MAPPING CUBES WITH HOLES ONTO CUBES WITH HANDLES

BY

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1. Introduction

In connection with some work by W. Haken [4] on the Poincaré conjecture in dimension 3, R. H. Bing raised the following question in [2]. If K_2 is any cube with 2 holes, does there always exist a continuous map f of K_2 onto a cube with 2 handles C_2 such that $f \mid \text{Bd } K_2$ is a homeomorphism onto Bd C_2? (We call such a map f a boundary preserving map of K_2 onto C_2.) In general, if K_n is a cube with n holes, does there always exist a boundary preserving map of K_n onto a cube with n handles C_n? For the case $n = 1$, J. Hempel in Theorem 5 of [5] answered the question in the affirmative. In Theorem 1 of this paper we show that the question has a negative answer for $n = 2$. It then follows, as a corollary to Theorem 1, that the question has a negative answer for $n \geq 2$. Theorem 2 gives a necessary and sufficient condition for the existence of a boundary preserving map of K_n onto C_n. Theorem 3 gives another sufficient condition for the existence of a boundary preserving map of K_2 onto C_2.

2. Terminology

Throughout this paper all sets which appear can be considered as polyhedral subsets of E^3. *A cube with n holes* K_n and *a cube with n handles* C_n are defined as on pages 90 and 95 of [2]. Any cube with holes or handles is to be thought of as a polyhedral subset of E^3. In analogy to the definition of 1-linked simple closed curves (scc's) in E^3 [9], we define disjoint scc's X, Y to be *1-linked in the 3-manifold M* if for each pair of compact orientable 2-manifolds M_X and M_Y in M such that Bd $M_X = X$ and Bd $M_Y = Y$, it follows that $M_X \cap M_Y \neq \emptyset$. At the end of Section 4 we note an analogy between the main result of this paper and the example of a boundary link $l_1 \cup l_2$ given in [9].

Suppose g is a map of K_n onto C_n. Then g is said to be a boundary preserving map of K_n onto C_n if g is continuous and $g \mid \text{Bd } K_n$ is a homeomorphism onto Bd C_n. It can be shown that if g is a boundary preserving map of K_n onto C_n, then there is a piecewise linear map f of K_n onto C_n and a product neighborhood θ_1 ($=$ Bd $K_n \times [0, 1]$) of Bd K_n in K_n and a product neighborhood θ_2 of Bd C_n in C_n such that (1) $f \mid \theta_1$ is a homeomorphism onto θ_2 and (2) $f(K_n - \theta_1) = C_n - \theta_2$. We will assume then that any boundary preserving map f of K_n onto C_n has been adjusted so that it is piecewise linear and satisfies (1) and (2) above.

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3. Description of T

In this section we describe a cube with 2 holes T which we show (Theorem 1 of Section 4) has no boundary preserving map of T onto the cube with 2 handles C_2. The example we will describe is Zeeman's example $\bar{E}^3 - C_1 - C_2$ of case 3 of [10] where we take the one point compactification of $\bar{E}_3$ and remove the interior of a regular neighborhood of $C_1 \cup C_2$.

Let T' be a solid cube in E^3 containing the two arcs J'_u, J'_l and the two disks D'_u, D'_l as indicated in Figure 1. The intersection of D'_u and D'_l consists of the two disjoint arcs A'_u and A'_l. Let R be a regular neighborhood of $J'_u \cup J'_l$ in T'. Then R is the union of two disjoint cubes R_u and R_l, containing J'_u and J'_l, respectively.

Assume R is taken so that $R \cap D'_u$ is a regular neighborhood in D'_u of $J'_u \cup (D'_u \cap J'_l)$ and $R \cap D'_l$ is a regular neighborhood in D'_l of $J'_l \cup (D'_l \cap J'_u)$. Assume also that $A_u = \mathrm{Cl}(A_u - R)$ and $A_l = \mathrm{Cl}(A'_l - R)$ are arcs.

