LOCALIZATION THEOREMS IN HARMONIC ANALYSIS ON R^n

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1. The theory of Rajchman and Zygmund on localization properties of trigonometric series (see for instance [6, pp. 330–344, 363–370]) has been extended to the multi-dimensional case by Berkovitz, Gosselin and Shapiro in [1], [2] and [4] (for a brief account of some of their results see [5, p. 83–85]). They discuss the localization problem for summability kernels of a very special type and their methods are very closely linked to the particular properties of the kernels. Our aim is to present a method, in one dimension basically the same as Zygmund's method in [6] (the proof of his Theorem 9.6), and which is applicable to a larger class of kernels.

The method can be applied to R^n as well as to Z^n. We choose to present it on R^n for the reason that the study on R^n contains all the essential ideas which are needed for the corresponding study on Z^n, on the other hand, it is more general due to the non-compactness of the character group of R^n. In order to facilitate a comparison of our results on R^n with the previously established results on Z^n we have, in the last section, formulated a localization theorem for Bochner-Riesz summability.

In the following we make freely use of the basic concepts of Fourier analysis of tempered distributions, as presented in any standard text-book on the subject.

2. Let q be a positive and continuous function on R^n, such that

$$p(x) = \sup_{y \in R^n} q(x + y)/q(y) \tag{1}$$

is finite for all $x \in R^n$, bounded on compact sets and $O(|x|^A)$, for some positive number A, as $|x| \to \infty$. Obviously p is a Borel measurable positive function. We introduce the space M_p of Borel measures μ on R^n with a finite norm

$$\|\mu\|_p = \int_{R^n} p(x) \, | d\mu(x) |,$$

the space B_q of Borel measurable functions g on R^n with a finite norm

$$\| g \|_q = \int_{R^n} | g(x) | \, g(x) \, dx,$$

and finally the space B^q of Borel measurable functions f on R^n with a finite norm

$$\| f \|^q = \operatorname{ess\,sup}_{x \in R^n} | f(x) | / q(-x),$$

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and having the property that $f(x)/q(-x) \to 0$, as $|x| \to \infty$, except in a set of Lebesgue measure 0.

Let us assume that $f \in B^q$, $g \in B_q$ and $\mu \in M_p$. Using (1), which gives

$$q(x+y) \leqq q(y)p(x), \tag{2}$$

for x and y in R^n, and using standard arguments in the theory of convolution of measures, we find that

$$f * g(x) = \int_{R^n} f(x-y)g(y)\,dy$$

exists for every $x \in R^n$, that

$$f * \mu(x) = \int_{R^n} f(x-y)\,d\mu(y) \quad \text{and} \quad g * \mu(x) = \int_{R^n} g(x-y)\,d\mu(y)$$

are well defined except in sets of Lebesgue measure 0, and define functions in B^q and B_q, respectively, which satisfy

$$\| f * \mu \|^q \leqq \| f \|^q \| \mu \|_p \quad \text{and} \quad \| g * \mu \|_q \leqq \| g \|_q \| \mu \|_p,$$

respectively, and finally that

$$(f * \mu) * g = f * (f * \mu),$$

for every point in R^n.

We had assumed that, for some $A > 0$,

$$p(x) = O(|x|^A),$$

as $|x| \to \infty$, hence by (2)

$$q(x) = O(|x|^A) \quad \text{and} \quad 1/q(x) = O(|x|^A),$$

as $|x| \to \infty$. This implies that

$$f(x) = O(|x|^A),$$

as $|x| \to \infty$, if $f \in B^q$, while

$$\int_{R^n} |f|\,(1+|x|)^{-A}\,dx < \infty,$$

if $f \in B_q$. Hence, if we consider f in one of these classes as a distribution on R^n, it belongs to the sub-class of tempered distributions. This has the consequence that its Fourier transform $\hat{f}$, formally defined by

$$\hat{f}(y) = \int_{R^n} e^{-ix\cdot y} f(x)\,dx,$$

where $x \cdot y$ denotes inner product, exists as a tempered distribution on the dual R^n.

For every fixed $t \in R^n$ we denote by T_t the translation operator, defined for every distribution f by the formal relation

$$(T_t f)(x) = f(x + t).$$

The difference operator Δ_t is defined at $T_t - T_0$. If f is a tempered distribution with Fourier transform $\hat{f}$, then the translate $T_t f$ is a tempered distribution with Fourier transform

$$\hat{f}(y)e^{it \cdot y}.$$

3. DEFINITION 1. Let m be a positive integer. A sequence $\{g_\nu\}_1^\infty$ of functions in B_q is said to belong to the class A_q^m if, for every sequence $(t_1, t_2, \cdots, t_m)$ of points in R^n,

$$I_\nu(C) = \int_C |(\Delta_{t_1}\Delta_{t_2} \cdots \Delta_{t_m} g_\nu)(x)| \, q(x) \, dx$$

1^0. tends to 0, as $\nu \to \infty$, for every compact $C \subset R^n$,
2^0. is bounded, as $\nu \to \infty$, if $C = R^n$.

