \; \theta \; G$.*

PROPOSITION 1.1. *The relation $X \; \theta \; Y$ has the following properties:*

(1) *If σ is an epimorphism of G upon H, then $N \; \theta \; G$ implies $N^\sigma \; \theta \; H$.*

(2) *If A and B are normal subgroups of G with $A \subseteq B$, if $A \; \theta \; G$ and $B/A \; \theta \; G/A$, then $B \; \theta \; G$.*

(3) *Products of θ-immersed normal subgroups are θ-immersed normal subgroups.*

Proof. (1) is an almost immediate consequence of the definition of immersion together with the remark that products of epimorphisms are epimorphisms.

Suppose next that A and B are normal subgroups of G with $A \subseteq B$ and $A \; \theta \; G$, $B/A \; \theta \; G/A$. If σ is an epimorphism of G upon H with $B^\sigma \neq 1$, then we distinguish two possibilities: If firstly $A^\sigma \neq 1$, then we deduce from $A \; \theta \; G$ the existence of a pair $(\mathfrak{A}, \mathfrak{B})$ in θ and a normal subgroup M of H with

$1 \subset M \subseteq A^\sigma \subseteq B^\sigma$ such that M is an $\mathfrak{A}$-group and $H/\mathfrak{c}_H M$ is a $\mathfrak{B}$-group. If secondly $A^\sigma = 1$, then σ induces an epimorphism λ of G/A upon H; and we deduce from $B/A \ \theta \ G/A$ and $1 \neq B^\sigma = B^\lambda$ the existence of a pair $(\mathfrak{A}, \mathfrak{B})$ and a normal subgroup M of H with $1 \subset M \subseteq B^\lambda = B^\sigma$ such that M is an $\mathfrak{A}$-group and $H/\mathfrak{c}_H M$ is a $\mathfrak{B}$-group. This proves $B \ \theta \ G$.

Denote by $\mathfrak{M}$ a set of θ-immersed normal subgroups of G and let P be the product of the normal subgroups in $\mathfrak{M}$. Consider an epimorphism σ of G upon H with $P^\sigma \neq 1$. Then there exists a normal subgroup N in the set $\mathfrak{M}$ with $N^\sigma \neq 1$. From $N \ \theta \ G$ we deduce the existence of a pair $(\mathfrak{A}, \mathfrak{B})$ in θ and a normal subgroup M of H with $1 \subset M \subseteq N^\sigma \subseteq P^\sigma$ such that M is an $\mathfrak{A}$-group and $H/\mathfrak{c}_H M$ is a $\mathfrak{B}$-group. Hence $P \ \theta \ G$.

We define the *θ-hypercenter* $\mathfrak{h}_\theta G$ as the product of all θ-immersed normal subgroups of G. This is a well determined characteristic subgroup of G [since $1 \ \theta \ G$].

COROLLARY 1.2. *The θ-hypercenter has the following properties*:

(1) *$\mathfrak{h}_\theta G$ is a θ-immersed characteristic subgroup of G.*

(2) $\mathfrak{h}_\theta[G/\mathfrak{h}_\theta G] = 1$.

(3) *$\mathfrak{h}_\theta G$ is the intersection of all normal subgroups X of G with $\mathfrak{h}_\theta(G/X) = 1$.*

(4) *$(\mathfrak{h}_\theta G)^\sigma \subseteq \mathfrak{h}_\theta(G^\sigma)$ for every homomorphism σ of G.*

Proof. (1) is an immediate consequence of Proposition 1.1, (3). There exists one and only one characteristic subgroup C of G with $h_\theta G \subseteq C$ and $C/\mathfrak{h}_\theta G = \mathfrak{h}_\theta[G/\mathfrak{h}_\theta G]$. It is a consequence of (1) that $\mathfrak{h}_\theta G \ \theta \ G$ and $C/\mathfrak{h}_\theta G \ \theta \ G/\mathfrak{h}_\theta G$. Thus it follows from Proposition 1.1, (2) that $C \ \theta \ G$; and this implies $C \subseteq \mathfrak{h}_\theta G$ because of the definition of the θ-hypercenter. Hence $\mathfrak{h}_\theta[G/\mathfrak{h}_\theta G] = C/\mathfrak{h}_\theta G = 1$, proving (2). —(4) is an immediate consequence of (1) and Proposition 1.1, (1).

To prove (3) we denote by J the intersection of all the normal subgroups X of G with $\mathfrak{h}_\theta(G/X) = 1$. It is an immediate consequence of (2) that $J \subseteq \mathfrak{h}_\theta G$. If X is a normal subgroup of G with $\mathfrak{h}_\theta(G/X) = 1$, then we deduce from (4) that

$$X\mathfrak{h}_\theta G/X \subseteq \mathfrak{h}_\theta(G/X) = 1.$$

Hence $\mathfrak{h}_\theta G \subseteq X$ for all these X, proving $\mathfrak{h}_\theta G \subseteq J$. Thus $J = \mathfrak{h}_\theta G$, proving (3).

Remark 1.3. Assume that θ consists of the one pair $(\mathfrak{U}, \mathfrak{T})$ only where as always

$$\mathfrak{T} = \text{trivial class, consisting of 1 only,}$$

$$\mathfrak{U} = \text{universal class, containing all groups.}$$

If X is a normal subgroup of the group Y such that X is a $\mathfrak{U}$-group and $Y/\mathfrak{c}_Y X$ is a $\mathfrak{T}$-group, then $Y/\mathfrak{c}_Y X = 1$ so that $Y = \mathfrak{c}_Y X$ and $X \subseteq \mathfrak{z}Y$. It follows from Corollary 1.2, (3) that

$$\mathfrak{h}_{(\mathfrak{U}, \mathfrak{T})} G = \mathfrak{h}_\theta G = \mathfrak{h}G = \text{hypercenter of} \quad G.$$

This may serve as a justification for the term θ-hypercenter.

Proposition 1.1, (1) may be expressed in the form: θ-immersion is epimorphism inherited. —Similarly we shall say that *θ-immersion is factor inherited,* if θ meets the following requirement:

If N is a θ-immersed normal subgroup of G and if T is a normal subgroup of the subgroup S of G with $T \subseteq N$, then $T\ \theta\ S$.

A justification for the expression "θ-immersion is factor inherited" will be found in the criterion of Proposition 1.5 below.

PROPOSITION 1.4. *If θ-immersion is factor inherited, then*

(a) *θ-immersed normal subgroups are θ-groups and*

(b) *if A, B are normal subgroups of G with $A \subseteq B$, then $B\ \theta\ G$ is necessary and sufficient for $A\ \theta\ G$ and $B/A\ \theta\ G/A$.*

Proof. If N is a θ-immersed normal subgroup of G, then letting $S = T = N$ in our definition we obtain $N\ \theta\ N$ so that N is a θ-group. —Assume next that A and B are normal subgroups of G with $A \subseteq B$. It is a consequence of Proposition 1.1, (2) that $A\ \theta\ G$ together with $B/A\ \theta\ G/A$ implies $B\ \theta\ G$. If conversely $B\ \theta\ G$, then we deduce $B/A\ \theta\ G/A$ from Proposition 1.1, (1); and we deduce $A\ \theta\ G$ by letting $N = B$, $S = G$ and $T = A$ in our definition of factor inheritance of θ-immersion.

The group theoretical property $\mathfrak{E}$ is said to be *factor inherited,* if every factor [= epimorphic image of a subgroup] of an $\mathfrak{E}$-group is an $\mathfrak{E}$-group.

PROPOSITION 1.5. *θ-immersion is factor inherited, if for every pair $(\mathfrak{A}, \mathfrak{B})$ in θ the properties $\mathfrak{A}$ and $\mathfrak{B}$ are factor inherited.*

Proof. We shall first treat two special cases and then reduce the general case to these two special cases.

Case 1. Assume that $N\ \theta\ G$ and that $U \subseteq G$.

Consider a normal subgroup V of U with $V(U \cap N)/V \neq 1$. This last condition is equivalent to $U \cap N \not\subseteq V$. There exist normal subgroups X of G with $(U \cap N) \cap X \subseteq V$, for instance $X = 1$; and among these there exists a maximal one, say W [Maximum Principle of Set Theory]. From $1 = WN/W$ we would deduce $N \subseteq W$ so that

$$U \cap N = U \cap N \cap W \subseteq V,$$

a contradiction. Hence $1 \neq WN/W$. From $N\ \theta\ G$ we deduce now the existence of a pair $(\mathfrak{A}, \mathfrak{B})$ in θ and of a normal subgroup K of G with

$$W \subset K \subseteq WN$$

such that K/W is an $\mathfrak{A}$-group and $(G/W)/\mathfrak{c}_{G/W}(K/W)$ is a $\mathfrak{B}$-group. Because of the maximality of W we have

$$U \cap N \cap W \subseteq V, \quad U \cap N \cap K \not\subseteq V.$$

Consequently $V(U \cap N \cap K)/V$ is a normal subgroup of U/V with

$$1 \subset V(U \cap N \cap K)/V \subseteq V(U \cap N)/V.$$

From $U \cap N \cap W \subseteq V \cap N \cap K$ we deduce that

$$V(U \cap N \cap K)/V \simeq (U \cap N \cap K)/(V \cap N \cap K)$$

is an epimorphic image of

$$(U \cap N \cap K)/(U \cap N \cap W) \cong W(U \cap N \cap K)/W \subseteq K/W.$$

Thus $V(U \cap N \cap K)/V$ is a factor of the $\mathfrak{A}$-group K/W and as such it is itself an $\mathfrak{A}$-group.

Denote by R the uniquely determined subgroup of U with

$$V \subseteq R \quad \text{and} \quad R/V = \mathfrak{c}_{U/V}[V(U \cap N \cap K)/V]$$

and denote by S the uniquely determined normal subgroup of G with

$$W \subseteq S \quad \text{and} \quad S/W = \mathfrak{c}_{G/W}(K/W).$$

Then $S \circ K \subseteq W$ so that

$$[U \cap S] \circ [V(U \cap N \cap K)] \subseteq V[U \cap N \cap W] = V.$$

Hence $U \cap S \subseteq R$ so that U/R is an epimorphic image of

$$U/(U \cap S) \simeq US/S \subseteq G/S \simeq (G/W)/(S/W) = (G/W)/\mathfrak{c}_{G/W}(K/W).$$

But the latter group is a $\mathfrak{B}$-group. Hence

$$(U/V)/\mathfrak{c}_{U/V}[V(U \cap N \cap K)/V] = (U/V)/(R/V) \simeq U/R$$

is likewise a $\mathfrak{B}$-group.

Thus we have shown that $N \,\theta\, G$ and $U \subseteq G$ imply $(U \cap N) \,\theta\, U$.

Case 2. Assume that $N \,\theta\, G$ and that A is a normal subgroup of G with $A \subseteq N$.

Consider a normal subgroup K of G with $KA/K \neq 1$. This is equivalent to $A \not\subseteq K$ and to $K \cap A \subset A$. There exist normal subgroups X of G with $K \subseteq X$ and $K \cap A = A \cap X$; for instance $X = K$. Among these there exists a maximal one, say M [Maximum Principle of Set Theory]. From $1 = MN/M$ we would deduce $A \subseteq N \subseteq M$ and $A = A \cap M = A \cap K \subset A$, a contradiction. Hence $MN/M \neq 1$ so that from $N \,\theta\, G$ we can deduce the existence of a pair $(\mathfrak{A}, \mathfrak{B})$ in θ and of a normal subgroup L of G with

$$M \subset L \subseteq MN$$

such that L/M is an $\mathfrak{A}$-group and $(G/M)/\mathfrak{c}_{G/M}(L/M)$ is a $\mathfrak{B}$-group. Because of the maximality of M we have

$$K \cap A = A \cap M \subset A \cap L.$$

From $K(L \cap A)/K = 1$ we would conclude that $L \cap A \subseteq A \cap K$, a contradiction. Hence

$$1 \subset K(L \cap A)/K \subseteq KA/K.$$

Furthermore

$$K(L \cap A)/K \simeq (L \cap A)/(K \cap A)$$
$$= (L \cap A)/(M \cap A) \simeq M(L \cap A)/M \subseteq L/M.$$

The normal subgroup $K(L \cap A)/K$ of G/K is therefore a factor of the $\mathfrak{A}$-group L/M and as such it is itself an $\mathfrak{A}$-group.

Denote by R the uniquely determined normal subgroup with

$$M \subseteq R \quad \text{and} \quad R/M = \mathfrak{c}_{G/M}(L/M);$$

and denote by S the uniquely determined normal subgroup with

$$K \subseteq S \quad \text{and} \quad S/K = \mathfrak{c}_{G/K}[K(L \cap A)/K].$$

Then $R \circ L \subseteq M$ and consequently

$$R \circ K(L \cap A) \subseteq K(A \cap M) = K(A \cap K) = K;$$

and this implies $R \subseteq S$. Consequently

$$(G/K)/\mathfrak{c}_{G/K}[K(L \cap A)/K] = (G/K)/(S/K) \simeq G/S$$

is an epimorphic image of

$$G/R \simeq (G/M)/(R/M) = (G/M)/\mathfrak{c}_{G/M}(L/M).$$

Since the latter group is a $\mathfrak{B}$-group, so is $(G/K)/\mathfrak{c}_{G/K}[K(L \cap A)/K]$.

Thus we have shown that every normal subgroup A of G with $A \subseteq N$ and $N\ \theta\ G$ likewise satisfies $A\ \theta\ G$.

The general case. Assume that $N\ \theta\ G$, that T is a normal subgroup of the subgroup S of G and that $T \subseteq N$.

Application of Case 1 shows that $N \cap S\ \theta\ S$. Naturally $T \subseteq N \cap S$; and since T is a normal subgroup of S, we may apply Case 2. Hence $T\ \theta\ S$; and thus we have shown that θ-immersion is factor inherited.

LEMMA 1.6. *If $\mathfrak{A}$ and $\mathfrak{B}$ are factor inherited group theoretical properties, and if N is a normal subgroup of G, then the requirements*

(a) $N\ (\mathfrak{A}, \mathfrak{B})\ G$

and

(b) $N\ (\mathfrak{A}, \mathfrak{U})\ G$ *and* $(\mathfrak{U}, \mathfrak{B})\ G$

are equivalent.

The simple proof may be left to the reader.

Remark 1.7. It is not difficult to construct a θ such that θ-immersion is factor inherited [use Proposition 1.5] and a finite group G with either of the following properties:

The Frattini subgroup ΦG of G is not part of $\mathfrak{h}_\theta G$.

The [ordinary] hypercenter $\mathfrak{h}G$ of G is not part of $\mathfrak{h}_\theta G$.

2. In the present section we are going to discuss the simplifications arising from the presence of sufficiently many minimal normal subgroups.

LEMMA 2.1. *Assume that M is a minimal normal subgroup of G with $M \theta G$. Then there exists a pair $(\mathfrak{A}, \mathfrak{B})$ in θ such that M is an $\mathfrak{A}$-group and such that $H/\mathfrak{c}_H(M^\sigma)$ [$\cong G/\mathfrak{c}_G M$] is a $\mathfrak{B}$-group for every epimorphism σ of G upon H with $M^\sigma \neq 1$ [and hence $M \simeq M^\sigma$].*

Proof. Because of the minimality of M there exists a pair $(\mathfrak{A}, \mathfrak{B})$ in θ such that

$$M \quad \text{is an } \mathfrak{A}\text{-group and} \quad G/\mathfrak{c}_G M \quad \text{is a } \mathfrak{B}\text{-group.}$$

Suppose next that K is a normal subgroup of G with $KM/K \neq 1$. Then $M \not\subseteq K$ so that $M \cap K \subset M$. Since $M \cap K$ is a normal subgroup of G and M is a minimal normal subgroup of G, we conclude that $M \cap K = 1$. Since M and K are normal subgroups of G, this implies $M \circ K = 1$ so that $K \subseteq \mathfrak{c}_G M$.

We note $M = M/(M \cap K) \simeq KM/K$. Suppose next that x is an element in G. Then Kx belongs to the centralizer of KM/K in G/K if, and only if, $x \circ M \subseteq K$. But $x \circ M \subseteq M$, since M is a normal subgroup of G. Hence $x \circ M \subseteq K$ if, and only if, $x \circ M \subseteq K \cap M = 1$; and this is equivalent with $x \circ M = 1$. It follows that

$$\mathfrak{c}_{G/K}(KM/K) = \mathfrak{c}_G M/K.$$

Hence

$$(G/K)/\mathfrak{c}_{G/K}(KM/K) = (G/K)/(\mathfrak{c}_G M/K) \simeq G/\mathfrak{c}_G M.$$

Since the latter group is a $\mathfrak{B}$-group, so is the [isomorphic] former one.

The normal subgroup N of G is termed *an $\mathfrak{m}$-immersed normal subgroup of G*, if there exists to every epimorphism σ of G upon H with $N^\sigma \neq 1$ a minimal normal subgroup M of H with $M \subseteq N^\sigma$.

This condition is certainly satisfied, if the normal subgroup N of G meets one of the following requirements:

(1) The minimum condition is satisfied by the normal subgroups of G which are part of N.

(2) The minimum condition is satisfied by the normal subgroups of N.

(3) The minimum condition is satisfied by the normal subgroups of G.

(4) N is finite.

PROPOSITION 2.2. *If θ-immersion is factor inherited, then the following properties of the $\mathfrak{m}$-immersed normal subgroup N of G are equivalent:*

(i) *$N \theta G$.*

(ii) *If σ is an epimorphism of G upon H and if M is a minimal normal subgroup of H with $M \subseteq N^\sigma$, then $M \theta H$.*

(iii) *If σ is an epimorphism of G upon H and if M is a minimal normal subgroup of H with $M \subseteq N^\sigma$, then there exists a pair $(\mathfrak{A}, \mathfrak{B})$ in θ such that M is an $\mathfrak{A}$-group and $H/\mathfrak{c}_H M$ is a $\mathfrak{B}$-group.*

Proof. Suppose first that $N \theta G$. If σ is an epimorphism of G upon H, then $N^\sigma \theta H$ by Proposition 1.1, (1). If M is a minimal normal subgroup of H with $M \subseteq N^\sigma$, then we recall that θ-immersion is factor inherited; and this implies in particular $M \theta H$. Hence (ii) is a consequence of (i). — By means of Lemma 2.1 we deduce (iii) from (ii). —Assume finally the validity of (iii); and consider an epimorphism σ of G upon H with $N^\sigma \neq 1$. Since the normal subgroup N of G is $\mathfrak{m}$-immersed, there exists a minimal normal subgroup M of H with $1 \subset M \subseteq N^\sigma$. By (iii) there exists a pair $(\mathfrak{A}, \mathfrak{B})$ in θ such that M is an $\mathfrak{A}$-group and $H/\mathfrak{c}_H M$ is a $\mathfrak{B}$-group. Consequently $N \theta G$, proving the equivalence of (i)–(iii).