Let T be the cube with 2 holes obtained by removing Int R (w.r.t. T') from

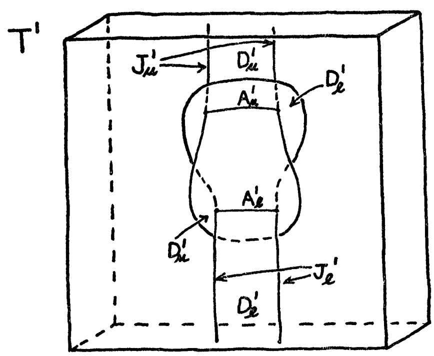

FIG. 1

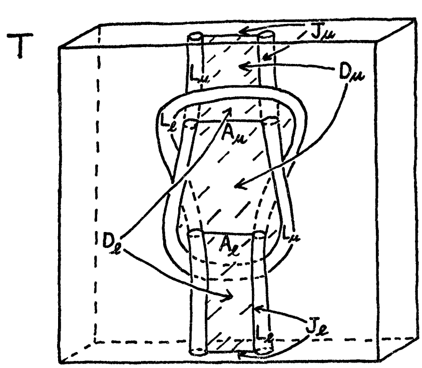

FIG. 2

T' (see Figure 2). Let

$$D_u = D'_u - (\text{Int } R), \quad D_l = D'_l - (\text{Int } R), \quad L_u = \text{Cl}(\text{Bd } R_u \cap \text{Int } T'), \quad \text{and}$$
$$L_l = \text{Cl}(\text{Bd } R_l \cap \text{Int } T').$$

Then $D_u(D_l)$ is a disk with 2 holes and let $J_u(J_l)$ be the scc of Bd D_u(Bd D_l) that does not intersect $A_l(A_u)$. Note that L_u, L_l are annuli on Bd T. (See Figure 2 for a picture of these subsets of T.) Let $D_u^*(D_l^*)$ be the disk obtained from the closure of the component of $D_u - A_u(D_l - A_l)$ not containing Bd $D_u - J_u$ (Bd $D_l - J_l$). Let $L_u^*(L_l^*)$ be the subannulus of $L_u(L_l)$ bounded by Bd $D_l - J_l$(Bd $D_u - J_u$).

4. Proof of Theorem 1

Some necessary parts to the proof of Theorem 1 are contained in the following six lemmas. The first three of these lemmas are concerned with some general topological properties needed for the investigation of our example T, and the last three lemmas are concerned with some specific properties of T.

Suppose M_1, M_2 are compact orientable 2-manifolds in E^3 such that Bd $M_1 \cap$ Bd $M_2 = \emptyset$. In this paper we use the definition of the linking number $\mathfrak{o}$(Bd M_1, Bd M_2) of Bd M_1, Bd M_2 as given on page 81 of [1] with the integers as the coefficient domain. The following lemma is proved in [1].

LEMMA 1. *If* $\mathfrak{o}(\text{Bd } M_1, \text{Bd } M_2) \neq 0$, *then* $\mathfrak{o}(\text{Bd } M_2, \text{Bd } M_1) \neq 0$ *and if* M'_1, M'_2 *are compact 2-manifolds such that* Bd M'_1 = Bd M_1, Bd M'_2 = Bd M_2, *then*

$$\mathfrak{o}(\text{Bd } M'_1, \text{Bd } M'_2) = \mathfrak{o}(\text{Bd } M_1, \text{Bd } M_2).$$

In [8], A Dehn surface of type (p, r) is defined and in [6], a conservative ε-alteration of a singular disk is defined. We may extend the term conservative ε-alteration to apply to Dehn surfaces of type (p, r). Using this terminology we have the following lemma.