We are now in a position to formulate and prove our first basic theorem. We recall that multiplication of a distribution and an infinitely differentiable function is a well-defined operation which gives a new distribution, and that two distributions are said to coincide in an open set if the support of their difference is contained in the complement of this set.

THEOREM 1. *Let $f \in B^q$ have a Fourier transform $\hat{f}$ with a compact support. We assume that, in some neighborhood of $y = 0$ on the dual R^n and for some non-negative integers N and m*

$$\hat{f} = \sum_{j=1}^N \hat{h}_j \hat{P}_j, \tag{3}$$

where $\hat{h}_j$ are Fourier transforms of functions $h_j \in B^q$ and where $\hat{P}_j$ are homogeneous polynomials of positive degree m on R^n. Furthermore we assume that $\{g_\nu\}_1^\infty \in A_q^m$. Then

$$\int_{R^n} f(-x)g_\nu(x) \, dx \to 0,$$

as $\nu \to \infty$.

Proof of Theorem 1. Let $\hat{\mu} \in \mathfrak{D}$, where $\mathfrak{D}$ denotes the class of infinitely differentiable functions with compact support. Since p is of at most polynomial growth, $\hat{\mu}$ is the Fourier-Stieltjes transform

$$\int_{R^n} e^{-ix \cdot y} \, d\mu(x)$$

of a measure $\mu \in M_p$, in fact this measure is absolutely continuous. If $f_0 \in B^q$, then the distribution $\hat{f}_0 \hat{\mu}$ is the Fourier transform of $f_0 * \mu$, which belongs to B^q by the properties stated in §2.

Let us choose this function $\hat{\mu}$ such that it takes the value 1 in an open neighborhood of $y = 0$. Then (3) shows that

$$\hat{f} = \hat{f}\hat{\mu} = \sum_{j=1}^{N} \hat{h}_j \hat{\mu} \hat{P}_j \tag{4}$$

in this neighborhood. By the above-mentioned results, the distributions $\hat{h}_j \hat{\mu}$, as well as the distributions $\hat{h}_j \hat{\mu}\hat{P}_j$ are Fourier transforms of functions in B^q. Thus we have exchanged (3) to a representation (4) where the factors in front of $\hat{P}_j$ still are Fourier transforms of functions in B^q, but now such that the factors have compact supports, and where the right hand member, too, is the Fourier transform of a function in B^q.

Hence there is no restriction to assume that these properties hold already for the representation (3). We then denote by h the function in B^q, for which

$$\hat{h} = \hat{f} - \sum_{j=1}^{N} \hat{h}_j \hat{P}_j .$$

The distribution $\hat{h}$ has then a compact support not containing $y = 0$.

Using the identity

$$1 = \left(\sum_{k=1}^{n} y_k^2\right)^m \cdot \left(\sum_{k=1}^{n} y_k^2\right)^{-m}, \qquad y \neq 0,$$

where y_k are the coordinates of R^n, it is easy to see that there exist a positive integer M, homogeneous polynomials $\hat{R}_l$, $l = 1, 2, \cdots, M$ of degree m, and rational functions $\hat{S}_l$, $l = 1, 2, \cdots, M$, infinitely differentiable except at $y = 0$, such that

$$1 = \sum_{l=1}^{M} \hat{S}_l \hat{R}_l, \qquad y \neq 0.$$

Let $\hat{v} \in \mathfrak{D}$ have support not containing $y = 0$ and coincide with 1 on an open set containing the support of $\hat{h}$. Then

$$\hat{h} = \hat{h}\hat{v} = \sum_{l=1}^{M} \hat{h}\hat{S}_l \hat{v} \hat{R}_l ,$$

hence

$$\hat{f} = \sum_{j=1}^{N} \hat{h}_j \hat{P}_j + \sum_{l=1}^{M} \hat{h}\hat{S}_l \hat{v}\hat{R}_l . \tag{5}$$

But $\hat{S}_l \hat{v} \in \mathfrak{D}$, hence, by the arguments used in the discussion of (4), $\hat{h}\hat{S}_l \hat{v}$ is a distribution with compact support, and it is the Fourier transform of a function in B^q. Thus we have exchanged (3) to a representation (5) which holds everywhere, and where the factors in front of the polynomials are distributions with compact supports and which are Fourier transforms of functions in B^q. Thus there is no restriction to assume that all these properties hold already for the representation (3). And this means, using the linearity, that it is enough to consider the case when (3) has the form

$$\hat{f} = \hat{h} \prod_{k=1}^{m} y_{p_k} ,$$

where $1 \leqq p_k \leqq n$, and where $\hat{h}$ has compact support and is the Fourier transform of a function $h \in B^q$.