Remark 2.3. Assume that p is a prime and that m and n are relatively prime integers with $1 < n < m$. Denote by $\mathfrak{A}$ the class of all elementary abelian groups of an order p^r with r a divisor of $m + n$. Denote by $\mathfrak{B}$ the class of all finite abelian groups. Let N and M be elementary abelian groups of orders p^n and p^m respectively. There exists a cyclic group Γ_N of automorphisms of order $p^n - 1$ which is transitive on the elements, not 1, of N—let F be the field of order p^n and let N be isomorphic to its additive group, Γ_N to its multiplicative group, acting in the canonical way. Likewise there exists a cyclic group Γ_M of automorphisms of M whose order is $p^m - 1$ and which is transitive on the elements, not 1, in M. Let $A = M \otimes N$ and let Γ be the group of all automorphisms of A which preserve M and N and which induce in M an automorphism in Γ_M and in N an automorphism in Γ_N. Finally let $G = A\Gamma$ be the product of A and Γ, formed in the holomorph of A.

Then A is an elementary abelian group of order p^{m+n}. Furthermore, A is a normal subgroup of G with $A = \mathfrak{c}_G A$; and

$$G/\mathfrak{c}_G A = G/A \simeq \Gamma \simeq \Gamma_M \otimes \Gamma_N$$

is a direct product of two cyclic groups of finite order and hence abelian. It follows that A $(\mathfrak{A}, \mathfrak{B})$ G.

Next we note that M and N are minimal normal subgroups of G. Since neither m nor n is a divisor of $m + n$, as m and n are relatively prime, we see that neither M nor N is $(\mathfrak{A}, \mathfrak{B})$ immersed in G.

The hypothesis that θ-immersion be factor inherited is consequently indispensable for the validity of Proposition 2.2. This leads us to the following definition:

The normal subgroup N of G is *strictly θ-immersed in G*, in symbols $N \bar{\theta} G$, if to every epimorphism σ of G upon H and to every minimal normal subgroup M of H with $M \subseteq N^\sigma$ there exists a pair $(\mathfrak{A}, \mathfrak{B})$ in θ such that M is an $\mathfrak{A}$-group and $H/\mathfrak{c}_H M$ is a $\mathfrak{B}$-group.

COROLLARY 2.4. *Assume that N is an $\mathfrak{m}$-immersed normal subgroup of G.*

(A) *If $N \bar{\theta} G$, then $N \theta G$.*

(B) *If θ-immersion is factor inherited, then $N \theta G$ and $N \bar{\theta} G$ are equivalent requirements.*

The validity of (A) is easily derived from the various definitions and that of (B) is contained in Proposition 2.2.

3. It will be convenient to make use of the concept of the commutator quotient. We remind the reader of its definition [which implicite, has been used before]: If X and Y are normal subgroups of the group G with $X \subseteq Y$, then the commutator quotient

$$(X:Y) = (X:Y)_G = \text{set of all elements } g \text{ in } G \text{ with } g \circ Y \subseteq X.$$

This is the uniquely determined normal subgroup of G which contains X and satisfies

$$(3.1) \qquad (X:Y)/X = \mathfrak{c}_{G/X}(Y/X).$$

Since X is a normal subgroup of G, we have clearly $X \circ Y \subseteq X$ and this is equivalent to saying $X \subseteq (X:Y)$. Furthermore the following three statements are evidently equivalent:

$$(3.2) \qquad Y/X \text{ is abelian;} \quad Y \circ Y \subseteq X; \quad Y \subseteq (X:Y).$$

In the present section we shall discuss the special situation arising in case the normal subgroup under consideration is finite and soluble. The basis of this discussion is the

LEMMA 3.3. *Assume that A and B are finite normal subgroups of the group G with $A \subset B$, and that B/A is an elementary abelian p-group and a minimal normal subgroup of G/A. Then every p-Sylow subgroup P of B has the following properties:*

(a) *$B = AP$ and $B/A \simeq P/(A \cap P)$.*

(b) *$G = A\mathfrak{n}_G P$.*

(c) *$A \cap P$ is a normal subgroup of $\mathfrak{n}_G P$ and $P/(A \cap P)$ is a minimal normal subgroup of $\mathfrak{n}_G P/(A \cap P)$.*

(d) *$([P \cap A]:P) = \mathfrak{n}_G P \cap (A:B)$.*

(e) *$G/(A:B) \simeq \mathfrak{n}_G P/([P \cap A]:P)$.*

(f) *The group of automorphisms, induced in B/A by G/A, is essentially the same as the group of automorphisms, induced in $P/(P \cap A)$ by $\mathfrak{n}_G P/(P \cap A)$.*

(g) *$[G/A]/\mathfrak{c}_{G/A}[B/A] \simeq [\mathfrak{n}_G P/(P \cap A)]/\mathfrak{c}_{\mathfrak{n}_G P/(P \cap A)}[P/(P \cap A)]$.*

Proof. (a) is an immediate consequence of the facts that B/A is a p-group and P a p-Sylow subgroup of the finite group B.

Since P is a Sylow subgroup of the finite group B, the Frattini argument proves $G = B\mathfrak{n}_G P$. Noting $B = AP$ [by (a)] and $P \subseteq \mathfrak{n}_G P$ we conclude that $G = B\mathfrak{n}_G P = AP\mathfrak{n}_G P = A\mathfrak{n}_G P$. This proves (b).

It is a consequence of the remark (3.2) and the commutativity of B/A that

(1) $A \subset B \subseteq (A{:}B)$.

Combine this with (b) to see that

(2) $G = (A{:}B)\mathfrak{n}_G P$.

Next we note that A is normalized by G and hence by $\mathfrak{n}_G P$. Since P too is normalized by $\mathfrak{n}_G P$, we may conclude that $A \cap P$ is a normal subgroup of $\mathfrak{n}_G P$. To prove the second half of (c) consider a normal subgroup W of $\mathfrak{n}_G P$ with

$$A \cap P \subset W \subseteq P \subseteq B.$$

Clearly A and W are normalized by $\mathfrak{n}_G P$. Furthermore

$$(A{:}B) \circ W \subseteq (A{:}B) \circ P \subseteq (A{:}B) \circ B \subseteq A$$

so that

$$(A{:}B) \circ AW \subseteq [(A{:}B) \circ A][(A{:}B) \circ W] \subseteq AW.$$

Thus we have shown that AW is normalized by $(A{:}B)$ and $\mathfrak{n}_G P$ and consequently by $(A{:}B)\mathfrak{n}_G P = G$ (by (2)). Hence AW is a normal subgroup of G and $AW \subseteq B$ so that AW/A is a normal subgroup of G/A which is part of the minimal normal subgroup B/A of G/A. If AW/A were equal to 1, then W would be part of A and this would imply $W \subseteq A \cap P \subset W$, a contradiction. Hence $AW/A = B/A$ so that $B = AW$. From $W \subseteq P \subseteq B = AW$ and Dedekind's Modular Law we conclude now that

$$P = W(A \cap P) = W$$

[because of $A \cap P \subset W$]. This shows that $P/(A \cap P)$ is a minimal normal subgroup of $\mathfrak{n}_G P/(A \cap P)$, completing the proof of (c).

The element x in G belongs to $\mathfrak{n}_G P \cap (A{:}B)$ if, and only if,

$$x \circ P \subseteq P \quad \text{and} \quad x \circ B \subseteq A.$$

Since $B = AP$ by (a), and since $x \circ A \subseteq A$ for every x as A is a normal subgroup of G, this pair of inequalities is equivalent to

$$x \circ P \subseteq P \quad \text{and} \quad x \circ P \subseteq A.$$

This pair of inequalities is equivalent to

$$x \circ P \subseteq A \cap P;$$

and this signifies that x belongs to $([A \cap P]{:}P)$. Hence x belongs to $\mathfrak{n}_G P \cap (A{:}B)$ if, and only if, x belongs to $([A \cap P]{:}P)$, proving our equation (d).

Next we note that by (2) and (d)

$$G/(A{:}B) = (A{:}B)\mathfrak{n}_G P/(A{:}B) \simeq \mathfrak{n}_G P/[\mathfrak{n}_G P \cap (A{:}B)] = \mathfrak{n}_G P/([P \cap A]{:}P),$$

proving (e).

To prove (f) and (g) we note first that by (3.1) the group of automorphisms, induced in B/A by G/A, is essentially the same as

$$[G/A]/\mathfrak{c}_{G/A}[B/A] = [G/A]/[(A{:}B)/A] \simeq G/(A{:}B),$$

that the group of automorphisms, induced in $P/(P \cap A)$ by $\mathfrak{n}_G P/(P \cap A)$, is essentially the same as

$$\begin{aligned}
&[\mathfrak{n}_G P/(P \cap A)/\mathfrak{c}_{\mathfrak{n}_G P/(P\cap A)}[P/(P \cap A)] \\
&\qquad = [\mathfrak{n}_G P/(P \cap A)]/[((P \cap A){:}P)_{\mathfrak{n}_G P}/(P \cap A)] \\
&\qquad \simeq \mathfrak{n}_G P/([P \cap A]{:}P)_{\mathfrak{n}_G P} = \mathfrak{n}_G P/[\mathfrak{n}_G P \cap ([P \cap A]{:}P)] \\
&\qquad = \mathfrak{n}_G P/([P \cap A]{:}P) \qquad \text{by (d);}
\end{aligned}$$

and combining these isomorphies with (e) we obtain (f) and (g).

THEOREM 3.4. *If θ-immersion is factor inherited, then the following properties of the finite, soluble normal subgroup N of G are equivalent:*

(i) $N \,\theta\, G$.

(ii) $P \,\theta\, \mathfrak{n}_G P$ *for every primary subgroup P of N.*

(iii) *If σ is an epimorphism of G upon H and if M is a minimal normal subgroup of H with $M \subseteq N^\sigma$, then there exists a pair $(\mathfrak{A}, \mathfrak{B})$ in θ such that M is an $\mathfrak{A}$-group and $H/\mathfrak{c}_H M$ is a $\mathfrak{B}$-group.*

(iv) (a) $N_p \,\theta\, G$ *for every prime p where N_p is the product of all normal p-subgroups of N.*
(b) $(N \cap U) \,\theta\, U$ *for every maximal subgroup U of G.*

Proof. (ii) is a consequence of (i), since θ-immersion is factor inherited.

Assume next the validity of (ii) and consider a pair A, B of normal subgroups of G such that

$$A \subset B \subseteq N \quad \text{and} \quad B/A \text{ is a minimal normal subgroup of } G/A.$$

Since N is soluble, so are B and B/A. Since B/A is characteristic simple, B/A is an elementary abelian p-group. Denote by P some p-Sylow subgroup of B. Then we deduce from Lemma 3.3 and the finiteness of N the following facts:

(a) $B/A \simeq P/(A \cap P)$.

(b) $A \cap P$ is a normal subgroup of $\mathfrak{n}_G P$ and $P/(A \cap P)$ is a minimal normal subgroup of $\mathfrak{n}_G P/(A \cap P)$.

(c) $[G/A]/\mathfrak{c}_{G/A}[B/A] \simeq [\mathfrak{n}_G P/(P \cap A)]/\mathfrak{c}_{\mathfrak{n}_G P/(P\cap A)}[P/(P \cap A)]$.

From (ii) we deduce $P \,\theta\, \mathfrak{n}_G P$. Denote by σ the canonical epimorphism of $\mathfrak{n}_G P$ upon $H = \mathfrak{n}_G P/(A \cap P)$. Then $P^\sigma = P/(P \cap A)$ is by (b) a minimal normal subgroup of H. Consequently there exists a pair $(\mathfrak{A}, \mathfrak{B})$ in θ such that P^σ is an $\mathfrak{A}$-group and $H/\mathfrak{c}_H(P^\sigma)$ is a $\mathfrak{B}$-group. Apply (a) and (c) to see that

(d) B/A is an $\mathfrak{A}$-group and $[G/A]/\mathfrak{c}_{G/A}[B/A]$ is a $\mathfrak{B}$-group.

This shows that (iii) is a consequence of (ii). —Since N is finite, (i) is a consequence of the [much stronger] condition (iii). Hence (i)–(iii) are equivalent.

(iv) is a consequence of (i), since θ-immersion is factor inherited, and since N_p is a characteristic subgroup of the normal subgroup N of G and as such N_p is a normal subgroup of G.

Assume conversely the validity of (iv) and consider a p-subgroup P of N. If firstly $G = \mathfrak{n}_G P$, then P is a normal subgroup of G so that $P \subseteq N_p$. Since $N_p \,\theta\, G$ and since θ-immersion is factor inherited, $P \,\theta\, \mathfrak{n}_G P$. If secondly $\mathfrak{n}_G P \subset G$, then all subgroups, conjugate to P in G, are contained in the finite normal subgroup N. Hence their number is finite; and this is equivalent to the finiteness of the index $[G:\mathfrak{n}_G P]$. Consequently there exists a maximal subgroup U of G with $\mathfrak{n}_G P \subseteq U$. Apply (iv.b) to show $(N \cap U) \,\theta\, U$. Since θ-immersion is factor inherited, and since $P \subseteq N \cap \mathfrak{n}_G P \subseteq N \cap U$, we conclude again $P \,\theta\, \mathfrak{n}_G P$. Hence (ii) is a consequence of (iv), proving the equivalence of (i)–(iv).

Remark 3.5. **A.** The hypothesis, that θ-immersion is factor inherited, has been used when deducing (ii) from (i); but this hypothesis has not been used when deducing (iii) from (ii) and when deducing (i) from (iii). This is our principal reason for inserting (iii) whose equivalence with (i), on the basis of factor inheritance of θ-immersion, is contained in Proposition 2.2.

B. No use has been made of the solubility of N when deducing (ii) from (i) and (i) from (iii). But this hypothesis is indispensable when deducing (iii) or (i) from (ii), as may be seen from the following

Example. Let $\mathfrak{A} = \mathfrak{U}$ and $\mathfrak{B} = \mathfrak{K}$ ($=$ commutativity) and let θ just consist of the one pair $(\mathfrak{U}, \mathfrak{K})$. Furthermore let $G = N$ be the simple group of order 60. Then

$$\mathfrak{n}_G P/\mathfrak{c}_G P \quad \text{is abelian for every primary subgroup} \quad P \quad \text{of} \quad G.$$

This property is much stronger than the property (ii) of Theorem 3.4. But the validity of $G \,\theta\, G$ would imply that $G/\mathfrak{z}G = G$ is abelian which is patently false.

COROLLARY 3.6. *If θ-immersion is factor inherited, then the following properties of the finite soluble group G are equivalent:*

(i) *G is a θ-group.*

(ii) *$P \,\theta\, \mathfrak{n}_G P$ for every primary subgroup P of G.*

(iii) *If M is a minimal normal subgroup of the epimorphic image H of G, then there exists a pair $(\mathfrak{A}, \mathfrak{B})$ in θ such that M is an $\mathfrak{A}$-group and $H/\mathfrak{c}_H M$ is a $\mathfrak{B}$-group.*

(iv) (a) *$G_p \,\theta\, G$ for every prime p (where G_p is the product of all normal p-subgroups of G).*
(b) *Every maximal subgroup of G is a θ-group.*

This is just the special case $N = G$ of Theorem 3.4.

The group theoretical property $\mathfrak{E}$ is said to meet the *Iwasawa-Schmidt requirement* if every finite group whose proper factors are $\mathfrak{E}$-groups is soluble. —If in particular $\mathfrak{E}$ is factor inherited, then this implies that finite $\mathfrak{E}$-groups are soluble. —The most notable examples of properties meeting the Iwasawa-Schmidt requirement are nilpotency [Iwasawa-Schmidt], supersolubility [Huppert] and dispersion [Baer [2]].

COROLLARY 3.7. *If θ-immersion is factor inherited, and if the group theoretical property θ meets the Iwasawa-Schmidt requirement, then the following properties of the finite normal subgroup N of G are equivalent:*

(i) $N \, \theta \, G$.

(ii) $P \, \theta \, \mathfrak{n}_G P$ *for every primary subgroup P of N.*

(iii) *If σ is an epimorphism of G upon H and if M is a minimal normal subgroup of H with $M \subseteq N^\sigma$, then there exists a pair $(\mathfrak{A}, \mathfrak{B})$ in θ such that M is an $\mathfrak{A}$-group and $H/\mathfrak{c}_H M$ is a $\mathfrak{B}$-group.*

We precede the proof of this result by a proof of the following special case:

(+) If $P \, \theta \, \mathfrak{n}_G P$ for every primary subgroup P of the finite group F, then F is a θ-group and hence soluble.

Proof. If (+) were false, then there would exist a finite group G of minimal order with the following properties:

(1) $P \, \theta \, \mathfrak{n}_G P$ for every primary subgroup P of G.

(2) G is not a soluble θ-group.

Since θ-immersion is factor inherited, property (1) is inherited by every subgroup S of G. It is a consequence of Proposition 1.1, (1) and the Frattini argument, that every epimorphic image of S meets requirement (1) too. Hence

(1*) $P \, \theta \, \mathfrak{n}_V P$ for every primary subgroup P of every factor V of G.

Because of the minimality of G we deduce from (1*) that

(3) every proper factor of G is a soluble θ-group.

But the group-theoretical property θ meets the Iwasawa-Schmidt requirement. Hence we may deduce from (3) the solubility of G. Apply Corollary 3.6 to see that G is a θ-group. This contradicts (2); and this contradiction proves the validity of the special case (+).

Proof of Corollary 3.7. We have noted in Remark 3.5, **B** that the implications (iii) $\rightarrow$ (i) $\rightarrow$ (ii) are valid without the hypothesis that N be soluble. Assume finally the validity of (ii). Then

$$P \, \theta \, \mathfrak{n}_N P \quad \text{for every primary subgroup} \quad P \quad \text{of} \quad N,$$

since θ-immersion is factor inherited. Since N is finite, we may apply the special case $(+)$. Hence N is a soluble θ-group. Now we may apply Theorem 3.4 to show $N \theta G$, completing the proof.

4. If $\mathfrak{D}$ is any factor inherited class of groups, then it will prove convenient to define a $\mathfrak{D}$-*formation* as a class $\mathfrak{F}$ of groups, meeting the following requirements:

(4.I) Every $\mathfrak{F}$-group is a $\mathfrak{D}$-group.

(4.II) If H is an epimorphic image of an $\mathfrak{F}$-group, then H is an $\mathfrak{F}$-group.

(4.III) If G is a $\mathfrak{D}$-group, if 1 is the intersection of all normal subgroups X of G with $\mathfrak{F}$-quotient group G/X, then G is an $\mathfrak{F}$-group.

In other words: A $\mathfrak{D}$-formation is an epimorphism inherited subclass of $\mathfrak{D}$ which is residually closed in $\mathfrak{D}$.