LEMMA 2. *Let D be a Dehn surface of type* $(0, r)$ *in the 3-manifold M such that a regular neighborhood of* Bd D *in M consists of r disjoint solid tori. Then there exists a nonsingular surface of type* $(0, r)$ *in M which is a conservative ε-alteration of D.*

Proof. Let $(\Delta_1, \cdots, \Delta_r)$ be the boundary components of D. Since a regular neighborhood of Bd D in M consists of r disjoint solid tori, it follows that there exist r disjoint solid tori $\Gamma_1, \cdots, \Gamma_r$ in M such that for $1 \leq i \leq r$, Δ_i is a longitude of Γ_i on Bd Γ_i. For $2 \leq i \leq r$, let h_i be a homeomorphism of Bd Γ_i onto itself which carries the boundary of a meridional disk Ψ_i of Γ_i onto Δ_i. Now add $\bigcup_{i=2}^r \Gamma_i$ to $M - \bigcup_{i=2}^r \text{Int } \Gamma_i$ by the identification $x \equiv h_i(x)$ for $x \in \text{Bd } \Gamma_i$. The resulting manifold M' now contains the singular disk $D' = D \cup (\bigcup_{i=2}^r \Psi_i)$. It then follows by Theorem IV. 3 of [6] that there is a nonsingular disk D'' which is a conservative ε-alteration of D' in M' and, if the ε is small enough, D'' contains $\bigcup_{i=2}^{r} \Psi_i$; hence $D'' - \text{Int}(\bigcup_{i=2}^{r} \Psi_i)$ is a non-singular Dehn surface of type $(0, r)$ which is a conservative ε-alteration of D in M.

LEMMA 3. *Suppose f is a boundary preserving map of T (or any cube with 2 holes K_2) onto C_2. Suppose further that X, Y are disjoint scc's on* Bd C_2 *which are not* 1*-linked in C_2. Then $f^{-1}(X), f^{-1}(Y)$ are not* 1*-linked in T.*

Proof. Let X, Y bound in C_2 the disjoint compact orientable 2-manifolds M_X, M_Y respectively.

Let h_1, h_2 be homeomorphisms of $M_X \times [0, 1]$, $M_Y \times [0, 1]$ into C_2 such that

(1) $h_1(M_X \times [0, 1]) \cap h_2(M_Y \times [0, 1]) = \emptyset$,
(2) $h_1(M_X \times \{1/2\}) = M_X$, $h_2(M_Y \times \{1/2\}) = M_Y$, and
(3) $h_1(X \times [0, 1]) \subseteq \text{Bd } C_2$, $h_2(Y \times [0, 1]) \subseteq \text{Bd } C_2$.

Let R_X be a regular neighborhood of $f^{-1}(M_X)$ contained in

$$f^{-1}(h_1(M_X \times [0, 1]))$$

and let R_Y be a regular neighborhood of $f^{-1}(M_Y)$ contained in

$$f^{-1}(h_2(M_Y \times [0, 1])).$$

Let R'_X be the component of R_X containing $f^{-1}(X)$ and let R'_Y be the component of R_Y containing $f^{-1}(Y)$. Let Z be an arc in Bd $T \cap R'_X$ which intersects and pierces $f^{-1}(X)$ at just one point. Now if $f^{-1}(X)$ does not separate Bd R'_X, then we may join the endpoints of Z by an arc Z' in $R'_X - f^{-1}(M_X)$. But then $f(Z \cup Z')$ can be adjusted slightly to form a scc in $h_1(M_X \times [0, 1])$ which intersects and pierces M_X at just one point, contradicting that, locally, M_X has two sides. Hence $f^{-1}(X)$ separates Bd R'_X into two components and, by a similar argument, $f^{-1}(Y)$ separates Bd R'_Y. The closure of a component of Bd $R'_X - f^{-1}(X)$ and a component of Bd $R'_Y - f^{-1}(Y)$ form the surfaces required to show $f^{-1}(X), f^{-1}(Y)$ are not 1-linked in T.