We choose $\delta > 0$ so small that

$$e^{i\delta y_j} \neq 1,$$

$j = 1, 2, \cdots, n$, on the support K of $\hat{h}$, except if $y_j = 0$. Then there exists a function $\hat{\mu} \in \mathfrak{D}$, such that

$$\hat{\mu}(y) \prod_{k=1}^{m} (\exp i\delta y_{p_k} - 1) = \prod_{k=1}^{m} y_{p_k}$$

on an open set, including K. With $\hat{h}_0 = \hat{h}\hat{\mu}$ we obtain

$$\hat{f} = \hat{h}_0 \prod_{k=1}^{m} (\exp i\delta y_{p_k} - 1).$$

$\hat{h}_0$ is then the Fourier transform of a function $h_0 \in B^q$, by the argument in the beginning of this proof.

We let $x_1, x_2, \cdots, x_n$ denote the coordinate basis in R^n which corresponds to $y_1, y_2, \cdots, y_n$ in the sense that

$$x \cdot y = \sum_{j=1}^{n} x_j y_j,$$

and introduce the operator

$$D = \Delta_{\delta x_{p_1}} \Delta_{\delta x_{p_2}} \cdots \Delta_{\delta x_{p_m}}.$$

It is then easy to see that $f = Dh_0$. Hence

$$f * g_\nu(x) = \int_{R^n} (Dh_0)(x - y) g_\nu(y)\, dy$$

$$= \int_{R^n} h_0(x - y)(Dg_\nu)(y)\, dy,$$

for every $x \in R^n$. For $x = 0$ we obtain from this

$$\left| \int_{R^n} f(-y) g_\nu(y)\, dy \right| \leqq \int_{R^n} \frac{| h_0(-y) |}{q(y)} \, | (Dg_\nu)(y) | \, q(y)\, dy.$$

Since we had assumed that $\{g_\nu\}_1^\infty \in A_q^m$, the right hand member tends to 0, as $\nu \to \infty$, which proves the theorem.

Remark. Theorem 1 can be generalized in the respect that it can be proved to hold for functions q of more general type. The condition that the function p, defined by (1), is of polynomial growth can thus be exchanged to the condition that it satisfies

$$\sum_{k=0}^{\infty} \frac{\log p(kx)}{k^2} < \infty,$$

for every $x \in R^n$. This can be seen by application of the theory in [3], which is possible, since p is necessarily sub-multiplicative. It is also possible to obtain corresponding theorems for other spaces than B^q and B_q.

4. Definition 2. A sequence $\{g_\nu\}_1^\infty$ in the class A_q^m is said to belong to the class a_q^m if, with the notations of Definition 1, $I_\nu(R^n)$ is bounded, as $\nu \to \infty$, uniformly for t_i, $i = 1, 2, \cdots, m$, in any fixed compact subset of R^n.

The following is our second basic theorem. This theorem is relevant for the case when $\hat{f}$ has non-compact support.

THEOREM 2. *We assume that the hypotheses of Theorem 1 are true, except that we strengthen the assumption on $\{g_\nu\}_1^\infty$ by demanding that it belongs to a_q^m, while we weaken the assumption on f by allowing the support of $\hat{f}$ to be non-compact. Then the conclusion of Theorem 1 still holds.*

In the proof of this theorem we need the following lemma.

LEMMA 1. *Let $g \in B_q$, let $\hat{h}$ be infinitely differentiable on R^n, with support in $\{y \mid |y| \leqq 1\}$ and satisfying $\hat{h}(0) = 1$, and let μ be a bounded Borel measure on R with its support compact and included in the set $\{u \mid |u| \geqq 1\}$ on R. We denote by h the summable and continuous function on R^n which has the Fourier transform $\hat{h}$.*

Then there exists a function $g_0 \in B_q$, such that the Fourier transforms of g and g_0 coincide on the set $\{y \mid |y| > 1\}$ and such that

$$|g_0(x)| \leqq \int_{R^n} \left| g(x) - \int_{-\infty}^{\infty} g(x - yu)\, d\mu(u) \right| |h(y)|\, dy,$$

for every $x \in R^n$.

Proof of Lemma 1. We introduce the function h_0 defined by

$$h_0(x) = \int_{-\infty}^{\infty} h(x/u) u^{-n}\, d\mu(u),$$

for every x. The integral is well defined, since we integrate over a compact set, not containing $u = 0$. The assumptions on $\hat{h}$ imply that $h(x)$ tends to 0 faster than any power of $|x|$, as $|x| \to \infty$. Hence the same is true for $h_0(x)$. Furthermore its Fourier transform $\hat{h}_0$ satisfies

$$\hat{h}_0(y) = \int_{-\infty}^{\infty} \hat{h}(yu)\, d\mu(u),$$

for every y. Hence $\hat{h}_0$ is infinitely differentiable and its support is contained in the set $\{y \mid |y| \leqq 1\}$.