In our applications $\mathfrak{D}$ may be the class of all groups or the class of all finite groups; and in this latter case the concept of $\mathfrak{D}$-formation coincides with Gaschütz's concept of formation.

If $\mathfrak{E}$ is any group theoretical property, then $\mathfrak{E}G$ is the intersection of all normal subgroups X of G with $\mathfrak{E}$-quotient group G/X. This is a well determined characteristic subgroup of G. Using this concept one may characterize a $\mathfrak{D}$-formation as a subclass $\mathfrak{F}$ of $\mathfrak{D}$, meeting the following requirement:

(4.IV) The epimorphism σ maps the $\mathfrak{D}$-group G upon an $\mathfrak{F}$-group if, and only if, $(\mathfrak{F}G)^\sigma = 1$.

PROPOSITION 4.1. *If $\mathfrak{D}$ is a factor inherited class of groups and $\mathfrak{F}$ is a $\mathfrak{D}$-formation, then the following properties of the normal subgroup N of the $\mathfrak{D}$-group G are equivalent*:

(i) $N \cap \mathfrak{F}G \subseteq \mathfrak{h}\mathfrak{F}G$.

(ii) $N \circ \mathfrak{F}G \subseteq \mathfrak{h}\mathfrak{F}G$.

(iii) *If σ is an epimorphism of G upon H with $N^\sigma \neq 1$, then there exists a normal subgroup K of H with $1 \subset K \subseteq N^\sigma$ such that $H/\mathfrak{c}_H K$ is an $\mathfrak{F}$-group.*

(iv) $[G/\mathfrak{h}\mathfrak{F}G]/\mathfrak{c}_{G/\mathfrak{h}\mathfrak{F}G}[N\mathfrak{h}\mathfrak{F}G/\mathfrak{h}\mathfrak{F}G]$ *is an $\mathfrak{F}$-group.*

Proof. Since N and $\mathfrak{F}G$ are normal subgroups of G, we have

$$N \circ \mathfrak{F}G \subseteq N \cap \mathfrak{F}G$$

so that (ii) is a consequence of (i).

We assume next the validity of (ii) and consider an epimorphism σ of G upon H with $N^\sigma \neq 1$. If firstly $(N \circ \mathfrak{F}G)^\sigma \neq 1$, then

$$1 \subset (N \circ \mathfrak{F}G)^\sigma \subseteq (\mathfrak{h}\mathfrak{F}G)^\sigma \subseteq \mathfrak{h}(\mathfrak{F}G)^\sigma.$$

It follows that

$$1 \neq (N \circ \mathfrak{F}G)^\sigma \cap \mathfrak{z}(\mathfrak{F}G)^\sigma = K \subseteq N^\sigma.$$

Thus K is a normal subgroup, not 1, of H which is part of N^σ; and we have furthermore $(\mathfrak{F}G)^\sigma \subseteq \mathfrak{c}_H K$. Apply property (4.IV) to see that $H/\mathfrak{c}_H K$ is an $\mathfrak{F}$-group. —If secondly

$$1 = (N \circ \mathfrak{F}G)^\sigma = N^\sigma \circ (\mathfrak{F}G)^\sigma,$$

then N^σ and $(\mathfrak{F}G)^\sigma$ centralize each other. Hence $(\mathfrak{F}G)^\sigma \subseteq \mathfrak{c}_H(N^\sigma)$; and application of property (4.IV) shows that $H/\mathfrak{c}_H(N^\sigma)$ is an $\mathfrak{F}$-group. Thus we see that (iii) is a consequence of (ii).

Assume now the validity of (iii); and assume by way of contradiction that $N \cap \mathfrak{F}G \not\subseteq \mathfrak{h}\mathfrak{F}G$. Then we denote by σ the canonical epimorphism of G upon $H = G/\mathfrak{h}\mathfrak{F}G$; and we see that by (4.IV)

$$1 \subset \mathfrak{h}\mathfrak{F}G(N \cap \mathfrak{F}G)/\mathfrak{h}\mathfrak{F}G = (N \cap \mathfrak{F}G)^\sigma \subseteq N^\sigma \cap (\mathfrak{F}G)^\sigma = N^\sigma \cap \mathfrak{F}H.$$

There exist normal subgroups X of H with $X \cap (N \cap \mathfrak{F}G)^\sigma = 1$; and among these there exists a maximal one, say J. From

$$J \cap (N \cap \mathfrak{F}G)^\sigma = 1 \subset (N \cap \mathfrak{F}G)^\sigma$$

we deduce $1 \subset JN^\sigma/J$. Application of (iii) shows the existence of a normal subgroup L of H with $J \subset L \subseteq JN^\sigma$ such that $[H/J]/\mathfrak{c}_{H/J}[L/J]$ is an $\mathfrak{F}$-group. From the maximality of J we deduce

$$1 \subset L \cap (N \cap \mathfrak{F}G)^\sigma = K \subseteq N^\sigma \cap \mathfrak{F}H.$$

The element x in H belongs to $(J:L)$ if, and only if, $x \circ L \subseteq J$; and this implies

$$x \circ K \subseteq (x \circ L) \cap (N \cap \mathfrak{F}G)^\sigma \subseteq J \cap (N \cap \mathfrak{F}G)^\sigma = 1.$$

Hence $(J:L) \subseteq \mathfrak{c}_H K$ so that $H/\mathfrak{c}_H K$ is an epimorphic image of the $\mathfrak{F}$-group

$$H/(J:L) \simeq [H/J]/[(J:L)/J] = [H/J]/\mathfrak{c}_{H/J}[L/J].$$

Thus we have shown the existence of a normal subgroup K of H with $1 \subset K \subseteq N^\sigma \cap \mathfrak{F}H$ such that $H/\mathfrak{c}_H K$ is an $\mathfrak{F}$-group. Denote by V and W the uniquely determined normal subgroups of G such that

$$\mathfrak{h}\mathfrak{F}G \subseteq V \cap W, \qquad V/\mathfrak{h}\mathfrak{F}G = K, \qquad W/\mathfrak{h}\mathfrak{F}G = \mathfrak{c}_H K.$$

From

$$G/W \simeq [G/\mathfrak{h}\mathfrak{F}G]/[W/\mathfrak{h}\mathfrak{F}G] = H/\mathfrak{c}_H K$$

we deduce that G/W is an $\mathfrak{F}$-group; and this implies

$$\mathfrak{F}G \subseteq W$$

by the definition of $\mathfrak{F}G$. Furthermore

$$V \circ \mathfrak{F}G \subseteq V \circ W \subseteq \mathfrak{h}\mathfrak{F}G;$$

and this implies

$$1 \subset K = V/\mathfrak{h}\mathfrak{F}G \subseteq \mathfrak{z}(\mathfrak{F}G/\mathfrak{h}\mathfrak{F}G) = 1,$$

a contradiction. Hence $N \cap \mathfrak{F}G \subseteq \mathfrak{h}\mathfrak{F}G$, showing that (i) is a consequence of (iii) and that therefore (i)–(iii) are equivalent.

It is clear that property (ii) is equivalent with

$$[N\mathfrak{h}\mathfrak{F}G/\mathfrak{h}\mathfrak{F}G] \circ [\mathfrak{F}G/\mathfrak{h}\mathfrak{F}G] = 1.$$

But this is equivalent to the statement

$$\mathfrak{F}G/\mathfrak{h}\mathfrak{F}G \subseteq \mathfrak{c}_{G/\mathfrak{h}\mathfrak{F}G}[N\mathfrak{h}\mathfrak{F}G/\mathfrak{h}\mathfrak{F}G].$$

It is a consequence of (4.IV) that the last statement is equivalent to (iv), showing the equivalence of (i)–(iv).

Remark 4.2. Condition (iii) of Proposition 4.1 asserts that N is $(\mathfrak{U}, \mathfrak{F})$-immersed in G.

The stabilizer $\mathfrak{s}_G N$ of the normal subgroup N of the group G is the set of all elements s in G with the property:

(4.$\mathfrak{s}$) If U, V are normal subgroups of G with $U \subset V \subseteq N$, then there exists a normal subgroup W of G with

$$s \circ W \subseteq U \subset W \subseteq V.$$

The elements in the stabilizer of N induce consequently the 1-automorphism in every principal factor of G which is part of N; but they need not induce the 1-automorphism in N nor are they subject to any requirements concerning G/N. Thus the stabilizer should not be mixed up with the group of stability of a normal subgroup; see Specht [p. 88].

LEMMA 4.3. *If N is a normal subgroup of G, then $\mathfrak{s}_G N$ is a normal subgroup of G.*

Proof. It is clear that 1 belongs to $\mathfrak{s}_G N$. Consider elements x, y in $\mathfrak{s}_G N$ and a pair U, V of normal subgroups of G with $U \subset V \subseteq N$. Then there exists a normal subgroup X of G with

$$x \circ X \subseteq U \subset X \subseteq V;$$

and there exists a normal subgroup Y of G with

$$y \circ Y \subseteq U \subset Y \subseteq X \subseteq V.$$

From the first condition we deduce that

$$x \circ Y \subseteq x \circ X \subseteq U.$$

Hence x and y both induce the 1-automorphism in Y/U. But then xy^{-1} likewise induces the 1-automorphism in Y/U; and this is equivalent to

$$xy^{-1} \circ Y \subseteq U \subset Y \subseteq V,$$

showing that xy^{-1} likewise belongs to $\mathfrak{s}_G N$. Hence $\mathfrak{s}_G N$ is a subgroup of G; and it is an immediate consequence of the defining condition (4.$\mathfrak{s}$) that it is a normal subgroup of G.

LEMMA 4.4. *The following properties of the normal subgroups A and B of G are equivalent, provided B is $\mathfrak{m}$-immersed in G:*

(i) $A \cap B \subseteq \mathfrak{h}A$.

(ii) $A \circ B \subseteq \mathfrak{h}A$.

(iii) $A \subseteq \mathfrak{s}_G B$.

Proof. Since A and B are normal subgroups of G, we have $A \circ B \subseteq A \cap B$ so that (ii) is a consequence of (i).

Assume next the validity of (ii) and consider a pair of normal subgroups U, V of G with $U \subset V \subseteq B$. We distinguish two possibilities.

Case 1. $V \cap \mathfrak{h}A \subseteq U$. Then we select $W = V$ and obtain

$$A \circ W = A \circ V \subseteq (A \circ B) \cap V \subseteq \mathfrak{h}A \cap V \subseteq U \subset V = W,$$

since V is a normal subgroup, so that W meets requirement (4.$\mathfrak{s}$) for every element in A.

Case 2. $V \cap \mathfrak{h}A \not\subseteq U$. Then we note that the characteristic subgroup $\mathfrak{h}A$ of the normal subgroup A is a normal subgroup of G and that

$$1 \subset U(V \cap \mathfrak{h}A)/U \subseteq U\mathfrak{h}A/U \subseteq \mathfrak{h}(UA/U),$$

since the canonical epimorphism of G upon G/U maps A upon UA/U and $\mathfrak{h}A$ upon $U\mathfrak{h}A/U$. It follows that

$$1 \neq [U(V \cap \mathfrak{h}A)/U] \cap \mathfrak{z}(UA/U).$$

If we denote by W the uniquely determined normal subgroup of G with $U \subseteq W$ and $W/U = [U(V \cap \mathfrak{h}A)/U] \cap \mathfrak{z}(UA/U)$, then we have

$$A \circ W \subseteq U \subset W \subseteq V$$

so that W meets the requirements of (4.$\mathfrak{s}$) for every element in A.

Thus we have shown in both cases that $A \subseteq \mathfrak{s}_G B$ and we have deduced (iii) from (ii).

Assume finally the validity of (iii) and assume by way of contradiction that $A \cap B \not\subseteq \mathfrak{h}A$. Then

$$B \cap \mathfrak{h}A \subset B \cap A \subseteq B.$$

Since B is $\mathfrak{m}$-immersed in G, there exists [as is easily seen] a normal subgroup W of G with the following properties:

$$B \cap \mathfrak{h}A \subset W \subseteq B \cap A \subseteq B$$

and

$$W/(B \cap \mathfrak{h}A) \text{ is a minimal normal subgroup of } G/(B \cap \mathfrak{h}A).$$

If a is an element in $A \subseteq \mathfrak{s}_G B$, then we let $U = B \cap \mathfrak{h}A$ and $V = W$ in the defining property (4.$\mathfrak{s}$). Consequently there exists a normal subgroup J of G with

$$a \circ J \subseteq B \cap \mathfrak{h}A \subset J \subseteq W.$$

But since $W/(B \cap \mathfrak{h}A)$ is a minimal normal subgroup of $G/(B \cap \mathfrak{h}A)$, we conclude that $J = W$ and that therefore $a \circ W \subseteq B \cap \mathfrak{h}A$. Hence

$$A \circ W \subseteq B \cap \mathfrak{h}A \subset W \subseteq B \cap A \subseteq A.$$

But this is equivalent to

$$1 \subset W/(B \cap \mathfrak{h}A) \subseteq \mathfrak{z}[A/(B \cap \mathfrak{h}A)] \subseteq \mathfrak{h}[A/(B \cap \mathfrak{h}A)] = \mathfrak{h}A/(B \cap \mathfrak{h}A).$$

Hence

$$B \cap \mathfrak{h}A \subset W \subseteq B \cap \mathfrak{h}A,$$

a contradiction proving that (i) is a consequence of (iii) and that (i)–(iii) are equivalent.

Remark 4.5. **A.** The hypothesis that B be $\mathfrak{m}$-immersed in G has been used only when deriving (i) from (iii).

B. When deriving (iii) from (ii) we have proven (4.$\mathfrak{s}$) in the following stronger form:

If A is a normal subgroup with $A \circ B \subseteq \mathfrak{h}A$, and if U, V are normal subgroups of G with $U \subset V \subseteq B$, then there exists a normal subgroup W of G with

$$A \circ W \subseteq U \subset W \subseteq V.$$

C. We may let $A = \mathfrak{s}_G B$ in Lemma 4.4 and we find consequently that

$$B \cap \mathfrak{s}_G B \subseteq \mathfrak{h}\mathfrak{s}_G B.$$

D. If we impose in Proposition 4.1 in addition to the other requirements the further condition that the normal subgroup N of G be $\mathfrak{m}$-immersed, then the conditions (i)–(iv) of Proposition 4.1 are, because of Lemma 4.4, equivalent to

(v) $\mathfrak{F}G \subseteq \mathfrak{s}_G N$.

Corollary 4.6. *The following properties of the* $\mathfrak{m}$*-immersed normal subgroup N of G are equivalent:*

(a) $G = N\mathfrak{s}_G N$.

(b) *There exists a normal subgroup S of G with $G = SN$ and $S \circ N \subseteq \mathfrak{h}S$.*

Proof. It is a consequence of Remark 4.5.**C** that $N \circ \mathfrak{s}_G N \subseteq \mathfrak{h}\mathfrak{s}_G N$. Hence choosing $S = \mathfrak{s}_G N$ we deduce (b) from (a).

If conversely S is a normal subgroup of G with $G = SN$ and $S \circ N \subseteq \mathfrak{h}S$, then we deduce $S \subseteq \mathfrak{s}_G N$ from Lemma 4.4 so that $G = NS = N\mathfrak{s}_G N$.

Lemma 4.7. (I) *If N is a hypercentral normal subgroup of G, then $N \subseteq \mathfrak{s}_G N$.*

(II) *The* $\mathfrak{m}$*-immersed normal subgroup N of G is hypercentral if, and only if, $N \subseteq \mathfrak{s}_G N$.*

Proof. Assume firstly that N is a hypercentral normal subgroup of G. Consider an element g in N and normal subgroups U, V of G with $U \subset V \subseteq N$. Then V/U is a normal subgroup, not 1, of the epimorphic image N/U of the hypercentral group N. It follows that

$$1 \neq [V/U] \cap \mathfrak{z}[N/U] \quad \text{is a normal subgroup of} \quad G/U.$$

Denote by W the uniquely determined normal subgroup of G with $U \subseteq W$ and $W/U = [V/U] \cap \mathfrak{z}[N/U]$. Then $g \circ W \subseteq N \circ W \subseteq U \subset W \subseteq V$; and g consequently belongs to $\mathfrak{s}_G N$ so that $N \subseteq \mathfrak{s}_G N$.

Assume secondly that N is an $\mathfrak{m}$-immersed normal subgroup of G. If N is hypercentral, then we deduce $N \subseteq \mathfrak{s}_G N$ from (I). If conversely $N \subseteq \mathfrak{s}_G N$, then

$$N = N \cap \mathfrak{s}_G N \subseteq \mathfrak{h}\mathfrak{s}_G N$$

by Lemma 4.4 [or Remark 4.5. **C**]. But the hypercenter and its subgroups are hypercentral so that N is hypercentral, proving (II).

COROLLARY 4.8. *The following properties of the* $\mathfrak{m}$*-immersed normal subgroup* N *of* G *are equivalent*:

(i) $N \subseteq \mathfrak{h}G$.

(ii) N *is hypercentral and* $G = N\mathfrak{s}_G N$.

(iii) $G = \mathfrak{s}_G N$.

Proof. Assume first the validity of $N \subseteq \mathfrak{h}G$. Then

$$G \cap N = N \subseteq \mathfrak{h}G;$$

and we deduce $G \subseteq \mathfrak{s}_G N$ from Lemma 4.4. Hence (iii) is a consequence of (i).

If (iii) is true, then $G = NG = N\mathfrak{s}_G N$ and $N \subseteq G = \mathfrak{s}_G N$ implies the hypercentrality of N by Lemma 4.7 (II). Hence (ii) is a consequence of (iii).

Assume finally the validity of (ii). Then we deduce $N \subseteq \mathfrak{s}_G N$ from Lemma 4.7 (I) and the hypercentrality of N so that $G = N\mathfrak{s}_G N = \mathfrak{s}_G N$. Apply Lemma 4.4 to see that

$$N = G \cap N \subseteq \mathfrak{h}G.$$

Hence (i) is a consequence of (ii), proving the equivalence of (i)–(iii).

Remark 4.9. Combination of condition (ii) of Corollary 4.8 with Corollary 4.6 produces the condition

(ii*) N is hypercentral and there exists a normal subgroup S of G with $G = SN$ and $S \circ N \subseteq \mathfrak{h}S$,

which is consequently likewise equivalent to the conditions (i)–(iii) of Corollary 4.8.

PROPOSITION 4.10. *If* $\mathfrak{D}$ *is a factor inherited class of groups and* $\mathfrak{F}$ *is a* $\mathfrak{D}$*-formation, if* N *is an* $\mathfrak{m}$*-immersed normal subgroup of the* $\mathfrak{D}$*-group* G *and* $N \cap \mathfrak{F}G$

is an $\mathfrak{m}$*-immersed normal subgroup of* $\mathfrak{F}G$*, then the following properties are equivalent*:

(i) $N \cap \mathfrak{F}G \subseteq \mathfrak{h}\mathfrak{F}G$.