LEMMA 4. *In T, J_u and J_l are* 1*-linked.*

Proof. Suppose J_u, J_l are not 1-linked in T. Let M_u, M_l be disjoint compact orientable 2-manifolds in T bounded by J_u, J_l, respectively. Now J_u belongs to the first commutator subgroup $(\pi_1(M_u))'$ of $\pi_1(M_u)$. If X is a scc in $T - (M_u \cup M_l)$, then $\mathfrak{o}(X, J_u) = 0$ and $\mathfrak{o}(X, J_l) = 0$; hence $X \in (\pi_1(T))'$. Since each loop in M_u is obviously homotopic to a loop in $T - (M_u \cup M_l)$, it follows that $J_u \in (\pi_1(T))''$. By [10],

$$\pi_1(T) = \{c, g, x : [c[g, x]] = x\},$$

where x can be taken to represent J_u. As suggested in [10], we may map $\pi_1(T)$ onto the permutation group S_3 on three elements by sending c, g to (12) and x to (123). Since $(123) \notin S_3'' = \{1\}$, it follows that $J_u \notin (\pi_1(T))''$, contradiction. Hence J_u, J_l are 1-linked in T.

LEMMA 5. *Suppose f is a boundary preserving map of T onto C_2 (recall the assumption made on f in Section 2) and X is a scc on* Bd C_2 *such that X does not bound a disk on* Bd C_2 *and either $X \cap f(J_u) = \emptyset$ or $X \cap f(J_l) = \emptyset$. Then X is not null homotopic in C_2.*

Proof. Suppose X is null homotopic in C_2 and disjoint from $f(J_u)$. Using Dehn's Lemma, we obtain a disk F such that Bd $F = X$ and Int $F \subseteq$ Int C_2. Let $R(F)$ be a regular neighborhood of F in $C_2 - f(J_u)$. Since C_2 is a cube with 2 handles and X does not bound a disk on Bd C_2, it follows that Cl $(C_2 - R(F))$ is either a cube with 1 handle or two disjoint cubes with 1 handle. Since $f(J_u)$ is null homologous in C_2 (using integer coefficients), it follows that $f(J_u)$ is null homologous in Cl $(C_2 - R(F))$ and hence bounds a disk M_u in Cl $(C_2 - R(F))$. Since $f(J_l)$ is null homologous in C_2, it bounds a compact orientable 2-manifold M_l in C_2 and, by adjusting M_l to be in general position with M_u, cutting M_l off on M_u, and pushing M_l to one side of M_u, it follows that we may assume $M_u \cap M_l = \emptyset$. Then $f(J_u)$, $f(J_l)$ are not 1-linked in C_2 and hence, by Lemma 3, J_u and J_l are not 1-linked in T, contradicting Lemma 4. Interchanging $f(J_u)$ and $f(J_l)$ gives a proof for the case $X \cap f(J_l) = \emptyset$.

Under the assumption that there exists a boundary preserving map of T onto C_2, the next lemma shows that we may obtain compact 2-manifolds E_u, E_l in C_2 with properties enough like those of D_u, D_l in T to imply (in Theorem 1) the contradiction that C_2 is not a cube with handles. In the next lemma we choose θ_1 so that $D_u^* \cup D_l^* \subseteq \theta_1$; hence $f \mid D_u^* \cup D_l^*$ is a homeomorphism (see Section 2 for a description of θ_1 and Section 3 for D_u^*, D_l^*).

LEMMA 6. *Suppose f is a boundary preserving map of T onto C_2. Then, in C_2, there exists a copy E_u of D_u and a compact orientable 2-manifold E_l such that*

(1) Bd $E_u = f($Bd $D_u)$, Bd $E_l = f(J_l)$,
(2) Int $E_u \cup$ Int $E_l \subseteq$ Int C_2,
(3) *E_u and E_l are in relative general position, and*
(4) $f(D_u^*) \subseteq E_u$, $f(D_l^*) \subseteq E_l$.