We put

$$g_0 = g - g * h_0,$$

which obviously is well defined and belongs to B_q. We denote by $\hat{g}$ and $\hat{g}_0$ the Fourier transforms of g and g_0, respectively. The Fourier transform of $g * h_0$ is $\hat{g}\hat{h}_0$, hence its support is contained in $\{y \mid |y| \leqq 1\}$. This implies that $\hat{g}$ and $\hat{g}_0$ coincide on the set $\{y \mid |y| > 1\}$.

Furthermore, by absolute convergence, we have for every $x \in R^n$

$$\begin{aligned} g_0(x) &= g(x) - \int_{R^n} g(x-y) \int_{-\infty}^{\infty} h(y/u)u^{-n}\, d\mu(u)\, dy \\ &= g(x) \int_{R^n} h(y)\, dy - \int_{R^n} \int_{-\infty}^{\infty} g(x-yu)h(y)\, d\mu(u)\, dy \\ &= \int_{R^n} h(y)\left\{g(x) - \int_{-\infty}^{\infty} g(x-yu)\, d\mu(u)\right\} dy \end{aligned}$$

which immediately gives the desired inequality.

Proof of Theorem 2. Let $\hat{\varphi} \in \mathfrak{D}$ have the property that its value is 1 in an open set, containing $\{y \mid |y| \leqq 1\}$. Then, using the same arguments as in the proof of Theorem 1, it is easy to see that

$$\hat{f} = \hat{f}\hat{\varphi} + \hat{f}(1-\hat{\varphi})$$

gives a partition of $\hat{f}$ into two parts, both satisfying the conditions of the theorem, the first part with compact support, the second with support included in $\{y \mid |y| > 1\}$. For the first part Theorem 1 is directly applicable. Hence, by linearity, there is no restriction of the validity of the proof to assume that we already from the outset have the support of $\hat{f}$ included in the set $\{y \mid |y| > 1\}$. It is then easy to see, using standard convolution arguments, that

$$\int_{R^n} f(-x)g_\nu(x)\, dx = \int_{R^n} f(-x)g_\nu^0(x)\, dx, \tag{6}$$

if $g^0 \in B_q$, and if the Fourier transforms of g_ν and g_ν^0 coincide on $\{y \mid |y| > 1\}$. This makes it possible to use Lemma 1 in order to estimate the left hand member of (6). We choose as μ in the lemma the discrete measure with the point masses

$$(-1)^{p+1}\binom{m}{p}$$

at the points $p = 1, 2, \cdots, m$ on R. Hence

$$g_\nu(x) - \int_{-\infty}^{\infty} g_\nu(x-yu)\, d\mu(u) = (-1)^m\{(\Delta_{-y})^m g_\nu\}(x),$$

for every $x \in R^n$. We can conclude from this, with h chosen as in the lemma, that we can find $\{g_\nu^0\}^\infty$, $g_\nu^0 \in B_q$ for every ν, such that (6) holds and such that

$$J_\nu(R^n) = \int_{R^n} |g_\nu^0(x)|\, q(x)\, dx \leqq \int_{R^n} |h(y)| \int_{R^n} |\Delta_{-y}^m g_\nu|\, q\, dx\, dy \tag{7}$$

and

$$J_\nu(C) = \int_C |g_\nu^0(x)|\, q(x)\, dx \leqq \int_{R^n} |h(y)| \int_C |\Delta_{-y}^m g_\nu|\, q\, dx\, dy \tag{8}$$

for compact sets C.

We have to prove that the left hand member of (6) tends to 0 as $\nu \to \infty$, hence it is enough to show that

$$\int_{R^n} | f(-x) g_\nu^0(x) | \, dx \to 0,$$

as $\nu \to \infty$. To prove this, it is enough to show that $J_\nu(R^n)$ is bounded while $J_\nu(C)$ tends to 0, for every compact C, if $\nu \to \infty$. By the assumption

$$\int_C | \Delta_{-y}^m g_\nu | \, q \, dx \to 0,$$

as $\nu \to \infty$, for every fixed $y \in R^n$. By this, (7), (8) and the Lebesgue theorem on dominated convergence, it is enough to prove that there exists a Borel measurable function k on R^n such that

$$\int_{R^n} | \Delta_{-y}^m g_\nu(x) | \, q(x) \, dx \leqq k(y)$$

for every $y \in R^n$, while

$$\int_{R^n} | h(y) | \, k(y) \, dy < \infty. \tag{9}$$

In order to do this we form a simple inequality which we for later use make slightly more general than what is needed right at the moment.

