HOMOTOPY NORMAL BUNDLES FOR LOCALLY FLAT IMMERSIONS AND EMBEDDINGS OF TOPOLOGICAL MANIFOLDS

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1. Introduction

Let M denote a topological n-manifold and N a topological $(n + k)$-manifold. For locally flat embeddings or immersions, $f : M \to N$, Fadell [2], [3] has constructed normal fiber spaces which satisfy a weak Whiteny duality theorem. Each of these normal fiber spaces is an appropriate path space similar to that described by Nash [12]. In Fadell's construction, the case of immersions is treated separately from that of embeddings and is, in fact, much more complicated. Indeed, in both cases the fibers are rather messy topological spaces.

Our objective is to exploit the theory of microbundle pairs and the corresponding version of the Kister-Mazur coring theorem, proven independently by Kuiper and Lashof [7] and the author [9], to obtain a simpler normal structure, the homotopy normal bundle of f. It will be shown that this h-normal bundle satisfies a weak Whitney duality (Theorem 4.5), a composition theorem (Theorem 4.7), and an appropriate "isotopy invariance" theorem (Theorem 4.9). Furthermore, we shall prove that if f has a tubular neighborhood, ν, then ν is fiber homotopy equivalent to the h-normal bundle (Theorem 4.11) and hence tubular neighborhoods are unique up to fiber homotopy equivalence. Finally we shall prove that the h-normal bundle is equivalent, in so far as is possible, to the normal structures previously given in both the smooth and topological categories.

In Section 2 we give a rather general form of the coring theorem which, with the results of Section 3 on Euclidean bundles, allows us, in Section 4, to define the h-normal bundle and verify its properties.

2. The coring theorem

The Kister-Mazur theorem [6] has been generalized by Kuiper and Lashof [7] and, in the topological case, by the author [9] as follows:

THEOREM 2.1. *In the topological and in the PL-category every* (R^{n+k}, R^n)-*microbundle,* (a, b), *over a locally finite simplicial complex contains a unique* (R^{n+k}, R^n)-*bundle,* (α, β), *in the sense of Steenrod* [12], *which is* (R^{n+k}, R^n)-*microbundle equivalent to* (a, b).

The coring theorem has been extended further, in [4] and [9], by weakening

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the restrictions on the base space and fiber and by noting that the argument holds for any number of radially invariant subspaces. In particular, consider a pair of topological spaces (X, Y). The open cone pair,

$$C_0(X, Y) = (C_0(X), C_0(Y))$$

is the pair

$$\left(\frac{X \times [0, \infty)}{X \times (0)}, \quad \frac{Y \times [0, \infty)}{Y \times (0)}\right).$$

$C_0(Y)$ is always given the induced topology from $C_0(X)$, which may be given either of two topologies. The first is the identification topology and the second (when X is locally compact) is the topology induced from the open cone $C_0(X^*, X)$, where X^* denotes the one point compactification of X. The latter will be referred to as the strong topology for the open cone.

We make the following definitions.

DEFINITION 2.2 *A $C_0(X, Y)$-microbundle, (a, b) is*

(i) *a pair of maps*

$$B \underset{i}{\rightarrow} (E_X, E_Y) \underset{p}{\rightarrow} B$$

such that $pi = \mathrm{id}_B$

(ii) *the open cone $C_0(X, Y)$, with cone point, $*$, as base point, and*

(iii) *a "local trivialization," $\{V_\alpha, g_\alpha\}_{\alpha \in A}$ such that V_α is an open subset of E_X,*

$$\bigcup_{\alpha \in A} p(V_\alpha) = B,$$

and g_α is a homeomorphism such that

$$\begin{array}{ccccc} & & V_\alpha \cap (E_X, E_Y) & & \\ & {\scriptstyle i|_{p(V_\alpha)}} \nearrow & & \searrow {\scriptstyle p|_{V_\alpha}} & \\ p(V_\alpha) & & \uparrow {\scriptstyle g_\alpha} & & p(V_\alpha) \\ & {\scriptstyle \sigma_*} \searrow & & \nearrow {\scriptstyle \pi_1} & \\ & & p(V_\alpha) \times C_0(X, Y) & & \end{array}$$

commutes, where $\sigma_*(x) = (x, *)$.

DEFINITION 2.3. Two $C_0(X, Y)$-microbundles, (a, b) and (a', b'), over B are said to be $C_0(X, Y)$-microbundle equivalent if there exist neighborhoods, U and U' of $i(B)$ and $i'(B)$, respectively, having $C_0(X, Y)$-microbundle structures, and a homeomorphism h, such that the following diagram commutes.

$$\begin{array}{ccc} h:(U, U \cap E_Y) & \rightarrow & (U', U' \cap E_Y') \\ p \downarrow \uparrow i & & p' \downarrow \uparrow i' \\ \mathrm{id}:B & \longrightarrow & B \end{array}$$

For convenience the term "microbundle" will signify either an equivalence class of microbundles or a particular representative of such a class. As in the theory of bundles, the basic properties respect equivalence classes and hence allow this convention. For example, a microbundle is said to be numerable if the cover of B, $\{p(V_\alpha)\}_{\alpha\in A}$, induced by the "local trivialization" is numerable in the sense of Dold [1], i.e. it admits a refinement by a locally finite partition of unity. We note that every microbundle over a paracompact space is numerable [1].