Proof. By Lemma 2, the singular Dehn surfaces $f(D_u)$, $f(D_l)$ of type (0, 3) may be replaced, in C_2, by nonsingular Dehn surfaces E_u, ${}_0E_l$ of type (0, 3) which are conservative ε-alterations of $f(D_u)$, $f(D_l)$, respectively. We may choose the ε of the ε-alteration small enough that $f(D_u^*) \subseteq E_u$ and $f(D_l^*) \subseteq {}_0E_l$. Since $f(L_u^*)$ intersects ${}_0E_l$ on one side of ${}_0E_l$, $E_l = {}_0E_l \cup f(L_u^*)$ is a compact orientable 2-manifold. (See Section 3 for a description of L_u^*.) By adjusting $E_l - f(D_l^*)$ slightly, so that Int $E_l \subseteq$ Int C_2 and E_u, E_l are in general position, the required surfaces E_u and E_l are obtained. Note that $E_u \cap E_l$ consists of the arc $f(A_l)$ and a finite number of disjoint scc's in $E_u - f(A_l)$.

THEOREM 1. *There does not exist a boundary preserving map of T onto C_2.*

Proof. Suppose f is a boundary preserving map of T onto C_2. Let E_u and E_l be as given in Lemma 5. Since C_2 is a cube with 2 handles, there is a disk F in C_2 such that Bd $F \subseteq$ Bd C_2, Int $F \subseteq$ Int C_2, Bd F does not bound a disk on Bd C_2, and F is in general position relative to E_u.

If $F \cap E_u$ contains a scc S which separates the two components of Bd $E_u - f(J_v)$ in E_u, then $\mathfrak{o}(S, f(J_l)) = 0$ using the disk S bounds in F. But, after a slight adjustment, S intersects and pierces E_l an odd number of times, hence $\mathfrak{o}(S, f(J_l)) \neq 0$ using E_l, and we have a contradiction to Lemma 1. If $F \cap E_u$ contains a scc S which separates $f(J_u)$ from Bd $E_u - f(J_u)$ in E_u, then $f(J_u)$ bounds a disk in C, contradicting Lemma 5. If $F \cap E_u$ contains any scc's which bound disks in E_u, they may be removed by cutting F off on E_u and pushing to one side of E_u. Hence we may assume $F \cap E_u$ consists of a finite collection of disjoint arcs with interiors in Int E_u and endpoints in Bd E_u.

Suppose an arc X in $F \cap E_u$ together with an arc Y in Bd E_u form a scc which bounds a disk F' in E_u such that Int $F' \cap F = \emptyset$. Now Y plus one of the two open arcs of Bd $F-$Bd Y form a scc Z which does not bound a disk on Bd C_2. But Z bounds a disk E in C_2 formed by the sum of the disk F' and the disk on F bounded by $(Z \cap \text{Bd } F) \cup X$. Then E may be adjusted slightly so that E is in general position relative to E_u, $E \cap E_u \subseteq F \cap E_u$ and the number of arcs $E \cap E_u$ which together with an arc in Bd E_u bound a disk in E_u is less than those of $F \cap E_u$. By applying the previous argument a finite number of times (and denoting the result by F again), it follows that we may assume F satisfies the following condition, which we refer to as Condition A: The intersection of F with E_u contains no arc that together with an arc in Bd E_u form a scc which bounds a disk in E_u.

Let $\mathfrak{A}$ be the collection of arcs in $F \cap E_u$ which intersect $f(J_u)$. Then each arc X of $\mathfrak{A}$ is one of the following two types:

(1) X has both endpoints in $f(J_u)$ and separates one component of Bd $E_u - f(J_u)$ from the other in E_u.
(2) X has one endpoint in $f(J_u)$ and the other in Bd $E_u - f(J_u)$.