Let $t_1, \cdots, t_m$ be arbitrary points in $\{x \mid | x | \leqq 1\}$ and s an arbitrary positive integer. Then

$$\begin{aligned} \int_{R^n} \left| \left(\prod_{k=1}^m \Delta_{st_k} \right) g_\nu \right| q \, dx &\leqq \int_{R^n} \prod_{k=1}^m \left(\sum_{r=0}^{s-1} T_{rt_k} \right) \left| \left(\prod_{k=1}^m \Delta_{t_k} \right) g_\nu \right| q \, dx \\ &= \int_{R^n} \left| \left(\prod_{k=1}^m \Delta_{t_k} \right) g_\nu \right| \prod_{k=1}^m \left(\sum_{r=0}^{s-1} T_{-rt_k} \right) q \, dx \\ &\leqq \int_{R^n} \left| \left(\prod_{k=1}^m \Delta_{t_k} \right) g_\nu \right| \prod_{k=1}^m \left(\sum_{r=0}^{s-1} p(-rt_k) \right) q \, dx \\ &\leqq s^m \left(\sup_{|x| \leqq s} p(x) \right)^m \int_{R^n} \left| \left(\prod_{k=1}^m \Delta_{t_k} \right) g_\nu \right| q \, dx. \end{aligned} \tag{10}$$

By the assumption we know that there exists a constant C, such that

$$\int_{R^n} | \Delta_{-y}^m g_\nu | \, q \, dx \leqq C$$

for every ν and every y satisfying $| y | \leqq 1$. (10) then shows that for any positive integer s

$$\int_{R^n} | \Delta_{-sy}^m g_\nu | \, q \, dx \leqq C s^m \left(\sup_{|x| \leqq s} p(x) \right)^m.$$

Due to the conditions on p we thus have constants D and E such that, for every $y \in R^n$

$$\int_{R^n} | \Delta^m_{-y} g_\nu | \, q \, dx \leqq D(1 + | y |)^E = k(y).$$

This function k obviously satisfies (9), and thus the theorem is proved.

5.. In this section, we shall discuss some sequences $\{g_\nu\}_1^\infty$ of particular interest and show that they belong to the class a_q^m under certain conditions on the parameters involved. The inequalities (10) show that it is enough to consider iterated differences $\Delta_{t_1}\Delta_{t_2} \cdots \Delta_{t_m}$, where all t_i satisfy $| t_i | \leqq 1$. We start with a simple lemma which we shall use repeatedly in the sequel.

LEMMA 2. *Let g be a distribution on R^n such that, for some set $(t_1, t_2, \cdots, t_m)$ in R^n, satisfying $0 < | t_i | \leqq 1$, $i = 1, 2, \cdots, m$, the partial derivative of order m in the directions of the vectors t_i*

$$\frac{\partial^m}{\partial t_1 \partial t_2 \ldots \partial t_m} g = Dg$$

is a measure μ. Let F be a closed subset of R^n and F_m the set of all points in R^n at distance $\leqq m$ from F. Then

$$\Delta_{t_1}\Delta_{t_2} \cdots \Delta_{t_m} g = \Delta g$$

is a measure λ and

$$\int_F q(x) \, | \, d\lambda(x) \, | \leqq \sup_{|x| \leqq m} p(x) \int_{F_m} q(x) \, | \, d\mu(x) \, |.$$

Proof of Lemma 2. It is of course no restriction to assume that g has compact support. Then it is easily seen, for instance using the Fourier transform, that

$$\Delta g = Dg * \sigma_1 * \sigma_2 * \cdots * \sigma_m ,$$

where the operations are convolution of measures and where for any $j = 1, 2, \cdots, m$, σ_j is the uniform mass distribution with total mass $| t_j |$ on the segment between 0 and t_j. Hence

$$\Delta g = Dg * \sigma$$

where σ is a measure with total mass $\leqq 1$ and with its support in $\{x \mid | x | \leqq m\}$. Hence by absolute convergence

$$\int_F q(x) \, | \, d\lambda(x) \, | \leqq \int_{x \in F} \int_{y \in R^n} q(x - y) p(y) \, | \, d\mu(x - y) \, | \cdot d\sigma(y)$$

$$\leqq \sup_{|x| < m} p(x) \int_{F_m} q(x) \, | \, d\mu(x) \, | \, .$$

In all remaining theorems we assume that

$$q(x) = (1 + |x|)^\alpha,$$

for every $x \in R^n$, where α is a real number.