(ii) $N \circ \mathfrak{F}G \subseteq \mathfrak{h}\mathfrak{F}G$.

(iii) *If* σ *is an epimorphism of* G *upon* H *and if* M *is a minimal normal subgroup of* H *with* $M \subseteq N^{\sigma}$*, then* $H/\mathfrak{c}_H M$ *is an* $\mathfrak{F}$*-group.*

(iv) $[G/\mathfrak{h}\mathfrak{F}G]/\mathfrak{c}_{G/\mathfrak{h}\mathfrak{F}G}[N\mathfrak{h}\mathfrak{F}G/\mathfrak{h}\mathfrak{F}G]$ *is an* $\mathfrak{F}$*-group.*

(v) $\mathfrak{F}G \subseteq \mathfrak{z}_G N$.

(vi) $N \cap \mathfrak{F}G$ *is hypercentral and* $\mathfrak{F}G = (N \cap \mathfrak{F}G)\mathfrak{z}_{\mathfrak{F}G}(N \cap \mathfrak{F}G)$.

(vii) $\mathfrak{F}G = \mathfrak{z}_{\mathfrak{F}G}(N \cap \mathfrak{F}G)$.

Proof. The equivalence of conditions (i), (ii) and (iv) is contained in Proposition 4.1. Since N is an $\mathfrak{m}$-immersed normal subgroup of G, we deduce from Proposition 2.2 the equivalence of our present condition (iii) and the condition (iii) of Proposition 4.1. Thus (i)–(iv) are equivalent.

If we substitute in Lemma 4.4 [use Remark 4.5, **D**] as follows

A	B
$\mathfrak{F}G$	N

then we obtain the equivalence of conditions (i), (ii) and (v). Hence conditions (i)–(v) are equivalent.

If we substitute in Corollary 4.8 as follows

N	G
$N \cap \mathfrak{F}G$	$\mathfrak{F}G$

and if we recall that $N \cap \mathfrak{F}G$ is an $\mathfrak{m}$-immersed normal subgroup of $\mathfrak{F}G$, then we obtain the equivalence of conditions (i), (vi) and (vii), completing the proof of the equivalence of conditions (i)–(vii).

Remark 4.11. The immersion requirements are certainly satisfied whenever the minimum condition is satisfied by the subnormal subgroups of G which are contained in N.

COROLLARY 4.12. *If* $\mathfrak{D}$ *is a factor inherited class of groups and* $\mathfrak{F}$ *is a* $\mathfrak{D}$*-formation, if the* $\mathfrak{D}$*-group* G *is* $\mathfrak{m}$*-immersed in* G *and if* $\mathfrak{F}G$ *is* $\mathfrak{m}$*-immersed in* $\mathfrak{F}G$*, then the following properties of* G *are equivalent*:

(i) $\mathfrak{F}G$ *is hypercentral.*

(ii) $H/\mathfrak{c}_H M$ *is an* $\mathfrak{F}$*-group for every minimal normal subgroup* M *of every epimorphic image* H *of* G.

(iii) $[G/\mathfrak{h}\mathfrak{F}G]/\mathfrak{z}[G/\mathfrak{h}\mathfrak{F}G]$ *is an* $\mathfrak{F}$*-group.*

(iv) $\mathfrak{F}G \subseteq \mathfrak{z}_G G$.

(v) $\mathfrak{F}G = \mathfrak{z}_{\mathfrak{F}G}\, \mathfrak{F}G$.

This is easily deduced from Proposition 4.10 by letting $N = G$.

In case the property $\mathfrak{F}$ under discussion is a property of [finite] soluble groups, it is possible to connect θ-immersion with properties of maximal subgroups. With this in mind we prove the

LEMMA 4.13. *If M is a minimal normal subgroup of the finite group G, and if $G/\mathfrak{c}_G M$ is soluble, then*

(1) *M is a primary elementary abelian group;*

(2) *there exists a subgroup S of G with $M \cap S = 1$ and $G = S\mathfrak{c}_G M$;*

(3) *if S is a subgroup of G with $M \cap S \subset M$ and $G = S\mathfrak{c}_G M$, then $M \cap S = 1$ and S is a maximal subgroup of MS;*

(4) *if T is a maximal subgroup of G with $M \not\subseteq T$, then*

(a) *$M \cap T = 1$ and $G = MT$,*

(b) *$M = \{m^T\}$ for every $m \neq 1$ in M,*

(c) *$o(M) = [G:T]$.*

Proof. Since M is a minimal normal subgroup of G, we have either $M \subseteq \mathfrak{c}_G M$ or else $M \cap \mathfrak{c}_G M = 1$. In the first case M is abelian; and in the second case

$$M = M/(M \cap \mathfrak{c}_G M) \simeq M\mathfrak{c}_G M/\mathfrak{c}_G M \subseteq G/\mathfrak{c}_G M$$

is soluble. Thus M is in either case a finite, soluble and characteristic simple group; and such a group is primary elementary abelian. This proves (1).

Because of (1) and the solubility of $G/\mathfrak{c}_G M$ we may deduce (2) from Baer [4, p. 650, Lemma 1, p. 656, Lemma 2 and p. 651, Proposition 2, (b)].

Assume next that S is a subgroup of G with $M \cap S \subset M$ and $G = S\mathfrak{c}_G M$. Then $S \subset MS = T$. Let X be a subgroup with $S \subseteq X \subset T$. Then $G = X\mathfrak{c}_G M$ and $M \not\subseteq X$, since otherwise $T = MS \subseteq X$. Hence

$$M \cap S \subseteq M \cap X \subset M.$$

Since $M \cap X$ is normalized by X and centralized by $\mathfrak{c}_G M$, it is normalized by G; and we deduce $1 = M \cap X = M \cap S$ from the minimality of M. Apply Dedekind's Modular Law on $S \subseteq X \subset T = MS$ to see that $X = S(X \cap M) = S$, proving that S is a maximal subgroup of $T = MS$. Thus we have verified (3).

(4) finally is an immediate consequence of (1) and Baer [3, p. 118, Lemma 1].

Remark 4.14. The requirement that G be finite is certainly too strong; but it seems likely that it does not suffice to require only the finiteness of M.

Suppose that $\mathfrak{J}$ is a set of positive integers, containing with any integer all its positive divisors; and denote by $\mathfrak{A} = \mathfrak{A}(\mathfrak{J})$ the class of all finite groups whose orders belong to $\mathfrak{J}$. It is clear that this class $\mathfrak{A}$ is factor inherited. Denote, furthermore, by $\mathfrak{B}$ some factor inherited class of finite soluble groups which is residually closed. It is a consequence of Proposition 1.5 that $(\mathfrak{A}, \mathfrak{B})$-immersion is factor inherited; and it is a consequence of Corollary 2.4, (B) that $(\mathfrak{A}, \mathfrak{B})$-immersion and strict $(\mathfrak{A}, \mathfrak{B})$-immersion are equivalent properties of finite normal subgroups. It is a consequence of Lemma 4.13 that every finite $(\mathfrak{A}, \mathfrak{B})$-group is soluble. Application of Gaschütz [p. 302, Satz 3.1] shows, therefore, that the class of finite $(\mathfrak{A}, \mathfrak{B})$-groups is a saturated formation. These assumptions concerning $\mathfrak{A}$, $\mathfrak{B}$ we retain throughout the remainder of §4.

PROPOSITION 4.15. *The following properties of the normal subgroup N of the finite group G are equivalent:*

(i) $N\ (\mathfrak{A}(\mathfrak{J}), \mathfrak{B})\ G$.

(ii) $\begin{cases} \text{(a) } \textit{If the normal subgroup } K \textit{ of } G \textit{ is part of } N, \textit{ and if the subgroup} \\ \quad\ \ S \textit{ of } G \textit{ is a maximal subgroup of } KS, \textit{ then } [KS:S] \textit{ belongs to } \mathfrak{J}. \\ \text{(b) } N \cap \mathfrak{B}G \subseteq \mathfrak{h}\mathfrak{B}G. \end{cases}$

Proof. We have noted already that $(\mathfrak{A}, \mathfrak{B})$-immersion is factor inherited and equivalent to strict $(\mathfrak{A}, \mathfrak{B})$-immersion of finite normal subgroups. Hence (i) is equivalent with

(i*) $N\ \overline{(\mathfrak{A}, \mathfrak{B})}\ G$.

Assume now the validity of (i*) and consider a subgroup S of G and a normal subgroup K of G with $K \subseteq N$ such that S is a maximal subgroup of $KS = T$. Then $S \subset T$ so that $K \not\subseteq S$. Denote by σ the canonical epimorphism of T upon $H = T/S_T$. Since the normal subgroup K of G is not part of S, it is not part of S_T either and this implies $K^\sigma \neq 1$. Consequently there exists a minimal normal subgroup M of H with $M \subseteq K^\sigma$. From $S^\sigma = S/S_T$ we deduce $(S^\sigma)_H = 1$ so that the maximal subgroup S^σ of H does not contain M. Since $(\mathfrak{A}, \mathfrak{B})$-immersion and strict $(\mathfrak{A}, \mathfrak{B})$-immersion are factor inherited, it follows from (i*) that $K^\sigma\ \overline{(\mathfrak{A}, \mathfrak{B})}\ H$. Since M is a minimal normal subgroup of H with $M \subseteq K^\sigma$, we conclude that

(a) M is an $\mathfrak{A}$-group and
(b) $H/\mathfrak{c}_H M$ is a $\mathfrak{B}$-group.

From (a) and our definition of the class $\mathfrak{A} = \mathfrak{A}(\mathfrak{J})$ we deduce that

(a′) $o(M)$ belongs to $\mathfrak{J}$.

Since $\mathfrak{B}$-groups are, by hypothesis, soluble, we deduce from (b) that

(b′) $H/\mathfrak{c}_H M$ is soluble.

Application of Lemma 4.13, (4.c) shows now that

(c) $o(M) = [H:S^\sigma] = [KS:S]$;

and thus we have derived (ii.a) from (i*).

The validity of (ii.b) is easily deduced from (i*) and Proposition 4.10.

Assume conversely the validity of (ii). Consider an epimorphism σ of G upon H and a minimal normal subgroup M of H with $M \subseteq N^\sigma$. Then we deduce from (ii.b) and Proposition 4.10 that

(d) $H/\mathfrak{c}_H M$ is a $\mathfrak{B}$-group.

But all $\mathfrak{B}$-groups are soluble so that

(d′) $H/\mathfrak{c}_H M$ is soluble.

Apply Lemma 4.13, (2)–(4) to show that

(e) there exists a subgroup S of H with the following properties:

$$M \cap S = 1, \qquad H = S\mathfrak{c}_H M, \qquad o(M) = [MS:S],$$
$$S \text{ is a maximal subgroup of } MS.$$

The inverse images $K = M^{\sigma^{-1}}$ and $T = S^{\sigma^{-1}}$ contain both the kernel L of σ. Clearly K is a normal subgroup of G with $L \subseteq K \subseteq NL$ so that by Dedekind's Modular Law $K = L(N \cap K)$ with $N \cap K$ a normal subgroup of G, contained in N. Likewise T is a maximal subgroup of

$$KT = (N \cap K)LT = (N \cap K)T.$$

Application of (ii.a) shows now that

$$[(N \cap K)T:T] = [KT:T]$$

belongs to $\mathfrak{J}$. But

$$o(M) = [MS:S] = [(K/L)(T/L):(T/L)] = [(KT/L):(T/L)] = [KT:T]$$

so that

(f) $o(M)$ belongs to $\mathfrak{J}$.

Recall the definition of $\mathfrak{A} = \mathfrak{A}(\mathfrak{J})$ and we have shown that

(f′) M is an $\mathfrak{A}$-group.

Combine (d) and (f′) to see that N is strictly $(\mathfrak{A}, \mathfrak{B})$-immersed in G. Hence (ii) implies (i*), proving the equivalence of (i), (i*) and (ii).

Corollary 4.16. *The finite group G is an $(\mathfrak{A}(\mathfrak{J}), \mathfrak{B})$-group if, and only if,*
(a) *$[G:S]$ belongs to $\mathfrak{J}$ for every maximal subgroup S of G and*
(b) *$\mathfrak{B}G$ is nilpotent.*

Proof. If we let $N = G$ in Proposition 4.15, then our conditions (a) and (b) are nothing but weak forms of conditions (ii.a) and (ii.b) respectively.

If there existed groups, meeting requirements (a) and (b) which are not $(\mathfrak{A}, \mathfrak{B})$-groups, then there would exist among these one, say G, of minimal order. We note:

(1) Conditions (a) and (b) are satisfied by G.
(2) G is not an $(\mathfrak{A}, \mathfrak{B})$-group.

If σ is an epimorphism of G upon H, and if S is a maximal subgroup of H, then the inverse image $S^{\sigma^{-1}} = T$ is a maximal subgroup of G. Hence $[H:S] = [G:T]$ belongs to $\mathfrak{J}$. Noting (4.IV) we conclude that $\mathfrak{B}H = (\mathfrak{B}G)^{\sigma}$ is nilpotent. Thus conditions (a), (b) are satisfied by every epimorphic image of G. Because of the minimality of G we conclude:

(3) Every proper epimorphic image of G is an $(\mathfrak{A}, \mathfrak{B})$-group.

$G \neq 1$ by (2). Consequently, there exist minimal normal subgroups of G. If $A \neq B$ are minimal normal subgroups of G, then G/A and G/B are, by (3), both $(\mathfrak{A}, \mathfrak{B})$-groups. Naturally, $A \cap B = 1$. Hence G is isomorphic to a subgroup of the direct product $(G/A) \otimes (G/B)$ of two $(\mathfrak{A}, \mathfrak{B})$-groups; and this implies that G itself is an $(\mathfrak{A}, \mathfrak{B})$-group, contradicting (2). Thus we have shown:

(4) There exists one and only one minimal normal subgroup M of G; and G/M is an $(\mathfrak{A}, \mathfrak{B})$-group.

Assume by way of contradiction that the Frattini subgroup $\Phi G \neq 1$. Then $G/\Phi G$ is, by (3), an $(\mathfrak{A}, \mathfrak{B})$-group. But we noted before that the class of finite $(\mathfrak{A}, \mathfrak{B})$-groups is a saturated formation. Hence G is an $(\mathfrak{A}, \mathfrak{B})$-group, contradicting (2). It follows that

(5) $\Phi G = 1$.

By (5), there exists a maximal subgroup S of G with $M \not\subseteq S$. Application of (b) and Corollary 4.12 shows that $G/\mathfrak{c}_G M$ is a $\mathfrak{B}$-group; and as such $G/\mathfrak{c}_G M$ is soluble. Application of Lemma 4.13, (4.c) shows that $o(M) = [G:S]$; and the latter number is by (a) a number in $\mathfrak{J}$ so that M is an $\mathfrak{A}$-group. We have shown:

(6) $M\ (\mathfrak{A}, \mathfrak{B})\ G$.

Combine (4), (6) with Proposition 1.4, (b) to see that $G\ (\mathfrak{A}, \mathfrak{B})\ G$, contradicting (2); and this contradiction completes the proof.

LEMMA 4.17. *Every finite minimal normal subgroup M of a group G has the following properties*:
(A) *If the maximal subgroup S of G does not contain M, then*
  (a) $M \cap S_G = 1$;
  (b) $S_G = S \cap \mathfrak{c}_G M$;
  (c) $M \cap S \simeq (M \cap S)\mathfrak{c}_G M/\mathfrak{c}_G M$.
  (d) S/S_G *and* $[G:S]$ *are finite.*
(B) *The following properties of M are equivalent:*
  (i) *M is a primary elementary abelian group.*
  (ii) *M is primary.*
  (iii) *M is soluble.*
  (iv) *If S is a maximal subgroup of G, then $M \cap S = 1$ or M.*
  (v) *If the maximal subgroup S of G does not contain M, then $[G:S]$ is a power of a prime and $M \cap S$ is nilpotent.*

Proof. If the maximal subgroup S of G does not contain M, then $M \cap S_G \subset M$; and $M \cap S_G = 1$ is a consequence of the minimality of M. But this implies that the normal subgroups M and S_G of G centralize each other so that $S_G \subseteq \mathfrak{c}_G M$. Hence $S_G \subseteq S \cap \mathfrak{c}_G M$. From the maximality of S and $M \subseteq S$ we deduce $G = MS$. Since $S \cap \mathfrak{c}_G M$ is certainly normalized by S and centralized by M, it is normalized by $MS = G$, hence a normal subgroup of G so that $S \cap \mathfrak{c}_G M \subseteq S_G$. This proves (b).

From (a) and (b) it follows that

$$(M \cap S)\mathfrak{c}_G M/\mathfrak{c}_G M \simeq (M \cap S)/(M \cap S \cap \mathfrak{c}_G M) = M \cap S,$$

proving (c). Next we note that

$$[G:S] = [MS:S] = [M:(M \cap S)] \leq o(M);$$

and the finiteness of M implies that of $[G:S]$. Applying (b) we see that

$$S/S_G = S/(S \cap \mathfrak{c}_G M) \simeq S\mathfrak{c}_G M/\mathfrak{c}_G M \subseteq G/\mathfrak{c}_G M;$$

and the latter group is finite as a group of automorphisms of the finite group M. This proves (d), showing the validity of (A).

Now we turn to the proof of (B). It is clear that (i) implies (ii) and that (ii) implies (iii). If M is soluble, then $M' \subset M$ so that $M' = 1$, since M is characteristic simple. If p is a prime divisor of $o(M)$, then it follows likewise that $M^p = 1$. Hence (i) is a consequence of (iii), showing the equivalence of (i)–(iii).

Assume next the validity of condition (i) and consider a maximal subgroup S of G with $M \not\subseteq S$. Then $G = MS$ and

$$[G:S] = [M:(M \cap S)] \quad \text{is a divisor of} \quad o(M).$$

But $o(M)$ is, by (i), a power of a prime so that $[G:S]$ is a power of a prime. Since M is abelian, so is $M \cap S$; and thus we have deduced (v) from (i).

Assume now that condition (v) is satisfied by M and that the maximal subgroup S of G does not contain M. Then $M \cap S$ is nilpotent and $[G:S]$ is a prime power p^n. Since $G = MS$, we have

$$p^n = [G:S] = [M:(M \cap S)].$$

This implies that M is the product of the nilpotent group $M \cap S$ and of a p-Sylow subgroup of M. But such a [finite] product is soluble by the Theorem of Kegel-Wielandt; see Scott [p. 381, 13.2.9]. Hence M is soluble. Since (i)–(iii) have been shown to be equivalent, M is abelian. Thus $M \cap S$ is normalized by S and centralized by M so that $M \cap S$ is a normal subgroup of $MS = G$. Since $M \cap S \subset M$, we deduce $M \cap S = 1$ from the minimality of M. We have deduced (iv) from (v).