Now assume $X_0 \in \mathfrak{A}$ is minimal in the sense that X_0 together with an arc Y_0 in Bd F form a scc which bounds a disk F_0 in F such that no element of $\mathfrak{A}$ is contained in $F_0 - X_0$. It follows from the proof of Lemma 6 that $f(L_l^*)$ intersects just one side of E_u. Let the side of E_u which intersects $f(L_l^*)$ be called its positive side. We now have the following two cases:

(a) F_0 lies on the positive side of E_u near X_0.
(b) F_0 lies on the negative side of E_u near X_0.

Call the minimal arc X_0 of $\mathfrak{A}$ an ix arc if X_0 satisfies conditions (i) and (x) above, where i = 1, 2 and x = a, b. Each of the four possible cases ix is now shown to lead to a contradiction.

Case I. X_0 is of type 1a. Since Bd $X_0 \subseteq f(J_u)$, if Bd $F_0 \cap f(L_l^*) \neq \emptyset$, then Bd $F_0 \cap$ Bd $f(L_l^*) \neq \emptyset$, and it follows by the general position of F_0 with E_u that there is an arc X in $F_0 \cap E_u$ with both endpoints in Bd $E_u - f(J_u)$. Since $X \cap X_0 = \emptyset$, $F_0 \subseteq F$, and X_0 separates the two components of Bd $E_u - f(J_u)$, it follows that X together with an arc in Bd $E_u - f(J_u)$ form a scc which bounds a disk in E_u, violating Condition A. Hence Bd $F_0 \cap f(L_l^*) = \emptyset$ and it follows that $F_0 \cap E_u = X_0$. We may adjust X_0 in E_u so that X_0 is in general position relative to E_l. Let $E_l^* =$ Cl $(E_l - f(D_l^*))$. Now by pulling F_0 off E_u along X_0 (that is X_0 is moved into the positive side of E_u), it follows that $\mathfrak{o}$(Bd F_0, Bd $E_l^*) = 0$ using F_0 (since $F_0 \cap$ Bd $E_l^* = \emptyset$) but

$$\mathfrak{o}(\text{Bd } F_0, \text{Bd } E_l^*) = +1 \text{ or } -1$$

using E_l^*, contradicting Lemma 1.

Case II. X_0 is of type 2a. In this case, by pulling F_0 off E_u along X_0 (and into the positive side of E_u), it follows that the endpoints of X_0 are separated in Bd C_2 by Bd $E_u - f(J_u)$. Hence Bd F_0 intersects and pierces Bd $E_u - f(J_u)$ an odd number of times. By pushing F_0 slightly into Int C_2, it follows that $\mathfrak{o}$(Bd F_0, Bd $E_u) = 0$ using F_0 but $\mathfrak{o}$(Bd F_0, Bd $E_u) \neq 0$ using E_u, contradicting Lemma 1

Case III. X_0 is of type 1b. We may adjust F_0 slightly so that it is in general position with respect to $f(D_l^*)$ and Bd F_0 intersects $f(A_l)$ at just one point. Since Bd $F_0 \cap f(L_l^*) = \emptyset$, as shown in Case I, it follows by the general position of F_0 with $f(D_l^*)$ that there is an arc X in $F_0 \cap f(D_l^*)$ with one endpoint Bd $F_0 \cap f(A_l)$ and the other in $f(J_l)$. Since $X \subseteq f(D_l^*)$, $X \cap$ Int $E_l^* = \emptyset$ and there is a homeomorphism h of C_2 onto itself fixed on Bd C_2, Bd E_l^* and X such that $h(E_l^*) \cap X_0 = \emptyset$. Let $E_l^{**} = h(E_l^*)$. It follows that

$$\text{Int } E_l^{**} \cap E_u \subseteq E_u - (f(A_l) \cup X_0),$$

and hence we may cut E_l^{**} off on E_u and then off $f(D_l^*)$, so that $M_l = f(D_l^*) \cup E_l^{**}$ forms a compact orientable 2-manifold with boundary $f(J_l)$ such that $M_l \cap E_u = f(A_l)$. Let R be a regular neighborhood of $M_l \cup f(L_l)$ in C_2 such that $R \cap E_u$ is a regular neighborhood of

$$f(A_l) \cup (\text{Bd } E_u - f(J_u))$$

in E_u. Let M_u be Cl $(E_u - R)$ together with the component of Bd $R - E_u$ not containing $f(L_l)$. It then follows that M_u and M_l are disjoint compact orientable 2-manifolds with boundaries $f(J_u)$ and $f(J_l)$, respectively. By Lemma 3, J_u and J_l are not 1-linked in T, contradicting Lemma 4.