THEOREM 3. *Let $h \in B_q$ and let m be a positive integer, $m \geqq n + \alpha$. We assume that every weak partial derivative of h of order m is a measure μ, except possibly at $x = 0$, and that these measures satisfy*

$$\int_{|x| \geqq 1/\rho} |x|^\alpha \, |d\mu(x)| = O(\rho^{m-n-\alpha}),$$

as $\rho \to \infty$. We put for $x \in R^n$

$$g_\nu(x) = h(x/R_\nu),$$

where $R_\nu \to \infty$, as $\nu \to \infty$, and assume that $\{g_\nu\}_1^\infty$ fulfills

$$I_\nu(C) \to 0,$$

as $\nu \to \infty$, uniformly in $t_1, t_m, \cdots, t_M$, $|t_j| \leqq 1$, for every compact C, using the notations of Definition 1. Then $\{g_\nu\}_1^\infty \in a_q^m$.

Proof of Theorem 3. By the remark in the beginning of this paragraph, it is enough to show that

$$\int_{|x| \geqq m+1} |\Delta_{t_1} \Delta_{t_2} \cdots \Delta_{t_m} g_\nu| \, q \, dx$$

is bounded, as $\nu \to \infty$, uniformly if $|t_j| \leqq 1, j = 1, 2, \cdots, m$. By Lemma 2 it is enough to show that

$$\int_{|x| \geqq 1} |Dg_\nu| \, q \, dx$$

is bounded, as $\nu \to \infty$, for every partial derivative Dg_ν of g_ν, uniformly for every choice of the m directions of the derivations. The uniformity of the last condition causes no problem, since it is a direct consequence of the fact that the derivatives of order m form a finite-dimensional vector space. Hence we have only to verify the boundedness for an arbitrary chosen set of directions. We denote by μ the corresponding *m-th* derivative of h. Then, with $\rho = R_\nu$,

$$\begin{aligned}\int_{|x| \geqq 1} |Dg_\nu| \, q \, dx &= \int_{|x| \geqq 1} (1 + |x|)^\alpha \rho^{n-m} \, |d\mu(x/\rho)| \\ &= O\left(\int_{|x| \geqq 1} |x|^\alpha \rho^{n-m} \, |d\mu(x/\rho)|\right) \\ &= O\left(\rho^{n-m+\alpha} \int_{|x| \geqq 1/\rho} |x|^\alpha \, |d\mu(x)|\right),\end{aligned}$$

and the right hand member is bounded, as $\rho \to \infty$, by the assumption.

THEOREM 4. *Let m be a positive integer and β a real number such that*

$$m \geqq \alpha + n, \quad \beta + 1 \geqq \alpha + n, \quad \beta + 1 > 0.$$

We assume that

$$g_\nu(x) = h(x/R_\nu),$$

for every $x \in R^n$, where $R_\nu \to \infty$, as $\nu \to \infty$. Here

$$h(x) = (1 - |x|^2)^\beta \quad \text{if } |x| \leqq 1,$$
$$= \quad 0 \quad \text{if } |x| > 1.$$

Then $\{g_\nu\}_1^\infty \in a_q^m$.

Proof of Theorem 4. It is immediately seen that $I_\nu(C) \to 0$ for every compact C, as $\nu \to \infty$, uniformly if $|t_j| \leqq 1$, using the notations of Definition 1. Hence it remains to show that

$$\int_{|x| \geqq m+1} |\Delta_{t_1}\Delta_2 \cdots \Delta_{t_m} h(x/\rho)| (1 + |x|)^\alpha \, dx,$$

is bounded, as $\rho \to \infty$, uniformly if $|t_j| \leqq 1$. We split this integral into two parts, integrating over $m + 1 \leqq |x| \leqq \rho - m - 1$ and $\rho - m - 1 \leqq |x| < \infty$, respectively. For the first of these parts we have, by Lemma 2, apart from a constant, the upper bound

$$I_1 = \int_{1 \leqq |x| \leqq \rho - 1} \left| \frac{\partial^m}{\partial t_1 \partial t_2 \cdots \partial t_m} h\left(\frac{x}{\rho}\right) \right| |x|^\alpha \, dx,$$

and for the second, we have apart from a constant, the upper bound

$$I_2 = \int_{\rho - 2m - 1 \leqq |x| \leqq \rho} h(x/\rho) \rho^\alpha \, dx.$$

It is easy to see, that

$$\frac{\partial^m}{\partial t_1 \partial t_2 \cdots \partial t_m} h(x) = O(l|x|),$$

if $|x| < 1$, uniformly if $|t_j| \leqq 1$, where

$$l(t) = \left| \frac{d}{dt} (1 - t)^{\beta - m + 1} \right| + 1.$$

Hence, we have, if $\rho \to \infty$,

$$I_1 = O\left(\int_{1 \leqq |x| \leqq \rho - 1} l(|x|/\rho) \rho^{-m} |x|^\alpha \, dx \right)$$
$$= O\left(\rho^{\alpha + n - m} \int_{1/\rho \leqq |x| \leqq 1 - 1/\rho} l(|x|) |x|^\alpha \, dx \right)$$
$$= O\left(\rho^{\alpha + n - m} \left\{ \int_{1/\rho \leqq |x| \leqq 1} |x|^\alpha \, dx + \int_{|x| \leqq 1 - 1/\rho} l(|x|) \, dx \right\} \right).$$