Assume finally that M is not primary. From $M \neq 1$ we deduce the existence of a prime divisor p of $o(M)$. If P is a p-Sylow subgroup of M, then

$$1 \subset P \subset M.$$

From the minimality of M we deduce that P is not a normal subgroup of G. Hence $\mathfrak{n}_G P \subset G$. Since every subgroup, conjugate to P in G, is contained in M, the number of subgroups, conjugate to P in G, is finite. Hence $[G:\mathfrak{n}_G P]$ is finite [and not 1]. Consequently there exists a maximal subgroup T of G

with $\mathfrak{n}_G P \subseteq T$. By the Frattini argument we have $G = M\mathfrak{n}_G P$; see Baer [3, p. 117, Lemma 1]. This implies $G = MT$. Now we note that

$$1 \subset P \subseteq M \cap \mathfrak{n}_G P \subseteq M \cap T \subset M$$

[by $G = MT$]. Hence (iv) is not satisfied by M, if (ii) is not satisfied by M. Consequently (ii) follows from (iv), completing the proof of the equivalence of (i)–(v).

Remark 4.18. **A.** If the finite minimal normal subgroup M of G is not abelian, then we have shown among other things the existence of a maximal subgroup of G which does not contain M; see (iv). But the abelian groups of type p^∞ provide examples of groups with minimal normal subgroups and without maximal subgroups.

B. It is easy to derive from Lemma 4.17, (B) a criterion for the solubility of finite groups.

PROPOSITION 4.19. *If the property $\mathfrak{F}$ of finite nilpotent groups is inherited by factors and direct products, then the following properties of the finite group G are equivalent*:

(i) *$\mathfrak{F}G$ is nilpotent.*

(ii) *If M is a minimal normal subgroup of the epimorphic image H of G, then $H/\mathfrak{c}_H M$ is an $\mathfrak{F}$-group.*

(iii) *If S is a maximal subgroup of G, then S/S_G is an $\mathfrak{F}$-group and $[G:S]$ is a prime power.*

Proof. If $\mathfrak{D}$ is the class of finite nilpotent groups, then $\mathfrak{F}$ is a $\mathfrak{D}$-formation. Application of Corollary 4.12 shows, therefore, the equivalence of properties (i) and (ii).

We assume next that G meets the equivalent requirements (i) and (ii). Since $\mathfrak{F}G$ is nilpotent and $G/\mathfrak{F}G$ is an $\mathfrak{F}$-group [as (4.IV) may be applied on the $\mathfrak{D}$-formation $\mathfrak{F}$], it follows that G is an extension of the nilpotent group $\mathfrak{F}G$ by the nilpotent group $G/\mathfrak{F}G$. Hence G is soluble. Consider a maximal subgroup S of G. Then $[G:S]$ is a prime power, as G is soluble; see, for instance, Lemma 4.17, (B.v). Let $H = G/S_G$ and $T = S/S_G$. Then T is a maximal subgroup of H with $T_H = 1$. Since $H \neq 1$, there exists a minimal normal subgroup M of H. Since G is soluble, so is M. Hence M is abelian [Lemma 4.17, (B)] so that $M \subseteq \mathfrak{c}_H M$. Since $T_H = 1$, we have $M \not\subseteq T$. Since T is a maximal subgroup of H, application of Lemma 5.12, (A.b) shows that $1 = T \cap \mathfrak{c}_H M$. Hence

$$T = T/[T \cap \mathfrak{c}_H M] \simeq T\mathfrak{c}_H M/\mathfrak{c}_H M \subseteq H/\mathfrak{c}_H M.$$

Hence T is, by (ii), isomorphic to a subgroup of an $\mathfrak{F}$-group; and as such $S/S_G = T$ is an $\mathfrak{F}$-group, proving (iii).

Since $\mathfrak{F}$ is a $\mathfrak{D}$-formation, and since the groups with property (ii) are [finite and] soluble, they form a saturated formation by Gaschütz [p. 302, Satz 3.1].

Assume now that G meets requirement (iii). Consider an epimorphism σ of G upon H and a maximal subgroup S of H. Then $T = S^{\sigma^{-1}}$ is a maximal subgroup of G which contains the kernel K of σ so that $K \subseteq T_G$ and $(T_G)^\sigma = S_G$. Hence $[H:S] = [G:T]$ and

$$S/S_G \simeq T/T_G$$

is an $\mathfrak{F}$-group by (iii); and we have shown:

(1) Every epimorphic image of G meets requirement (iii).

Assume now by way of contradiction that (i) and (ii) are not satisfied by G. Since G is finite, there exists consequently an epimorphic image H of G with the following properties:

(2) The equivalent conditions (i) and (ii) are not satisfied by H; but they are satisfied by every proper epimorphic image of H.

If the Frattini subgroup ΦH were not 1, then $H/\Phi H$ would satisfy (ii). But the class of finite groups with property (ii) is a saturated formation. Thus this would imply that H meets requirement (ii), contradicting (2). Consequently

(3) $\Phi H = 1$.

Since $H \neq 1$ by (2), there exists a minimal normal subgroup M of H. If A were a minimal normal subgroup of H with $M \neq A$, then $M \cap A = 1$. But H/M and H/A meet requirement (ii) by (2). Property (ii) is a saturated formation. Hence $H = H/(M \cap A)$ satisfies (ii), contradicting (2). We have shown:

(4) There exists one and only one minimal normal subgroup M of H.

Suppose that there exists a maximal subgroup S of H with $1 \subset M \cap S \subset M$. Then $M \not\subseteq S_H$ so that $S_H = 1$ by (4). Apply (1) to see that $S = S/S_H$ is an $\mathfrak{F}$-group. Hence $M \cap S$ is an $\mathfrak{F}$-group and consequently nilpotent. Apply (1) to see that

$$[M:M \cap S] = [MS:S] = [H:S]$$

is a prime power. Apply Lemma 4.17, (B) to see that $M \cap S = 1$ or $M \cap S = M$, a contradiction. Hence

(5) $M \cap S = 1$ or M for every maximal subgroup S of H.

A second application of Lemma 4.17, (B) shows that

(5*) M is a primary elementary abelian group.

By (3) there exists a maximal subgroup S of H with $M \not\subseteq S$. We deduce $M \cap S = 1$ from (5) and $S_H = 1$ from (4). Hence S is an $\mathfrak{F}$-group by (1).

Application of Lemma 4.17, (A.b) shows that $S \cap \mathfrak{c}_H M = 1$. From (5*) we deduce

$$M \subseteq \mathfrak{c}_H M \quad \text{and} \quad H = SM = S\mathfrak{c}_H M$$

because of the maximality of S. Consequently

$$H/\mathfrak{c}_H M = S\mathfrak{c}_H M/\mathfrak{c}_H M \simeq S/[S \cap \mathfrak{c}_H M] = S$$

is an $\mathfrak{F}$-group. We note:

(6) $H/\mathfrak{c}_H M$ is an $\mathfrak{F}$-group.

Combine (6), (4) and (2) to see that H meets requirement (ii). This contradicts (2); and this contradiction shows that (i) and (ii) are satisfied by G, proving the equivalence of (i)–(iii).

5. The considerations of this section are based on the following lemma which is a collection of more or less well known results. The proof, obtained essentially by application of Schur's Lemma and the theory of finite fields, is included for the convenience of the reader.

LEMMA 5.1. *If the minimal normal subgroup M of G is finite, and if $M \circ G' = 1$, then M and $\Gamma = G/\mathfrak{c}_G M$ have the following properties:*

(1) *M is a primary elementary abelian group.*

(2) *Γ is cyclic [and finite].*

(3) *If the order $o(M) = p^r$, then r is both the minimum and the greatest common divisor of all the positive integers n with $p^n \equiv 1$ modulo $o(\Gamma)$.*

Proof. By hypothesis M and G' centralize each other. Hence $G' \subseteq \mathfrak{c}_G M$ so that

(2′) Γ is abelian.

But Γ is essentially the same as the group of automorphisms, induced in M by G. Hence M induces an abelian group of automorphisms in M so that $M/\mathfrak{z}M$ is abelian. But M is characteristic simple. In particular therefore $\mathfrak{z}M = 1$ or $\mathfrak{z}M = M$. In either case M is abelian. Since M is a characteristic simple, finite, and abelian group, $M^p = 1$ for every prime divisor p of $o(M)$. This proves (1).

Let $o(M) = p^r$ [for p a prime and r a positive integer]. Identify Γ with the group of automorphisms, induced in M by G. Since M is abelian, the endomorphisms of M form a ring. Hence a ring Λ of endomorphisms is spanned by Γ. Since Γ is abelian, Λ is a commutative ring; and the group identity of Γ is the ring identity of Λ. Since M is finite, so is Λ.

If $\sigma \neq 0$ is an endomorphism in Λ, then $M^\sigma \neq 1$ and the kernel $K(\sigma)$ of σ is different from M. Furthermore

$$K(\sigma)^\lambda = K(\sigma^\lambda) = K(\sigma) \quad \text{for every} \quad \lambda \text{ in } \Gamma.$$

Hence $K(\sigma)$ is a normal subgroup of G and we deduce $K(\sigma) = 1$

from $K(\sigma) \subset M$ and the minimality of M. But M is finite so that σ is an automorphism of M. Since Λ is a finite ring and every element, not 0, in Λ is an automorphism of M,

(4) Λ is a finite field.

The multiplicative group of a finite field is cyclic. Since Γ is a subgroup of the multiplicative group of Λ, we conclude that Γ is cyclic, proving (2).

Every element $t \neq 1$ in M defines a mapping $\sigma \rightarrow t^{\sigma}$ of Λ into M. This mapping is single-valued and additive. Since every $\sigma \neq 0$ in Λ is an automorphism of M, this mapping is a monomorphism of the additive group Λ_{+} of Λ into M. Clearly t^{Λ} is a subgroup of M which contains $t^{1} = t \neq 1$. But $\Gamma \subseteq \Lambda$ so that t^{Λ} is a normal subgroup, not 1, of G. Apply the minimality of M to see that $t^{\Lambda} = M$. Hence $M \simeq \Lambda_{+}$; and this implies in particular

(5) $o(M) = p^{r}$ is the number of elements in the field Λ.

If we denote by Λ^{*} the multiplicative group of the field Λ, then we deduce from (5) that

(6) $o(\Lambda^{*}) = p^{r} - 1$.

Noting that Γ is a subgroup of Λ^{*}, it follows that $o(\Gamma)$ is a divisor of $p^{r} - 1$. Hence

(7) $p^{r} \equiv 1$ modulo $o(\Gamma)$.

The field Λ contains, by (5), exactly p^{r} elements. Mapping every element in Λ upon its p-th power produces an automorphism β of Λ whose order is r. This automorphism β induces in the subgroup Γ of Λ^{*} an automorphism of order $s \leq r$. The elements in Λ, fixed by β^{s}, form a subfield of Λ which contains exactly p^{s} elements. But Λ is spanned by Γ so that Γ is not part of a proper subfield of Λ. Hence $s = r$ and we have shown:

(8) Raising every element in Γ into its p-th power is an automorphism of order r of Γ.

Consequently r is both the minimum and the greatest common divisor of all the positive integers n with

$$\omega^{p^n} = \omega \quad \text{for every element } \omega \text{ in } \Gamma.$$

Since Γ is, by (2), a finite cyclic group, this condition is equivalent to

$$p^{n} \equiv 1 \quad \text{modulo} \quad o(\Gamma);$$

and this proves (3).

If N is a normal subgroup of G, if σ is an epimorphism of G upon H, and if M is a minimal normal subgroup of H with $M \subseteq N^{\sigma}$, then we say that M *is a principal factor of* G, *contained in* [or covered by] N. The group $H/\mathfrak{c}_H M$ is essentially the same as the group of automorphisms, induced in M by H;

and we term it *the group of automorphism, induced by G in its principal factor M*. If M happens to be an elementary abelian group of [finite] order $o(M) = p^r$, then we term p *the characteristic* and r *the rank of M*.

LEMMA 5.2. *If P is a finite normal p-subgroup of the group G, if G induces in every principal factor, contained in P, an abelian group of automorphisms, if k is a common multiple of all the ranks of principal factors of G, contained in P, then*

(a) *the groups of automorphisms, induced by G in the principal factors contained in P, are cyclic groups of orders dividing $p^k - 1$ and*

(b) $x^{p^k-1} \circ P = 1$ *for every x in G with* $(o(x\mathfrak{c}_G P), p) = 1$.

Proof. Suppose that σ is an epimorphism of G upon H and that M is a minimal normal subgroup of H with $M \subseteq P^\sigma$. By hypothesis $H/\mathfrak{c}_H M$ is abelian and this is equivalent to $H' \subseteq \mathfrak{c}_H M$ and hence to $M \circ H' = 1$. Naturally M is an elementary abelian p-group so that $o(M) = p^r$. By hypothesis k is a multiple of r. An immediate application of Lemma 5.1 shows that

(1) $H/\mathfrak{c}_H M$ is cyclic [and finite] and r is the greatest common divisor of all the positive integers n with

$$p^n \equiv 1 \quad \text{modulo} \quad o(H/\mathfrak{c}_H M).$$

Since k is a multiple of r, it follows in particular that

(2) $p^k \equiv 1$ modulo $o(H/\mathfrak{c}_H M)$.

Consider an element x in H. Then x induces in M an automorphism λ; and it is a consequence of (2) that $\lambda^{p^k-1} = 1$. If y is an element in M, then

$$y = y^{\lambda^{p^k-1}} = y^{x^{p^k-1}} = x^{-(p^k-1)}yx^{p^k-1}$$

and consequently

(3) $y \circ x^{p^k-1} = 1$ for every x in H and every y in M.

Consider now an element x in G with $(o(x\mathfrak{c}_G P), p) = 1$. There exist normal subgroups $N(i)$ of G with

$$1 = N(0), \quad N(i) \subset N(i+1), \quad N(j) = P,$$

$$N(i+1)/N(i) \text{ is a minimal normal subgroup of } G/N(i).$$

The element x induces in every $N(i + 1)/N(i)$ an automorphism whose order is, by (3), a divisor of $p^k - 1$. Hence x^{p^k-1} induces the 1-automorphism in every $N(i + 1)/N(i)$. It is well known and easily verified that consequently x^{p^k-1} induces in P a p-automorphism. But x induces in P an automorphism of order prime to p. The automorphism, induced by x^{p^k-1} in P, has consequently at the same time order a power of p and order prime to p so that x^{p^k-1} induces the 1-automorphism in P. Thus we have shown:

(4) $x^{p^k-1} \circ P = 1$ for every x in G with $(o(x\mathfrak{c}_G P), p) = 1$.

This proves (b); and (a) is contained in (1), (2).

THEOREM 5.3. *If $e(p)$ is, for every prime p, a positive integer, then the following properties of the finite normal subgroup N of G are equivalent:*

(i) (a) *G induces in every principal factor, contained in N, an abelian group of automorphisms.*
(b) *If a principal factor of G, contained in N, is a primary elementary abelian group of characteristic p and rank r, then $e(p)$ is a multiple of r.*

(ii) (a) *G induces in every principal factor, contained in N, an abelian group of automorphisms.*
(b) *$x^{p^{e(p)}-1} \circ y = 1$ for every prime p, every p-element y in $N \cap G'$ and every x in G with $(o(x\mathfrak{c}_G(N \cap G')), p) = 1$.*

(iii) (a) *N is soluble.*
(b) *If the principal factor M of G is contained in $N \cap G'$, if M is an elementary abelian p-group, then the group of automorphisms, induced in M by G, is cyclic of order dividing $p^{e(p)} - 1$.*

Note. Because of Proposition 4.10 [(iii), (vi)] we may substitute for conditions (i.a) and (ii.a) the requirement:

$$N \cap G' \text{ is nilpotent and } G' = (N \cap G')_{\mathfrak{s}G'}(N \cap G').$$

Proof. Noting that N is finite and that the property $\mathfrak{K}$ of being abelian meets all the desired requirements we deduce from Proposition 4.10, [(i), (iii)] that the conditions (i.a) and (ii.a) are equivalent to

(A) $N \cap G' \subseteq \mathfrak{h}(G')$.

If (A) is satisfied, then we have clearly:

(A′) N is soluble.

(A″) Every principal factor of G which is contained in N is a primary elementary abelian group.

(A‴) $N \cap G'$ is nilpotent [and consequently a direct product of primary groups].

Assume now the validity of (i). It is a consequence of (A‴) that the set P of all the p-elements in $N \cap G'$ is a characteristic subgroup of the normal subgroup $N \cap G'$. Thus P is a normal p-subgroup of G. If a principal factor of G is contained in P, then it is likewise contained in N and enjoys consequently property (i.a). We may apply Lemma 5.2, (b) to show that

(B) $x^{p^{e(p)}-1} \circ P = 1$ for every x in G with $(o(x\mathfrak{c}_G P), p) = 1$.

It is clear now that (ii) is a consequence of (i).

Assume next the validity of (ii). We have already deduced (iii.a) [= (A')] from (ii.a). Consider an epimorphism σ of G upon H and a minimal normal subgroup M of H with $M \subseteq N^\sigma \cap H'$. We noted already that N is soluble. Hence M is a finite, soluble, characteristic simple group and consequently an elementary abelian p-group. Every element of order a power of p in M is the σ-image of an element of order a power of p in $N \cap G'$; and every element of order prime to p in $H/\mathfrak{c}_H M$ is the σ-image of an element of order prime to p in $G/\mathfrak{c}_G(N \cap G')$. Apply (ii.b) to show that

$$x^{p^{e(p)}-1} \circ M = 1 \quad \text{for every } x \text{ in } H \text{ with } (o\,(x\mathfrak{c}_H M), p) = 1.$$

It is a consequence of (ii.a) and Lemma 5.2, (a) that

$$H/\mathfrak{c}_H M \text{ is cyclic of order prime to } p.$$

Combine these two results to see that

$$H/\mathfrak{c}_H M \text{ is cyclic of an order dividing } p^{e(p)} - 1.$$

This shows that (iii) is a consequence of (ii).

Assume finally the validity of (iii). If the principal factor M of G is contained in N, then M is finite, soluble, characteristic simple; and this implies:

(+) Every principal factor of G which is contained in N is a primary elementary abelian group.

Consider now an epimorphism σ of G upon H and a minimal normal subgroup M of H with $M \subseteq N^\sigma$. If firstly $M \cap H' = 1$, then

$$M \circ H \subseteq M \cap H' = 1$$

so that M and H centralize each other; and this implies $M \subseteq \mathfrak{z}H$ so that M because of its minimality is cyclic of order a prime. If secondly $M \cap H' \neq 1$, then we deduce $M \subseteq H'$ from the minimality of M. Hence $M \subseteq N^\sigma \cap H'$. The principal factor M of G is consequently contained in $N \cap G'$. Because of (+) the principal factor M of G is a primary elementary abelian p-group and thus its [finite] order is $o(M) = p^r$. We may apply (iii.b) to see that $H/\mathfrak{c}_H M$ is cyclic of order dividing $p^{e(p)} - 1$. But then Lemma 5.1, (3) shows that r is a divisor of $e(p)$. This shows that (i) is a consequence of (iii), completing the proof of the equivalence of (i)–(iii).