Case IV. X_0 is of type 2b. Let F_0' be the closure of the component of $(F_0 - E_u) \cup X_0$ containing X_0. Note that F_0' is a disk which intersects E_u on the negative side only and $F_0' \cap E_u$ consists of X_0 and a finite collection of disjoint arcs in $E_u - X_0$ each with endpoints in Bd $E_u - f(J_u)$. Since $E_l^* \cap E_u$ consists of $f(A_l)$ and disjoint scc's in $E_u - f(A_l)$, it follows that we may adjust Int E_l^* near $E_u - f(A_l)$ so that

$$(E_l^* \cap E_u) - f(A_l) \subseteq (E_u - F_0') \cup X_0 .$$

By pulling F_0' off E_u (into the negative side of E_u) away from the arcs in $F_0' \cap E_u - X_0$, we may assume

$$F_0' \cap E_u = X_0$$

as well as

$$F_0' \cap E_l^* \subseteq \text{Int } F_0' \cup X_0$$

(since $E_l^* \cap E_u - f(A_l) \subseteq (E_u - F_0') \cup X_0$ and E_l^* intersects E_u on the positive side near $f(A_l)$). We may adjust F_0' near E_u so that $X_0 \cap f(A_l) = \emptyset$. Since $F_0' \cap E_l^* \subseteq \text{Int} F_0' \cup X_0$ and Bd $E_l^* \cap F_0' = \emptyset$, there exists a homeomorphism h of C_2 onto itself which is fixed on Bd C_2 and Bd E_l^* such that $h(E_l^*) \cap X_0 = \emptyset$. Letting $E_l^{**} = h(E_l^*)$, the rest of the proof is the same as Case III.

These four cases now imply $F \cap f(J_u) = \emptyset$, and the existence of F contradicts Lemma 5 (where the X of Lemma 5 is taken to be Bd F). Hence there is no boundary preserving map f of T onto C_2 and the proof of Theorem 1 is complete.

COROLLARY. *For each $n \geq 2$ there is a cube with n holes T_n with no boundary preserving map onto the cube with n handles C_n.*

Proof. For $n \geq 2$, let T_n be the T of Section 3 together with $n-2$ disjoint cubes with 1 handle $H_1, H_2, \cdots, H_{n-2}$ such that for each i,

$$H_i \cap T = \text{Bd } H_i \cap \text{Bd } T = \text{a disk } D_i .$$

Suppose f is a boundary preserving map of T_n onto C_n. Using Dehn's Lemma, replace each $f(D_i)$ by a nonsingular disk D_i' in C_n such that $D_i' \cap D_j' = \emptyset$ for $i \neq j$. It follows that each $f(\text{Bd } H_i - D_i) \cup D_i'$ bounds a cube with one handle H_i' in C_n such that $H_i' \cap H_j' = \emptyset$ for $i \neq j$. Then, filling in the hole of each H_i and H_i' by a cube (see [2] for a discussion of this process), we obtain from T_n a T_n' homeomorphic to T and from C_n a C_n' homeomorphic to C_2. It now follows that f may be extended across the filled in holes to a boundary preserving map of $T_n' = T$ onto $C_n' = C_2$, contradicting Theorem 1.