The first integral in this expression has the upper bound $O(1 + \rho^{-\alpha-n})$, if $\alpha \neq -n$, $O(\log \rho)$ if $\alpha = -n$, hence $O(1 + \rho^{-\alpha-n+1})$ in all cases. The second integral is $O(1 + \rho^{m-\beta-1})$. Hence

$$I_1 = O(\rho^{\alpha+n-m}(1 + \rho^{-\alpha-n+1} + \rho^{m-\beta-1}))$$
$$= O(\rho^{\alpha+n-m} + \rho^{1-m} + \rho^{\alpha+n-\beta-1}) = O(1)$$

by the assumption on m, β and α.

Since furthermore by the assumptions

$$I_2 = \int_{1-(2m+1)/\rho \leqq |x| \leqq 1} h(x)\rho^{\alpha+n}\, dx$$
$$= O\left(\rho^{\alpha+n} \int_{1-(2m+1)/\rho}^{1} (1-t)^{\beta}\, dt\right) = O(\rho^{\alpha+n-\beta-1}) = O(1),$$

the theorem is proved.

THEOREM 5. *Let m be a positive integer and let β be a real number such that*

$$m > \alpha + n, \quad \beta + 1 \geqq \alpha + n, \quad \beta + 1 > 0.$$

We assume that

$$g_\nu(x) = k(x)(h(x/R_\nu') - h(x/R_\nu'')),$$

where R_ν' and $R_\nu'' \to \infty$, as $\nu \to \infty$. Here h is defined in Theorem 4, whereas k has m continuous derivatives, except at $x = 0$, and satisfies

$$k(\lambda x) = k(x),$$

for every $\lambda > 0$. Then $\{g_\nu\}_1^\infty \in a_q^m$.

Proof of Theorem 5. It is easy to see that $I_\nu(C) \to 0$ for every compact C, uniformly if all $|t_j| \leqq 1$, as $\nu \to \infty$. (As in the earlier proofs, we use the notations of Definitions 1 and 2.) Hence it remains to show that

$$\int_{|x| \geqq m+1} | \Delta_{t_1} \Delta_{t_2} \cdots \Delta_{t_m} (k(x)h(x/\rho)) | \, | x |^{\alpha}\, dx$$

is bounded, as $\rho \to \infty$, uniformly if $|t_j| \leqq 1$. We can split this integral into two parts in exactly the same way as we did in the proof of Theorem 4, and, as in this proof, there remains to estimate the two integrals

$$I_1 = \int_{1 \leqq |x| \leqq \rho - 1} \left| \frac{\partial^m}{\partial t_1 \partial t_2 \cdots \partial t_m} (k(x)h(x/\rho)) \right| | x |^{\alpha}\, dx$$

and

$$I_2 = \int_{\rho - 2m - 1 \leqq |x| \leqq \rho} k(x)h(x/\rho)\rho^{\alpha}\, dx.$$

The second integral is treated in exactly the same way as before and we then obtain that it is $O(1)$ as $\rho \to \infty$.

Since all partial derivatives of k of order $\leqq m$ are $O(|x|^{-m})$, we have that, for $0 < |a| < 1$,

$$\frac{\partial^m}{\partial t_1 \partial t_2 \cdots \partial t_m} (k(x)h(x)) = O(|x|^{-m} l(|x|)),$$

where l was defined by (11). Hence

$$\frac{\partial^m}{\partial t_1 \partial t_2 \cdots \partial t_m} (k(x)h(x/\rho)) = \frac{\partial^m}{\partial t_1 \partial t_2 \cdots \partial t_m} (k(x/\rho)h(x/\rho))$$
$$= O(|x|^{-m} l(|x|/\rho)),$$

if $0 < |x| < \rho$ as $\rho \to \infty$, and this holds of course uniformly if $|t_j| \leqq 1$. Hence

$$I_1 = O\left(\rho^{\alpha+n-m} \int_{1/\rho \leqq |x| \leqq 1-1/\rho} l(|x|) \, |x|^{\alpha-m} \, dx\right),$$

and using the corresponding estimates in the proof of Theorem 4, we see that

$$I_1 = O(\rho^{\alpha+n-m}(1 + \rho^{-\alpha+m-n} + \rho^{m-\beta-1}))$$
$$= O(\rho^{\alpha+n-m} + 1 + \rho^{\alpha+n-\beta-1}) = O(1).$$

6. We now turn to the function f in Theorems 1 and 2. We shall give a more explicit sufficient condition which guarantees that $\hat{f}$ in a neighborhood of $y = 0$ has a representation of the form (2).