To obtain a formulation of Theorem 5.3 which is more in accord with the terminology employed in Section 3, let $e(p)$ be for every prime p a non-negative integer. Then

$$\mathfrak{A}_e'(p) = \text{class of finite elementary abelian } p\text{-groups with rank dividing } e(p),$$

$$\mathfrak{B}_e'(p) = \mathfrak{K}\ [= \text{class of abelian groups}],$$

$$\theta_e' = \text{family of pairs } (\mathfrak{A}_e'(p), \mathfrak{B}_e'(p));$$

$$\mathfrak{A}_e''(p) = \text{class of elementary abelian } p\text{-groups},$$

$$\mathfrak{B}_e''(p) = \text{class of cyclic groups of order dividing } p^{e(p)} - 1,$$

$$\theta_e'' = \text{family of pairs } (\mathfrak{A}_e''(p), \mathfrak{B}_e''(p)).$$

If $e(p) = 0$ for some prime p, then $\mathfrak{A}'_e(p)$ is the class of all finite elementary abelian p-groups and $\mathfrak{B}''_e(p)$ is the class of all cyclic groups.

COROLLARY 5.4. *Assume that N is a finite normal subgroup of G.*

(a) *$N \bar{\theta}'_e G$, $N \theta''_e G$ and $N \bar{\theta}''_e G$ are equivalent properties.*

(b) *$N \bar{\theta}'_e G$ implies $N \theta'_e G$.*

(c) *If $e(p) \leq 2$ for every p, then $N \theta G$ and $N \bar{\theta}'_e G$ are equivalent properties.*

Proof. It is an immediate consequence of the equivalence of conditions (i) and (iii) of Theorem 5.3 that $N \bar{\theta}'_e G$ and $N \bar{\theta}''_e G$ are equivalent properties. Since $\mathfrak{A}''_e(p)$ and $\mathfrak{B}''_e(p)$ are factor inherited properties, we deduce from Proposition 1.5 that θ''_e-immersion is factor inherited; and it follows therefore from Corollary 2.4, (B) that $N \bar{\theta}''_e G$ and $N \theta''_e G$ are equivalent properties. This completes the proof of (a); and (b) is an immediate consequence of Corollary 2.4, (A). —If finally $e(p) \leq 2$ for every p, then the properties $\mathfrak{A}'_e(p)$ and $\mathfrak{B}'_e(p)$ are likewise factor inherited: it is a consequence of Proposition 1.5 that θ'_e-immersion is factor inherited and hence it is a consequence of Corollary 2.4, (B) that $N \bar{\theta}'_e G$ and $N \theta'_e G$ are equivalent properties.

Discussion 5.5. **A.** Corollary 5.4 shows that a certain class of immersions may be defined in two essentially different ways. We shall consequently say θ_e instead of $\bar{\theta}'_e$ and θ''_e. In short,

$$\theta_e = \bar{\theta}'_e = \theta''_e.$$

If $e(p) \leq 2$ for every prime p, then we have

$$\theta_e = \theta'_e$$

too. It is a consequence of Remark 2.3 that the latter equality does not hold true without the hypothesis $e(p) \leq 2$.

B. Condition (ii.b) of Theorem 5.3 is a "conditional identity".

C. If G happens to be a torsion group, then condition (ii.b) of Theorem 5.3 is equivalent to the following more elegant property:

$x^{p^{e(p)-1}} \circ y = 1$ for every prime p, every p-element y in $N \cap G'$ and every p'-element x in G.

COROLLARY 5.6. *The following properties of the finite, soluble, normal subgroup N of G are equivalent:*

(i) *$N \theta_e G$.*

(ii) *$S \theta_e \mathfrak{n}_G S$ for every primary subgroup S of N.*

(iii) *If S is a p-subgroup of N and if x is an element in $\mathfrak{n}_G S$ with $(o(x\mathfrak{c}_{\mathfrak{n}_G S} S), p) = 1$, then $x^{p^{e(p)-1}} \circ S = 1$; and $\mathfrak{n}_G S$ induces in every principal factor, contained in S, an abelian group of automorphisms.*

Proof. We noted before that θ_e-immersion is the same as θ''_e-immersion and that the latter is factor inherited [Remark 5.5, **A**]. Consequently we may apply Theorem 3.4 to prove the equivalence of (i) and (ii). It is a consequence of Lemma 5.2, (b) that (iii) is a consequence of (ii). If finally

(iii) is satisfied, then condition (ii) of Theorem 5.3 is satisfied by the normal and finite p-subgroup S of $\mathfrak{n}_G S$ so that $S\ \theta_e\ \mathfrak{n}_G S$ for every primary subgroup S of N. Thus (ii) and (iii) are likewise equivalent.

COROLLARY 5.7. *The following properties of the finite group G are equivalent:*

(i) *G is a $\bar{\theta}_e'$-group.*

(ii) *G is a θ_e''-group.*

(iii) *G induces in every principal factor an abelian group of automorphisms and*

$$x^{p^{e(p)-1}} \circ y = 1$$

for every prime p, every p-element y in G' and every p'-element x in G.

(iv) *G is soluble and $S\ \theta_e\ \mathfrak{n}_G S$ for every primary subgroup S of G.*

This is easily deduced from Corollary 5.4 and Corollary 5.6.

COROLLARY 5.7*. *If $e(p) \leq 2$ for every prime p, then the finite group G is a θ_e'-group if, and only if, G is a θ_e''-group.*

This is an immediate consequence of Corollary 5.7 and Corollary 5.4, (c).

PROPOSITION 5.8. *The classes θ_e of finite groups are saturated formations.*

Proof. Let

$\mathfrak{E}_e(p)$ be the class of finite elementary abelian p-groups,
$\mathfrak{D}_e(p)$ be the class of finite abelian groups D with $D^{p^{e(p)-1}} = 1$,
Δ_e the family of pairs $(\mathfrak{E}_e(p), \mathfrak{D}_e(p))$.

All the classes $\mathfrak{E}_e(p)$ and $\mathfrak{D}_e(p)$ are factor inherited and residual so that in particular they are formations in the sense of Gaschütz. According to Gaschütz [p. 302, Satz 3.1] the class Δ_e is a saturated formation. But it is easily deduced from Lemma 5.1 that $\Delta_e = \theta_e''$.

Remark 5.9. Naturally $\theta_e = \theta_e'' = \bar{\theta}_e'$ and the second definition involves classes of groups that are not factor inherited.

COROLLARY 5.10. *If $0 < e(p) \leq 2$, and if N is a finite normal subgroup of G, then $N\ \theta_e'\ G$ if, and only if, the following three conditions are satisfied:*

(a) $G' = (N \cap G')\mathfrak{z}_{G'}(N \cap G')$.

(b) $x^{p^{e(p)-1}} \circ y = 1$ *for every prime p, every p-element y in $N \cap G'$ and every x in G with $(o(x\mathfrak{c}_G(N \cap G')), p) = 1$.*

(c) *If $e(2) = 2$, then $x \circ y = 1$ for every 2-element x in $N \cap G'$ and every 3-element y in $N \cap G'$.*

Proof. The properties

$$N\ \bar{\theta}_e'\ G, \quad N\ \theta_e'\ G \quad \text{and} \quad N\ \theta_e''\ G$$

of the finite normal subgroup N of G are equivalent because of $0 < e(p) \leq 2$ and Corollary 5.4, (a) and (c).

If firstly $N \, \bar{\theta}'_e \, G$, then we deduce the validity of (b) from Theorem 5.3, (ii.b). From Theorem 5.3, (ii.a) and Proposition 4.10, [(iii), (vi)] we deduce

$$N \cap G' \subseteq \mathfrak{h}G',$$

$$N \cap G' \text{ is nilpotent and } G' = (N \cap G')\mathfrak{z}_{G'}(N \cap G');$$

note that finite hypercentral groups are nilpotent. Our conditions (a) and (c) are immediate consequences.

Assume conversely the validity of conditions (a)–(c). To prove the nilpotency of $N \cap G'$ consider a pair of different primes p, q and a p-element x and a q-element y, both in $N \cap G'$. We may assume without loss of generality that $p < q$. We have certainly $xy = yx$ [because of (c)] if we have at the same time $p = 2$, $q = 3$, $e(2) = 2$. Assume next that we do not have simultaneously $p = 2$, $q = 3$, $e(2) = 2$. If q were a divisor of $p^{e(p)} - 1$, then we recall that q is certainly not a divisor of $p - 1 < p < q$ and that $0 < e(p) < 3$. It would follow that $e(p) = 2$ and that q is a divisor of $p + 1 \leq q$. Hence $p + 1 = q$ so that [as p and q are primes] $p = 2$, $q = 3$, $e(2) = 2$, the case we excluded. Consequently q is prime to $p^{e(p)} - 1$; and this implies that $o(y)$ and $p^{e(p)} - 1$ are relatively prime. Hence

$$\{y\} = \{y^{p^{e(p)}-1}\}.$$

But x and $y^{p^{e(p)}-1}$ commute by (b) so that we have again $xy = yx$. Primary elements of relatively prime order in $N \cap G'$ commute consequently; and this is well known to be equivalent with the nilpotency of the finite group $N \cap G'$. Combine this with condition (a) and it follows from Corollary 4.8 that

$$N \cap G' \subseteq \mathfrak{h}G'.$$

Combine this with Proposition 4.1 and Proposition 2.2 to see that

G induces in every principal factor, contained in N, an abelian group of automorphisms.

Because of (b) condition (ii) of Theorem 5.3 is consequently satisfied by the finite normal subgroup N of G. Hence $N \, \bar{\theta}'_e \, G$ as we wanted to show.

Remark 5.11. Let $e(2) = 2$ and $e(p) = 1$ for odd primes p. Then the symmetric group of degree 4 is an example of a group G with the properties (a), (b) for $N = G$, though $N \cap G' = G'$ is not nilpotent. This shows that condition (c) is indispensable.

PROPOSITION 5.12. *If $e(p)$ is for every prime p a non-negative integer, then the following properties of the finite normal subgroup N of the group G are equivalent:*

(i) *If σ is an epimorphism of G upon H and if the minimal normal subgroup M of H is part of N^σ, then M is an elementary abelian group of prime power order p^r with r a divisor of $e(p)$ and $H/\mathfrak{c}_H M$ is abelian.*

(ii) (a) *If K is a normal subgroup of G and $K \subseteq N$, if S is a maximal subgroup of KS, then $[KS:S]$ is [finite and] a prime power p^r with r a divisor of $e(p)$.*
(b) $N \cap G' \subseteq \mathfrak{h}G'$.

(iii) (a) *If K is a normal subgroup of G and $K \subseteq N$, if S is a maximal subgroup of KS, then $[KS:S]$ is [finite and] a prime power p^r with r a divisor of $e(p)$ and S/S_{KS} is abelian.*
(b) *If σ is an epimorphism of G upon H, if the minimal normal subgroup M of H is part of N^σ, then there exists a subgroup S of H with $M \cap S = 1$ and $H = S\mathfrak{c}_H M$.*

Proof. In the notation used throughout this section condition (i) is equivalent to strict θ_e'-immersion of N in G; in short: (i) is equivalent with

(i*) $N\ \bar{\theta}_e'\ G$;

and it is a consequence of Corollary 5.4, (a) that (i*) is equivalent with

(i**) $N\ \theta_e''\ G$.

Assume first the validity of (i). Then (ii.b) may be deduced from Proposition 4.10. Next we note that θ_e''-immersion is factor inherited [Proposition 1.5]. Suppose now that K is a normal subgroup of G with $K \subseteq N$ and that the subgroup S of G is a maximal subgroup of $KS = T$. Denote by σ the canonical epimorphism of T upon $H = KS/S_T$. Then S^σ is a maximal subgroup of $H = K^\sigma S^\sigma$ so that $K^\sigma \neq 1$. Consequently there exists a minimal normal subgroup M of H with $M \subseteq K^\sigma$. Since $\bar{\theta}_e'$-immersion and θ_e''-immersion are equivalent properties of finite normal subgroups [Corollary 5.4], and since θ_e''-immersion is factor inherited, it follows that M is an elementary abelian group of prime power order p^r with r a divisor of $e(p)$. From $(S^\sigma)_H = 1$ we deduce that $M \not\subseteq S^\sigma$. Application of Lemma 4.17, (B) shows now that $M \cap S^\sigma = 1$. Hence

$$p^r = o(M) = [M:(M \cap S^\sigma)] = [MS^\sigma:S^\sigma] = [H:S^\sigma] = [T:S],$$

since M is not part of the maximal subgroup S^σ of H. This shows the validity of (ii.a); and we have derived (ii) from (i).

Assume next the validity of (ii). Consider a normal subgroup K of G with $K \subseteq N$ and a subgroup S of G which is a maximal subgroup of $T = KS$. Then it is a consequence of (ii.a) that $[T:S] = p^r$ with r a divisor of $e(p)$. Denote by σ the canonical epimorphism of T upon $T/S_T = H$. Then S^σ is a maximal subgroup of H and $H = K^\sigma S^\sigma$ so that $1 \subset K^\sigma$. Consequently there exists a minimal normal subgroup M of H with $M \subseteq K^\sigma$. Next we note that the property:

(+) the group X induces an abelian group of automorphism in every principal factor, contained in its finite normal subgroup Y,

is factor inherited by Proposition 1.5; and that this property is equivalent with

$Y \cap X' \subseteq \mathfrak{h}X'$ by Proposition 4.10. Consequently it follows from (ii.b) that $H/\mathfrak{c}_H M$ is abelian. We note that $(S^\sigma)_H = 1$ and that therefore $M \not\subseteq S^\sigma$. Apply Lemma 4.17, (A. b) to see that

$$1 = (S^\sigma)_H = S^\sigma \cap \mathfrak{c}_H M.$$

Hence

$$S^\sigma = S^\sigma/[S^\sigma \cap \mathfrak{c}_H M] \simeq S^\sigma \mathfrak{c}_H M/\mathfrak{c}_H M \subseteq H/\mathfrak{c}_H M$$

is abelian; and this implies the commutativity of S/S_T. Thus we have derived (iii.a) from (ii).

Consider next an epimorphism β of G upon B and a minimal normal subgroup C of B with $C \subseteq N^\beta$. We deduce from (ii.b) and Proposition 4.10 that $B/\mathfrak{c}_B C$ is abelian; and this implies by Lemma 5.1 that $B/\mathfrak{c}_B C$ is cyclic. There exists therefore a cyclic subgroup E of B with $B = E\mathfrak{c}_B C$. If the cyclic group E is infinite, then $E \cap C = 1$, since C is finite. If E is finite, then one recalls that by Lemma 5.1 the orders $o(C)$ and $o(B/\mathfrak{c}_B C)$ are relatively prime. It follows that the finite cyclic group E possesses a direct factor F whose order is prime to $o(C)$ with $B = F\mathfrak{c}_B C$. Clearly $F \cap C = 1$. This shows that (iii.b) likewise is a consequence of (ii).

Assume now the validity of (iii) and consider an epimorphism σ of G upon H and a minimal normal subgroup M of H with $M \subseteq N^\sigma$. By (iii.b) there exists a subgroup S of H with

$$M \cap S = 1 \quad \text{and} \quad H = S\mathfrak{c}_H M.$$

Then the same automorphisms are induced by H and S in M, so that M is a minimal normal subgroup of SM. Furthermore, $m^S = m^H$ for every m in M and $M = \{m^S\}$ for $m \neq 1$ in M. Suppose that X is a subgroup with $S \subset X \subseteq SM$. Then we deduce from Dedekind's Modular Law that $X = S(X \cap M)$ and that $X \cap M \neq 1$. It follows that $\{(X \cap M)^S\} = M$; and this implies

$$X = S(X \cap M) = S\{(X \cap M)^S\} = SM.$$

This shows that S is a maximal subgroup of SM. Let $T = S^{\sigma^{-1}}$ and $K = N \cap M^{\sigma^{-1}}$. Then K is a normal subgroup of G, since N and $M^{\sigma^{-1}}$ are normal subgroups of G; and clearly $K \subseteq N$. If J is the kernel of σ, then $J \subseteq T \cap M^{\sigma^{-1}}$ and $KJ = M^{\sigma^{-1}}$ because of $M \subseteq N^\sigma$ so that $K^\sigma = M$. Assume that X is a subgroup with $T \subset X \subseteq TK$. Then $S = T^\sigma \subset X^\sigma$, since the kernel J of σ is part of T. But $S \subset X^\sigma \subseteq T^\sigma K^\sigma = SM$ implies $X^\sigma = SM$, since S is a maximal subgroup of SM. From $J \subseteq X \subseteq TK$ we deduce now $X = TK$ so that T is a maximal subgroup of TK. Application of (iii.a) shows now that $[KT:T]$ is a prime power p^r with r a divisor of $e(p)$ and T/T_{KT} is abelian. Since the kernel J of σ is part of T, it is also part of T_{KT}. Hence

$$(KT)^\sigma = SM, \quad T^\sigma = S, \quad (T_{KT})^\sigma = S_{SM}.$$

Consequently

$$o(M) = [M:(S \cap M)] = [SM:S] = [KT:T] = p^r,$$

$$T/T_{KT} \simeq T^\sigma/[T_{KT}]^\sigma = S/S_{SM} = S/[S \cap \mathfrak{c}_{SM} M] \simeq S\mathfrak{c}_{SM} M/\mathfrak{c}_{SM} M$$

by Lemma 4.17, (A.b). But

$$H/\mathfrak{c}_H M = S\mathfrak{c}_H M/\mathfrak{c}_H M \simeq S/[S \cap \mathfrak{c}_H M]$$

is an epimorphic image of the abelian group $S/[S \cap \mathfrak{c}_{SM} M] \simeq T/T_{KT}$. Hence

$H/\mathfrak{c}_H M$ is abelian.

Since the minimal normal subgroup M is a finite p-group, it is likewise abelian [Lemma 4.17, (B)]. Thus, we have derived (i) from (iii) and proven the equivalence of (i)–(iii).

Corollary 5.13. *If $e(p)$ is for every prime p a non-negative integer, then the following properties of the finite group G are equivalent:*

(i) *G is a θ_e-group.*

(ii) (a) *If S is a maximal subgroup of G, then $[G:S] = p^r$ with p a prime and r a divisor of $e(p)$.*
(b) *G' is nilpotent.*

(iii) *If S is a maximal subgroup of G, then*
(a) *$[G:S] = p^r$ with p a prime and r a divisor of $e(p)$ and*
(b) *S/S_G is abelian.*

Proof. We recall that as a consequence of Corollary 5.4 [and the definition of θ_e in Discussion 5.5, **A**] condition (i) is equivalent to either of the following conditions:

(i*) G is a $\bar{\theta}_e'$-group;
(i**) G is a θ_e''-group.