By [10], $\pi_1(T) = \{c, g, x : [c[g, x]] = x\}$ and it follows that there is a homomorphism of $\pi_1(T)$ onto the free group on two generators, $\pi_1(C_2)$. In [9], N. Smythe gives an example of 1-linked scc's l_1, l_2 in S^3 that form a homology boundary link. Let ${}_0l_1, {}_0l_2$ be disjoint scc's in the xy-plane and let $R(l_1)$, $R(l_2)$, $R({}_0l_1)$, and $R({}_0l_2)$ be regular neighborhoods in S^3 of $l_1, l_2, {}_0l_1$, and ${}_0l_2$, respectively. Assume

$$R(l_1) \cap R(l_2) = \emptyset \quad \text{and} \quad R({}_0l_1) \cap R({}_0l_2) = \emptyset.$$

Then it follows that there is no boundary preserving map of the connected elementary figure (see [3])

$$S^3 - (\mathrm{Int}\, R(l_1) \cup \mathrm{Int}\, R(l_2))$$

onto the connected elementary figure

$$S^3 - (\mathrm{Int}\, R({}_0l_1) \cup \mathrm{Int}\, R({}_0l_2))$$

but there is a homomorphism of

$$\pi_1\, (S^3 - (\mathrm{Int}\, R(l_1) \cup R(l_2)))$$

onto the free group on two generators

$$\pi_1\, (S^3 - (\mathrm{Int}\, R({}_0l_1) \cup \mathrm{Int}\, R({}_0l_2)))\,.$$

We have obtained in Theorem 1 the analogous result for the connected elementary figure T with connected boundary.

5. The existence of boundary preserving maps

In this section we give some conditions which imply the existence of a boundary preserving map of K_n onto C_n. We say the disjoint scc's $l_1, \cdots, l_n$ in K_n form a boundary link [9] in K_n if they bound disjoint compact orientable 2-manifolds $M_1, \cdots, M_n$, respectively, in K_n. In Theorem 5 of [5], J. Hempel shows that there is a boundary preserving map of any K_1 onto C_1, and, to prove this, Hempel observes that any K_1 has a scc l_1 which is a boundary link in K_1 and Bd $K_1 - l_1$ is connected. The "if" portion of the next theorem is a straightforward generalization of Hempel's Theorem 5; the "only if" portion is a straightforward generalization of our Lemma 3.

THEOREM 2. *There exists a boundary preserving map of K_n onto C_n if and only if there exists a boundary link $l_1, \cdots, l_n$ in K_n such that* Bd $K_n - \bigcup_{i=1}^n l_i$ *is connected.*

Note that Theorem 2 together with Theorem 1 imply that if l_1, l_2 are scc's on Bd T such that Bd $T - l_1 \cup l_2$ is connected, then l_1, l_2 are 1-linked (not a boundary link) in T.

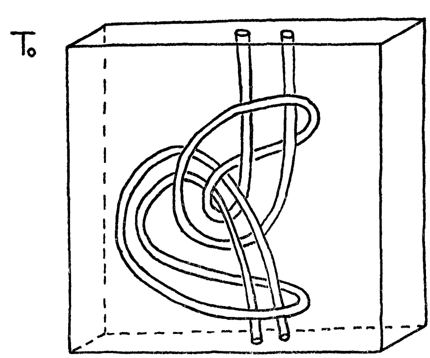

FIG. 3

We say K_n is reducible [7] if there is a disk D in K_n such that Bd $D \subseteq K_n$ and Bd D does not bound a disk on Bd K_n. It follows that if K_2 is reducible, then there is a boundary link l_1, l_2 in K_2 such that Bd $K_2 - l_1 \cup l_2$ is connected. Hence we have the next theorem.

THEOREM 3. *If K_2 is reducible, then there is a boundary preserving map of K_2 onto C_2.*

Figure 3 illustrates a cube with 2 holes T_0 that provides a counterexample to the converse of Theorem 3. It is easy to show that T_0 satisfies the hypothesis of the "if portion" of Theorem 2, but it can be shown (by a long geometric proof similar to that of Theorem 1) that T_0 is not reducible.

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