THEOREM 6. *Let m be a positive integer. We assume that $h \in B^q$ and that the Borel measure μ satisfies*

$$\int_{R^n} (1 + |x|)^{|\alpha|+m} \, |d\mu(x)| < \infty,$$

and has the property that its Fourier transform $\hat{\mu}$ vanishes at $y = 0$ together with its $m - 1$ first derivatives. Then $f = h * \mu$ fulfills the assumptions in Theorem 2.

Proof of Theorem 6. It is obviously enough to show that we have a representation

$$\hat{\mu}(y) = \textstyle\sum_{j=1}^N P_j(y)\hat{\mu}_j(y)$$

in R^n, where P_j are homogeneous polynomials of degree m, and where $\hat{\mu}_j$ are the Fourier-Stieltjes transforms of measures $\hat{\mu}_j \in M_p$, $j = 1, 2, \cdots, N$, where $p(x) = (1 + |x|)^{|\alpha|}$.

We denote by $(x_1, x_2, \cdots, x_n)$ and $(y_1, y_2, \cdots, y_n)$ the coordinates in the definition of the Fourier-Stieltjes transform

$$\mu(y) = \int_{R^n} e^{-ix\cdot y} \, d\mu(x).$$

Let μ_0 be the projection of this measure into the coordinate plane $x_1 = 0$.

Obviously

$$\int_{R^n} (1 + | x |)^{|\alpha|+m} | d\mu_0(x) | < \infty.$$

Furthermore, it is easy to see that there exists a uniquely determined measure ν_1 such that

$$\int_{R^n} (1 + | x |)^{|\alpha|+m-1} | d\nu_1(x) | < \infty$$

and such that if the measures are considered as distributions

$$iy_1 \hat{\nu}_1(y) = \hat{\mu}(y) - \hat{\mu}_0(y) = \hat{\mu}(y) - \hat{\mu}(0, y_2, \cdots, y_n).$$

By a similar projection of μ_0 into $x_2 = 0$ we obtain a representation

$$iy_2 \hat{\nu}_2(y) = \hat{\mu}(0, y_2, \cdots, y_n) - \hat{\mu}(0, 0, y_3, \cdots, y_n)$$

of a similar kind. Proceeding in this way we obtain

$$\begin{aligned}\hat{\mu}(y) = \hat{\mu}(y) - \hat{\mu}(0) &= (\hat{\mu}(y) - \hat{\mu}(0, y_2, \cdots, y_n)) + \cdots \\ &+ (\hat{\mu}(0, \cdots, y_n) - \hat{\mu}(0, 0, \cdots, 0)) \\ &= \textstyle\sum_{k=1}^{n} iy_k \hat{\nu}_k(y),\end{aligned}$$

where $\hat{\nu}_k$ are Fourier-Stieltjes transforms of measures $\hat{\nu}_k$ satisfying

$$\int_{R^n} (1 + | x |)^{|\alpha|+m-1} | d\nu_k(x) | < \infty.$$

It is also easy to see that $\hat{\nu}_k$ vanish together with their $m - 2$ first derivatives at $y = 0$.

The same argument can then be repeated for the measures ν_k, and proceeding in this way we arrive at the statement in the theorem.

7. We say that a locally bounded function f on R^n is Bochner-Riesz summable of order β, $\beta > -1$, to the value a, if

$$\lim_{R\to\infty} \int_{R^n} f(x)(1 - | x |^2/R^2)\, dx = a.$$

The following theorem is a direct corollary of Theorems 2, 4, 5 and 6.

THEOREM 7. *Let $q(x) = (1 + | x |^\alpha)$, where α is real and let m be a positive integer. We assume that f fulfils the conditions of Theorem 6.*

Then f is Bochner-Riesz summable of order β to 0 *if*

$$m \geqq \alpha + n, \quad \beta + 1 \geqq \alpha + n, \quad \beta > -1.$$

Let k fulfil the conditions of Theorem 5. *Then fk is Bochner-Riesz summable of order β if*

$$m > \alpha + n, \quad \beta + 1 \geqq \alpha + n, \quad \beta > -1.$$

This theorem is very convenient to use in order to compare our results with the results in [1], [2], [4], [5], and [6]. Such a comparison reveals that our general methods give, for the case of Bochner-Riesz summability, results which are as refined as the results in the papers cited. The only essential property of the Bochner-Riesz kernel which is relevant in this problem is thus the degree of differentiability which enters in a significant way in the proof of our Theorems 4 and 5.

To conclude we wish to point out that our results can, of course, be formulated as theorems on the summability of the Fourier integral of f at the point $y = 0$. Of course, it is quite inessential that this point is chosen, to extend the results to other points, we have only to multiply f with the corresponding character. Theorems on equisummability can then be obtained exactly as in the papers cited.

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