Restatement of (i*) in explicit form gives:

(i.+) If M is a minimal normal subgroup of the epimorphic image H of G, then M is a primary elementary abelian group, $o(M) = p^r$ with p a prime and r a divisor of $e(p)$ and $H/\mathfrak{c}_H M$ is abelian.

If we let $G = N$ in Proposition 5.12, then we see that either of the conditions (ii) and (iii) is a consequence of (i.+)—as a matter of fact they are very weak forms of the specializations obtained in this fashion.

If we choose next in Proposition 4.19 for $\mathfrak{F}$ the class of finite abelian groups, then we have $\mathfrak{F}G = G'$; and we see the equivalence of our condition (ii.b) with (iii.b) together with the requirement that maximal subgroups have prime power index. It follows that our conditions (ii) and (iii) are equivalent.

Assume next the validity of the equivalent conditions (ii) and (iii). It is evident that

(1) conditions (ii) and (iii) are satisfied by every epimorphic image of G.

Assume now by way of contradiction that G is not a θ_e-group. Then there exist epimorphic images of G which are not θ_e-groups; and among these there exists one of minimal order, say H. We note:

(2) H is not a θ_e-group.
(3) Every proper epimorphic image of H is a θ_e-group.

If the Frattini subgroup ΦH of H were not 1, then $H/\Phi H$ is by (3) a θ_e-group. But θ_e is by Proposition 5.8 a saturated formation. Hence H would be a θ_e-group, contradicting (2). Consequently

(4) $\Phi H = 1$.

Since $H \neq 1$ by (2), there exists a minimal normal subgroup M of H. If N is a minimal normal subgroup of H with $M \neq N$, then $M \cap N = 1$. It is a consequence of (3) that H/M and H/N are θ_e-groups. Since θ_e is a formation, this implies that $H = H/(M \cap N)$ is a θ_e-group, contradicting (2). Consequently

(5) there exists one and only one minimal normal subgroup M of H.

It is a consequence of (1), (ii.b) and Proposition 4.10 that $H/\mathfrak{c}_H M$ is abelian. Since H is [by (1) and (ii.b)] soluble, the minimal normal subgroup M of H is soluble. Apply Lemma 4.17, (B) to see that M is a primary elementary abelian group and that $M \cap S = 1$ or M for every maximal subgroup S of H. By (4) there exists a maximal subgroup T of H with $M \not\subseteq T$ so that $M \cap T = 1$. Clearly $H = MT$ so that

$$o(M) = [M:M \cap T] = [MT:T] = [H:T].$$

Apply [(1) and] (ii.a) to see that $o(M) = [H:T] = p^r$ with p a prime and r a divisor of $e(p)$. We note that

(6) $o(M) = p^r$ for p a prime and r a divisor of $e(p)$ and $H/\mathfrak{c}_H M$ is abelian.

Combine (3), (5) and (6) to see that H is a θ_e-group, contradicting (2). This contradiction completes the proof of the equivalence of conditions (i)–(iii).

6. A group is termed *supersoluble* [= hypercyclic] if every epimorphic image, not 1, possesses a cyclic normal subgroup, not 1. It is well known that the class of finite supersoluble groups is factor inherited, residual and enjoys both the saturation and the Iwasawa-Schmidt requirement; see Huppert.

Likewise we term the normal subgroup N of G *supersolubly immersed in* G, if to every epimorphism σ of G upon H with $N^\sigma \neq 1$ there exists a cyclic normal subgroup C of H with $1 \subset C \subseteq N^\sigma$.

THEOREM 6.1. *The following properties of the finite normal subgroup N of G are equivalent:*

(i) *N is supersolubly immersed in G.*

(ii) *Every principal factor of G, contained in N, is cyclic of order a prime.*

(iii) *N is soluble; and if the principal factor F of G, contained in $N \cap G'$, is an elementary abelian p-group, then the group of automorphisms, induced in F by G, is cyclic of order dividing $p - 1$.*

(iv) *Every primary subgroup P of N is supersolubly immersed in its normalizer $\mathfrak{n}_G P$.*

(v) *G induces in every principal factor, contained in N, an abelian group of automorphisms; and*

$$x^{p-1} \circ y = 1$$

for every prime p, every p-element y in $N \cap G'$ and every element x in G with

$$(o(x\mathfrak{c}_G(N \cap G')), p) = 1.$$

(vi)

$$G' = (N \cap G')\mathfrak{z}_{G'}(N \cap G')$$

and

$$x^{p-1} \circ y = 1$$

for every prime p, every p-element y in $N \cap G'$ and every element x in G with

$$(o(x\mathfrak{c}_G(N \cap G')), p) = 1.$$

Proof. The equivalence of conditions (i) and (ii) is contained in Proposition 2.2. The group of automorphisms of a cyclic group of order p is cyclic of order $p - 1$. Consequently (iii) is a consequence of (ii).

Denote by 1 the function, defined on the primes with constant value 1. Then condition (iii) is equivalent to the formula

$$N\ \theta_e''\ G;$$

and application of Theorem 5.3 shows that the principal factors of G, contained in N, are elementary abelian groups of rank 1. Hence N is supersolubly immersed in G and we have verified the equivalence of (i)–(iii).

We noted before that supersolubility meets the Iwasawa-Schmidt requirement; and that supersoluble immersion is factor inherited, may be deduced from Proposition 1.5. Thus we may apply Corollary 3.7 to show the equivalence of conditions (iv) and (i)–(iii).

That (v) is equivalent to (i)–(iv) is contained in Theorem 5.3, [(ii), (iii)] and the equivalence of (i) and (vi) may be deduced from Corollary 5.10 [both with $e = 1$].

Remark 6.2. **A.** Denote by E a simple, finite, non abelian group and denote by e the least common multiple of the orders of the elements in E. Apply Dirichlet's Prime Number Theorem to show the existence of an infinity of primes p with

$$p \equiv 1 \quad \text{modulo } e.$$

If p is any such prime, then there exist finite elementary abelian p-groups N, possessing groups of automorphisms, isomorphic to E. Consequently there exist extensions G of [their normal subgroup] N by E [$= G/N$] with $N = \mathfrak{c}_G N$. From $E^e = 1$ we deduce $(G/N)^{p-1} = 1$. Since N is commutative, this implies

$$x^{p-1} \circ y = 1$$

for every y in N and every element x in G. Noting again that N is an abelian p-group, we see that the second half of condition (v) is satisfied by N. But this abelian normal p-subgroup N is not supersolubly immersed in G, since the group of automorphisms, induced in N by G, is not even soluble. This shows the impossibility of omitting [or essentially weakening] the first half of condition (v).

B. Suppose that N is a finite normal subgroup of G and that principal factors of G which are contained in N are never abelian. Then the second half of condition (iii) is trivially satisfied [by default]; and this puts in evidence the impossibility of omitting in (iii) the requirement that N be soluble.

C. For further criteria for supersoluble immersion see Baer [1].

COROLLARY 6.3. *The following properties of the finite group G are equivalent*:

(i) *G is supersoluble.*

(ii) *Every principal factor of G is cyclic of order a prime.*

(iii) *G is soluble; and if the principal factor F of G [is contained in G' and] is an elementary abelian p-group, then the group of automorphisms, induced in F by G, is cyclic of order dividing $p - 1$.*

(iv) *Every primary subgroup of G is supersolubly immersed in its normalizer.*

(v) $x^{p-1} \circ y = 1$ *for every prime p, every p-element y in G' and every p'-element x in G.*

This is the special case of Theorem 6.1, obtained by letting $N = G$; note that the first half of Theorem 6.1, (vi) reduces to the trivial condition $G' = G'$. —For the equivalence of conditions (i) and (iv) of this corollary see earlier Baer [1, p. 366, Theorem 4.1] and Berkovič.

7. Denote by δ a partial order of the primes. If the prime p is, under this partial order δ, a predecessor of the prime q, then we say $p\ \delta\ q$; and this relation δ is subject to the rules

$$p\ \bar{\delta}\ p, \quad p\ \delta\ q \text{ and } q\ \delta\ r \text{ imply } p\ \delta\ r.$$

It will be convenient to term the set $\mathfrak{s}$ of primes a *δ-segment*, if x belongs to $\mathfrak{s}$ whenever there exists a prime y in $\mathfrak{s}$ with $x\ \delta\ y$.

The finite group G is termed *δ-dispersed*, if for every δ-segment $\mathfrak{s}$ the set $G_{\mathfrak{s}}$ of $\mathfrak{s}$-elements in G is a characteristic subgroup of G.

Furthermore for every prime p

$\mathfrak{A}_\delta(p) =$ the class of all finite elementary abelian p-groups,

$\mathfrak{B}_\delta(p)$ = the class of all finite groups whose orders are divisible only by p and the primes x with $p\ \delta\ x$.

Finally let Δ_δ be the family of pairs $(\mathfrak{A}_\delta(p), \mathfrak{B}_\delta(p))$.

Since the properties $\mathfrak{A}_\delta(p)$ and $\mathfrak{B}_\delta(p)$ are inherited by subgroups, epimorphic images and direct products, we deduce from Proposition 1.5 that

(7.1) *Δ_δ-immersion is factor inherited*

and it is a consequence of Gaschütz [p. 302, Satz 3.1] that

(7.2) *the class of finite Δ_δ-groups is a saturated formation.*

Next we note the

PROPOSITION 7.3. *The finite group G is a Δ_δ-group if, and only if, G is δ-dispersed.*

For a proof cp. Baer [2, p. 3, Satz 1.1].

COROLLARY 7.4. *Δ_δ meets the Iwasawa-Schmidt requirement.*

This is, by Proposition 7.3, equivalent to the theorem that a finite group is soluble, if its proper subgroups are δ-dispersed; see Baer [2, p. 6, Bemerkung 2.2].

PROPOSITION 7.5. *The following properties of the finite normal subgroup N of G are equivalent:*

(i) $N\ \Delta_\delta\ G$.

(ii) *If σ is an epimorphism of the subgroup S of G upon a finite group H, if $H/[S \cap N]^\sigma$ is a finite δ-dispersed group, then H is δ-dispersed.*

(iii) $N\ \Delta_\delta\ \{N, g\}$ *for every g in G.*

(iv) *If P is a primary subgroup of N, then $P\ \Delta_\delta\ \mathfrak{n}_G P$.*

This is a slight extension of Baer [2, p. 21/22, Zusatz 4.5 and p. 17, Satz 4.1]. Note that here G need not be finite.

Proof. Assume first that $N\ \Delta_\delta\ G$. Then we deduce from (7.1) that

$$[S \cap N]^\sigma\ \Delta_\delta\ H,$$

whenever σ is an epimorphism of the subgroup S of G upon H. If furthermore $H/[S \cap N]^\sigma$ is a finite δ-dispersed group, then H is finite, since N is finite, and $H/[S \cap N]^\sigma$ is a Δ_δ-group by Proposition 7.3. From

$$[S \cap N]^\sigma\ \Delta_\delta\ \mathrm{H} \quad \text{and} \quad H/[S \cap N]^\sigma\ \Delta_\delta\ H/[S \cap N]^\sigma$$

and (7.1) together with Proposition 1.4, (b) we deduce

$$H\ \Delta_\delta\ H.$$

Since H is finite, we may apply Proposition 7.3 to prove that H is δ-dispersed. Hence (ii) is a consequence of (i).

Assume next the validity of (ii). Consider an element g in G. Since N is finite, g induces in N an automorphism of positive order n. Then g^n is centralized by N and g so that g^n belongs to $\mathfrak{z}\{N, g\}$. Since N is finite, there exists a positive multiple m of n such that

$$\{g^m\} \cap N = 1 \quad \text{and} \quad \{g^m\} \subseteq \mathfrak{z}\{N, g\}.$$

Denote by σ the canonical epimorphism of $\{N, g\}$ upon $H = \{N, g\}/\{g^m\}$. Since m is positive, H is finite and

$$H/N^\sigma = [\{N, g\}/\{g^m\}]/[\{N, g^m\}/\{g^m\}] \simeq \{N, g\}/\{N, g^m\} = N\{g\}/N\{g^m\}$$
$$\simeq \{g\}/[\{g\} \cap N\{g^m\}]$$

is a finite cyclic group and hence δ-dispersed. Application of condition (ii) shows that H is δ-dispersed; and this is equivalent to $H \, \Delta_\delta \, H$ by Proposition 7.3. Application of (7.1) shows $N^\sigma \, \Delta_\delta \, H$. From $\{g^m\} \subseteq \mathfrak{z}\{N, g\}$ we deduce $\{g^m\} \, \Delta_\delta \, \{N, g\}$. By a combination of (7.1) and Proposition 1.4, (b) we obtain $\{N, g\} \, \Delta_\delta \, \{N, g\}$; and this implies $N \, \Delta_\delta \, \{N, g\}$ by (7.1). Hence (iii) is a consequence of (ii).

That (i) is a consequence of (iii), is easily [and directly] verified. Hence (i)–(iii) are equivalent. —Because of Corollary 7.4 we may apply Corollary 3.7 to prove the equivalence of (i) and (iv), completing the proof.

Remark 7.6. **A.** Condition (ii) implies in particular that N itself is δ-dispersed and hence soluble.

B. Noting that $NS/N \simeq S/(S \cap N)$ is δ-dispersed whenever S is δ-dispersed one deduces from (ii) the prima facie weaker condition:

(ii*) If S is a finite δ-dispersed subgroup of G, then NS is δ-dispersed.

In case G is a torsion group, (iii) is an immediate consequence of (ii*). Hence (ii*) is equivalent to (i)–(iv), provided G is a torsion group. But without this condition it is very easy to construct examples showing that (ii*) is actually weaker.

C. Denote by E a finite, non-abelian, simple group and by p a prime, not dividing $o(E)$. Let δ be any partial ordering of the primes such that $p \, \delta \, x$ for every prime divisor x of $o(E)$. There exists a finite elementary abelian p-group N such that E is [isomorphic to] a group of automorphisms of N. Form $G = EN$ in the holomorph of N. Then N is a normal subgroup of G and clearly $N \, \Delta_\delta \, G$. But $G/\mathfrak{c}_G N \simeq E$ is simple, non-abelian and therefore certainly not δ-dispersed.

D. The remark **C** suggests the following variation of Δ_δ. Let

$$\mathfrak{A}'_\delta(p) = \mathfrak{A}_\delta(p) \quad \text{for every prime} \quad p,$$

$$\mathfrak{B}'_\delta(p) = \text{class of } \delta\text{-dispersed } \mathfrak{B}_\delta(p)\text{-groups for every prime } p;$$

and let Δ'_δ be the family of pairs $(\mathfrak{A}'_\delta(p), \mathfrak{B}'_\delta(p))$. Then $N \, \Delta'_\delta \, G$ implies $N \, \Delta_\delta \, G$, though the converse is, by **C**, in general false. Δ'_δ-immersion is, by Proposition 1.5, factor inherited. From Proposition 7.3 one deduces easily that a finite group G is δ-dispersed if, and only if, G is a Δ'_δ-group. Using Proposition 1.4, (b) one may prove that the finite normal subgroup N of G is Δ'_δ-immersed in G if, and only if, $\mathfrak{z}N \; \Delta'_\delta \; G$ and $G/\mathfrak{c}_G N$ is δ-dispersed.

Thus we see that the following three classes of finite groups are identical:

$$\Delta_\delta\text{-groups}; \quad \Delta'_\delta\text{-groups}; \quad \delta\text{-dispersed groups}.$$

But in general Δ_δ-immersion and Δ'_δ-immersion of finite normal subgroups are different relations. In other words: different immersion concepts may define the same class of finite groups.

8. The following considerations shall provide an extension of the methods of §**5**, suitable for application in the situation considered in §**4**. The basis of this discussion is the following

LEMMA 8.1. *If the property $\mathfrak{F}$ of finite groups is epimorphism inherited, and if G is a group, meeting the requirement:*

(8.1.$\mathfrak{F}$) *$H/\mathfrak{F}H$ is an $\mathfrak{F}$-group for every epimorphic image H of G,*

then G has the following properties:

(A) *$(\mathfrak{F}G)^\sigma = \mathfrak{F}(G^\sigma)$, for every homomorphism σ of G.*

(B) *The following properties of the prime p and the finite normal subgroup N of G are equivalent:*

(i) *If σ is an epimorphism of G upon H, if M is a minimal normal subgroup of H with $M \subseteq N^\sigma$ and p a divisor of the order $o(M)$, then $H/\mathfrak{c}_H M$ is an $\mathfrak{F}$-group.*

(ii) *If $S \neq 1$ is a p-subgroup of $N \cap \mathfrak{F}G$, then $S \not\subseteq \{S \circ \mathfrak{F}G\}$.*

(iii) *If A, B are normal subgroups of G with $A \subset B \subseteq N \cap \mathfrak{F}G$, if B/A is a minimal normal subgroup of G/A, and if p is a divisor of $[B:A]$, then $\mathfrak{F}G \subseteq (A:B)$.*

(iv) *If A, B are normal subgroups of G with $A \subset B \subseteq N$, if B/A is a minimal normal subgroup of G/A, and if p is a divisor of $[B:A]$, then $\mathfrak{F}G \subseteq (A:B)$.*

(v) $\begin{cases} \text{(a) } N \cap \mathfrak{F}G \text{ is } p\text{-soluble.} \\ \text{(b) If } \sigma \text{ is an epimorphism of } G \text{ upon } H, \text{ if } P \text{ is a normal } p\text{-subgroup} \\ \text{of } H \text{ with } P \subseteq N^\sigma \cap \mathfrak{F}H, \text{ then } P \subseteq \mathfrak{h}\mathfrak{F}H. \end{cases}$

Terminological reminder. The finite group X is *p-soluble* for p a prime, if every principal factor of X is either a p-group or a p'-group.

Proof. If σ is an epimorphism of G upon H, then σ induces an epimorphism of the $\mathfrak{F}$-group $G/\mathfrak{F}G$ upon $H/(\mathfrak{F}G)^\sigma$. Since $\mathfrak{F}$ is epimorphism inherited, $H/(\mathfrak{F}G)^\sigma$ is likewise an $\mathfrak{F}$-group; and this implies $\mathfrak{F}H \subseteq (\mathfrak{F}G)^\sigma$.

By (8.1.$\mathfrak{F}$), the group $H/\mathfrak{F}H$ is an $\mathfrak{F}$-group. Let $J = (\mathfrak{F}H)^{\sigma^{-1}}$ be the inverse image of $\mathfrak{F}H$. This is a normal subgroup of G with $G/J \simeq H/\mathfrak{F}H$. Hence G/J is an $\mathfrak{F}$-group so that $\mathfrak{F}G \subseteq J$. Consequently $(\mathfrak{F}G)^\sigma \subseteq J^\sigma = \mathfrak{F}H$, completing the proof of (A).

The following well known auxiliary proposition will be needed below:

(8.1.1) *If X is a subgroup of the group Y, then $\{X \circ Y\} = \{\{X^Y\} \circ Y\}$ is a normal subgroup of Y.*

The proof may be indicated for the convenience of the reader. It x is an element in X and a, b are elements in Y, then

$$x^b \circ a = (x \circ b)^{-1}(x \circ a^{b^{-1}}b)$$

belongs to $\{X \circ Y\}$; see, for instance M. Hall [p. 150, (10.2.1.3)]. It follows that $\{X \circ Y\} = \{\{X^Y\} \circ Y\}$; and this is a normal subgroup of Y, since $\{X^Y\}$ is a normal subgroup of Y.

To prove (B), consider a prime p and a finite normal subgroup N of G. Assume the validity of (i) and suppose that S is a p-subgroup with $1 \subset S \subseteq N$. Let $B = \{S^G\}$. This is a normal subgroup of G with $1 \subset S \subseteq B \subseteq N$. Among the normal subgroups X of G with $X \subset B$ there exists a maximal one, say A, since $B \subseteq N$ is finite. Then B/A is a minimal normal subgroup of G/A. Clearly $B/A \subseteq N/A$. If S were contained in A, then $\{S^G\} \subseteq A \subset B = \{S^G\}$, a contradiction. Hence $S \not\subseteq A$ and this is equivalent to $1 \subset AS/A \subseteq B/A$. But $AS/A \simeq S/(A \cap S)$ is a p-group. Hence p is a divisor of $[B:A]$. Application of (i) [with σ the canonical epimorphism of G upon G/A] shows that $[G/A]/\mathfrak{c}_{G/A}[B/A]$ is an $\mathfrak{F}$-group. It follows that B/A is centralized by $\mathfrak{F}(G/A)$. Apply (A) to see that

$$\mathfrak{F}(G/A) = A\mathfrak{F}G/A$$

and it follows that

$$S \circ \mathfrak{F}G \subseteq B \circ \mathfrak{F}G \subseteq A.$$

Hence $\{S \circ \mathfrak{F}G\} \subseteq A$. We noted and used before that $S \not\subseteq A$. Hence a fortiori $S \not\subseteq \{S \circ \mathfrak{F}G\}$. This proves (ii) in the slightly stronger form:

(ii*) *If $S \neq 1$ is a p-subgroup of N, then $S \not\subseteq \{S \circ \mathfrak{F}G\}$.*

Assume next the validity of (ii) and consider normal subgroups A, B of G with the following properties:

$$A \subset B \subseteq N \cap \mathfrak{F}G,$$

$$B/A \text{ is a minimal normal subgroup of } G/A,$$

$$p \text{ is a divisor of } [B:A].$$

There exist normal subgroups X of G with $B = AX$ [like $X = B$] and among these there exists a minimal one, say L, since $B \subseteq N$ is finite. From

$$B/A = AL/A \simeq L/(A \cap L)$$

we deduce that $[L:A \cap L] = [B:A]$ is a multiple of p. If S is a p-Sylow subgroup of L, then $(A \cap L)S/(A \cap L)$ is a p-Sylow subgroup of $L/(A \cap L)$ which is different from 1. Hence $S \not\subseteq A \cap L$. Clearly $\{S^G\} \subseteq L \subseteq B$. Since $S \not\subseteq A \cap L$, but $S \subseteq L$, we have $S \not\subseteq A$ so $\{S^G\} \not\subseteq A$ and $A \subset A\{S^G\} \subseteq B$. From the minimality of B/A we deduce $B = A\{S^G\}$ and from the minimality of L combined with $\{S^G\} \subseteq L$ we deduce $L = \{S^G\}$. Since $S \subseteq L \subseteq B \subseteq N \cap \mathfrak{F}G$, application of (ii) gives

$$S \not\subseteq \{S \circ \mathfrak{F}G\} = \{\{S^G\} \circ \mathfrak{F}G\} = \{L \circ \mathfrak{F}G\}$$

by (8.1.1). It follows that $\{L \circ \mathfrak{F}G\}$ does not contain L. But L is a normal subgroup. Hence $\{L \circ \mathfrak{F}G\} \subset L$. Because of the minimality of L and the normality of $\{L \circ \mathfrak{F}G\}$ we have

$$A \subseteq A\{L \circ \mathfrak{F}G\} \subset B = AL;$$

and this implies $A = A\{L \circ \mathfrak{F}G\}$ because of the minimality of B/A. Hence $L \circ \mathfrak{F}G \subseteq A$; and this implies

$$B \circ \mathfrak{F}G = (AL) \circ \mathfrak{F}G \subseteq A(L \circ \mathfrak{F}G) = A$$

so that $\mathfrak{F}G \subseteq (A{:}B)$: we have deduced (iii) from (ii).

Assume next the validity of (iii) and consider normal subgroups A, B of G with the properties:

$$A \subset B \subseteq N,$$

$$B/A \text{ is a minimal normal subgroup of } G/A,$$

$$p \text{ is a divisor of } [B{:}A].$$

Case 1. $B \subseteq A\mathfrak{F}G$. Then $U = A \cap \mathfrak{F}G$ and $V = B \cap \mathfrak{F}G$ are normal subgroups of G with $U \subseteq V \subseteq N \cap \mathfrak{F}G$. Application of Dedekind's Modular Law shows $B = A[B \cap \mathfrak{F}G] = AV$ and

$$B/A = AV/A \simeq V/(V \cap A) = V/U$$

so that in particular p is a divisor of $[V{:}U] = [B{:}A]$. If X is a normal subgroup of G with $U \subseteq X \subseteq V$, then $A \subseteq AX \subseteq AV = B$. We deduce from the minimality of B/A that $A = AX$ or $B = AX$. In the first case we have $X \subseteq A$ so that $X = U$; and in the second case we have [by Dedekind's Modular Law]

$$V = B \cap \mathfrak{F}G = AX \cap \mathfrak{F}G = X(A \cap \mathfrak{F}G) = XU = X.$$

Consequently V/U is a minimal normal subgroup of G/U with $U \subset V \subseteq N \cap \mathfrak{F}G$. Thus (iii) is applicable, proving $\mathfrak{F}G \subseteq (U{:}V)$. This is equivalent to $V \circ \mathfrak{F}G \subseteq U$. Consequently

$$B \circ \mathfrak{F}G = AV \circ \mathfrak{F}G \subseteq A(V \circ \mathfrak{F}G) \subseteq AU = A$$

so that $\mathfrak{F}G \subseteq (A{:}B)$.

Case 2. $B \not\subseteq A\mathfrak{F}G$. We have $A \subseteq B \cap A\mathfrak{F}G \subseteq B$ and because of the minimality of B/A we have $A = B \cap A\mathfrak{F}G$ or $B = B \cap A\mathfrak{F}G$. The second alternative is ruled out in our Case 2. Hence

$$A = B \cap A\mathfrak{F}G = A(B \cap \mathfrak{F}G)$$

by Dedekind's Modular Law. Consequently

$$B \circ \mathfrak{F}G \subseteq B \cap \mathfrak{F}G \subseteq A \quad \text{and hence} \quad \mathfrak{F}G \subseteq (A:B).$$

This completes the derivation of (iv) from (iii).

Assume next the validity of (iv). In order to derive (i) we consider normal subgroups J, L of G with the following properties:

$$J \subset L \subseteq JN,$$
$$L/J \text{ is a minimal normal subgroup of } G/J,$$
$$p \text{ is a divisor of } [L:J].$$

Let $A = J \cap N$ and $B = L \cap N$. These are normal subgroups of G with the following properties:

$$A \subseteq B \subseteq N;$$
$$L = JB,$$
$$B/A = (L \cap N)/(J \cap N) \simeq J(L \cap N)/J = L/J$$

by Dedekind's Modular Law. Hence

$$p \quad \text{is a divisor of} \quad [B:A] = [L:J] \quad \text{so that} \quad A \subset B;$$

and one verifies as usual that

$$B/A \quad \text{is a minimal normal subgroup of} \quad G/A.$$

Consequently (iv) is applicable so that $\mathfrak{F}G \subseteq (A:B)$; and this is equivalent to

$$B \circ \mathfrak{F}G \subseteq A.$$

Hence

$$L \circ \mathfrak{F}G = JB \circ \mathfrak{F}G \subseteq J(B \circ \mathfrak{F}G) \subseteq JA = J$$

so that by (A)

$$\mathfrak{F}(G/J) = J\mathfrak{F}G/J \subseteq \mathfrak{c}_{G/J}(L/J).$$

Hence $[G/J]/\mathfrak{c}_{G/J}[L/J]$ is an $\mathfrak{F}$-group by (8.1.$\mathfrak{F}$) and the fact that $\mathfrak{F}$ is epimorphism inherited. Thus (i) is a consequence of (iv); and we have proven the equivalence of conditions (i)–(iv).

Assume next the validity of the equivalent conditions (i)–(iv). Consider normal subgroups A, B of G such that

$$A \subset B \subseteq N \cap \mathfrak{F}G$$

$$B/A \text{ is a minimal normal subgroup of } G/A,$$

$$p \text{ is a divisor of } [B:A].$$

Apply (iii) to see that $\mathfrak{F}G \subseteq (A:B)$. Hence

$$B \circ B \subseteq B \circ \mathfrak{F}G \subseteq A$$

so that B/A is abelian. Since B/A is characteristic simple, finite, abelian, and the order of B/A is divisible by p, it follows that B/A is an elementary abelian p-group. Thus we have shown that a principal factor of G which is contained in $N \cap \mathfrak{F}G$ is either a p-group or of order prime to p; and the p-solubility of $N \cap \mathfrak{F}G$ is an easy consequence.

Consider next an epimorphism σ of G upon H and a normal p-subgroup P of H with $P \subseteq N^\sigma \cap \mathfrak{F}H$. We note that property (i) is, mutatis mutandis, inherited by all epimorphic images of G. It follows that (ii) is satisfied by the finite normal subgroup N^σ of H. Define inductively P_i by

$$P = P_0, \qquad \{P_i \circ \mathfrak{F}H\} = P_{i+1}.$$

Since P is a normal subgroup of H, all the P_i are normal subgroups of H and we have

$$P = P_0 \supseteq P_1 \supseteq \cdots \supseteq P_i \supseteq P_{i+1} \supseteq \cdots$$

Since P is finite, this chain terminates after a finite number of steps. Hence there exists a positive integer k such that

$$P_k = P_{k+1} = \{P_k \circ \mathfrak{F}H\}.$$

Since P_k is a p-group and N^σ meets requirement (ii), since $P_k \subseteq P \subseteq N^\sigma \cap \mathfrak{F}H$, it follows that $P_k = 1$. We deduce by complete induction that $P_{k-i} \subseteq \mathfrak{z}_i\, \mathfrak{F}H$. Hence $P = P_0 \subseteq \mathfrak{z}_k\, \mathfrak{F}H \subseteq \mathfrak{h}\mathfrak{F}H$. Thus we have deduced (v) from (i)–(iv).

Assume the validity of (v) and consider normal subgroups A, B of G with the properties:

$$A \subset B \subseteq N \cap \mathfrak{F}G,$$

$$B/A \quad \text{is a minimal normal subgroup of} \quad G/A,$$

$$p \quad \text{is a divisor of} \quad [B:A].$$

Since $N \cap \mathfrak{F}G$ is p-soluble by (v.a), it follows that B/A is a p-group. We deduce $B/A \subseteq \mathfrak{h}\mathfrak{F}(G/A)$ from (v.b). Since $B/A \neq 1$, this implies $1 \subset (B/A) \cap \mathfrak{z}\mathfrak{F}(G/A)$; and because of the minimality of B/A we obtain

$$B/A \subseteq \mathfrak{z}\mathfrak{F}(G/A).$$

This is, by (A), equivalent to

$$A\mathfrak{F}G/A = \mathfrak{F}(G/A) \subseteq \mathfrak{c}_{G/A}(B/A),$$

$$B \circ \mathfrak{F}G \subseteq B \circ A\mathfrak{F}G \subseteq A \quad \text{and hence} \quad \mathfrak{F}G \subseteq (A:B).$$

Thus we have derived (iii) from (v) and proven the equivalence of (i)–(v).

Discussion 8.2 [of Lemma 8.1]. **A.** It is clear that (iv) is just a weak form of (i) and (iii) a weak form of (iv). Hence condition (iv) could be dispensed with; it has been inserted mainly for convenience during the proof.

B. The requirement (8.1.$\mathfrak{F}$) is certainly satisfied by G, if

(a) $G/(X \cap Y)$ is an $\mathfrak{F}$-group whenever G/X and G/Y are $\mathfrak{F}$-groups and if

(b) the minimum condition is satisfied by the normal subgroups of G.

C. For the purposes of our proof we could substitute for (8.1.$\mathfrak{F}$) the follow ing weaker requirement:

If H is an epimorphic image of G, if $\mathfrak{F}_f H$ is the intersection of all normal subgroups X of H with H/X a finite $\mathfrak{F}$-group, then $H/\mathfrak{F}_f H$ is an $\mathfrak{F}$-group.

D. To put condition (ii) into a more striking form, define inductively $\mathfrak{F}^{(i)}S$ for S a subgroup of G by

$$\mathfrak{F}^{(0)}S = S, \qquad \mathfrak{F}^{(i+1)}S = \mathfrak{F}^{(i)}S \cap \{\mathfrak{F}^{(i)}S \circ \mathfrak{F}G\}.$$

The $\mathfrak{F}^{(i)}S$ form a descending chain of subgroups of S.

Now one sees without any difficulty that condition (ii) is by (ii*) equivalent to the following requirement:

(ii**) $\mathfrak{F}^{(i)}S = 1$ *for almost all i whenever S is a p-subgroup of N.*

E. If σ is an epimorphism of G upon H, and if the kernel of σ is contained in N, then

(8.2.**E**) $(N \cap S)^\sigma = N^\sigma \cap S^\sigma$ for every subgroup S of G.

This is easily verified, since an element in $N^\sigma \cap S^\sigma$ has the form $d = n^\sigma = s^\sigma$ with n and s in N and S respectively, and since $s = (sn^{-1})n$ with sn^{-1} in the kernel of σ and hence in N. Combine (8.2.**E**) and Lemma 8.1, (A) to find

$$(N \cap \mathfrak{F}G)^\sigma = N^\sigma \cap \mathfrak{F}H.$$

After these preparations one sees the identity of condition (iii) with the following property:

(iii*) *If σ is an epimorphism of G upon H, if the kernel of σ is part of N, if M is a minimal normal subgroup of H with $M \subseteq N^\sigma \cap \mathfrak{F}H$, and if $o(M)$ is a multiple of p, then $H/\mathfrak{c}_H M$ is an $\mathfrak{F}$-group.*

F. Conditions (iv) and (iii) put into evidence the fact that the finite normal subgroup N of G meets the equivalent requirements (i)–(v) if, and only if, these conditions (i)–(v) are satisfied by $N \cap \mathfrak{F}G$.

Proposition 8.3. *Assume that $\mathfrak{A}_p$ and $\mathfrak{B}_p$ are for every prime p properties of finite groups, that $\mathfrak{A}_p$-groups, not 1, are of order a multiple of p, and that every $\mathfrak{B}_p$ is epimorphism inherited. Assume furthermore that the group G meets the following requirement:*

(8.3 +) *If H is an epimorphic image of G, and if p is a prime, then $H/\mathfrak{B}_p H$ is a $\mathfrak{B}_p$-group.*

Then the following properties of the finite normal subgroup N of G are equivalent:

(i) *If σ is an epimorphism of G upon H, if M is a minimal normal subgroup of H with $M \subseteq N^\sigma$ and if p is a divisor of $o(M)$, then M is an $\mathfrak{A}_p$-group and $H/\mathfrak{c}_H M$ is a $\mathfrak{B}_p$-group.*

(ii) (a) *If the principal factor M of G is contained in N, and if p is a divisor of $o(M)$, then M is an $\mathfrak{A}_p$-group.*
(b) *If S is a p-group with $1 \subset S \subseteq N \cap \mathfrak{B}_p G$, then $S \not\subseteq \{S \circ \mathfrak{B}_p G\}$*

(iii) (a) *If the principal factor M of G is contained in N, and if p is a divisor of $o(M)$, then M is an $\mathfrak{A}_p$-group.*
(b) *If the kernel of the epimorphism σ of G upon H is contained in N, and if M is a minimal normal subgroup of H with $M \subseteq N^\sigma \cap \mathfrak{B}_p H$ and if $o(M)$ is a multiple of p, then $H/\mathfrak{c}_H M$ is a $\mathfrak{B}_p$-group.*

(iv) (a) *If the principal factor M of G is contained in N, and if p is a divisor of $o(M)$, then M is an $\mathfrak{A}_p$-group.*
(b) *$N \cap \mathfrak{B}_p G$ is p-soluble for every prime p.*
(c) *If σ is an epimorphism of G upon H, if P is a normal p-subgroup of H with $P \subseteq N^\sigma \cap \mathfrak{B}_p H$, then $P \subseteq \mathfrak{h}\mathfrak{B}_p H$.*

This is immediately deduced from Lemma 8.1, (B) and Discussion 8.2, **E.**

If we denote by θ the family of ordered pairs $(\mathfrak{A}_p, \mathfrak{B}_p)$, then condition (i) of Lemma 8.1, (B) implies $N\, \theta\, G$, though the converse cannot be expected to be true in general. Thus we shall say that N is *prime directed θ-immersed in G* whenever the equivalent conditions (i)–(iv) of Proposition 8.3 are satisfied by N. If, however, N is soluble, then θ-immersion in the strict sense and prime directed θ-immersion are identical properties of N. By application of Theorem 3.4 one obtains the

COROLLARY 8.4. *Assume that $\mathfrak{A}_p$ and $\mathfrak{B}_p$ are for every prime p properties of finite groups and that every $\mathfrak{B}_p$ is epimorphism inherited. Assume furthermore that θ is the family of ordered pairs $(\mathfrak{A}_p, \mathfrak{B}_p)$ and that θ-immersion is factor inherited. If the group G meets requirements (8.3.+), then the following properties of the soluble finite normal subgroup N of G are equivalent:*

(i) *$N\, \bar{\theta}\, G$.*

(ii) *If $\mathfrak{P}$ is a p-subgroup of N, then*

(a) *every principal factor of $\mathfrak{n}_G P$ which is contained in P is an $\mathfrak{A}_p$-group and*

(b) *$P \subseteq \mathfrak{h}\mathfrak{B}_p G$.*

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GARTENSTRASSE 11
6243 FALKENSTEIN IM TAUNUS, GERMANY