Employing the Ehresmann-Feldbau definition of a fiber bundle [13, p. 18], it is possible to prove the following theorem [9, Thm. 1.0.4].

THEOREM 2.4. *If either*

(a) *X is a locally compact Hausdorff space and $C_0(X, Y)$ is given the strong topology, or*

(b) *X and B are countably paracompact T_4 spaces and $C_0(X, Y)$ is given the identification topology,*

then every numerable $C_0(X, Y)$-microbundle, (a, b), contains a unique numerable $C_0(X, Y)$-bundle, (α, β), which is $C_0(X, Y)$-microbundle equivalent to (a, b).

Remark. If, in addition to condition (a), X were locally connected, the theorem would indicate the existence of a unique numerable Steenrod bundle with (topological) group consisting of the homeomorphisms of $C_0(X, Y)$ leaving the cone point fixed. Moreover, if X is a compact, locally connected, Hausdorff space this conclusion would also hold for the identification topology, since it is equivalent to the strong topology. As an example of this, consider the pair (S^{n+k-1}, S^{n-1}). Since $C_0(S^{n+k-1}, S^{n-1})$ is homeomorphic to (R^{n+k}, R^n) and S^{n+k-1} is a compact, locally connected, Hausdorff space, Theorem 2.4 extends Theorem 2.1 in the topological category to give the existence of a unique numerable Euclidean bundle.

The proof of Theorem 2.4 [9] consists of an extension of Kister's expansion theorem [6, Thm. 1] to open cone pairs and a strengthening of his existence and uniqueness theorems using the theory of numerable bundles as developed by Dold [1]. There are, of course, numerous technical difficulties to be overcome in extending Kister's theorems. However, since the arguments are quite "standard" there does not seem to be sufficient justification for the inclusion of the, rather lengthy, proof.

3. Euclidean bundles

If α is any cone bundle with canonical section, i_α, let α' denote the bundle obtained by removing the image of the section, i.e. if $X\backslash Y$ denotes the complement of Y in X, $E_{\alpha'} = E_\alpha\backslash i_\alpha(B)$. Two cone bundles, α and β, over B are said to be fiber homotopy equivalent if there exist fiber maps

$$f : (E_\alpha, E_{\alpha'}, i_\alpha(\beta)) \leftrightarrow (E_\beta, E_{\beta'}, i_\beta(\mathrm{B})) : g$$

such that fg and gf are homotopic to the appropriate identity maps, through homotopies of the triples which respect the fiber structure.

If (α, β) is a $C_0(X, Y)$-bundle there is an associated $C_0(X\backslash Y)$-bundle whose total space is given by $(E_X\backslash E_Y) \cup i_\alpha(B)$. In particular, associated to every $C_0(S^{n+k-1}, S^{n-1})$-bundle, (α, β), there is a $C_0(S^{n+k-1}\backslash S^{n-1})$-bundle, $\nu_{(\alpha,\beta)}$, the homotopy normal bundle of β in α. Note that $S^{n+k-1}\backslash S^{n-1}$ has the homotopy type of S^{k-1} so that the fibers of $\nu_{(\alpha,\beta)}$ are of the correct homotopy type.

With these definitions we can prove the following weak Whitney sum theorem, compare Milnor [10, Lemma 2.1.5].

THEOREM 3.1. *Suppose that (α, β) is a numerable Euclidean bundle. Then α is fiber homotopy equivalent to $\beta \oplus \nu_{(\alpha,\beta)}$.*

The proof of this theorem is an application of the following lemmas and an extension of a theorem of Dold [1].

Let $\rho'_t : R^{n+k} \to R^{n+k}$ be given by

$$\rho'_t(x, y) = (x,(1 - t)y),$$

where $x = (x_1, \cdots, x_n)$ and $y = (x_{n+1}, \cdots, x_{n+k})$. We may assume, without loss of generality, that B is arcwise connected for if it is not the lemma may be proven over each arc component separately. Fix a point, $b \in B$, and choose a local coordinate patch containing it.

LEMMA 3.2. *Suppose that (α, β) is a numerable Euclidean bundle over B. There is a (fiberwise) strong deformation retraction ρ_t, of E_α onto E_β, which is a fibered homeomorphism for $0 \leqq t < 1$ and which agrees with ρ'_t over b, with respect to the chosen coordinates.*

The proof of this lemma is an easy application of the techniques developed by Dold [1], as is the proof of the next lemma.

Let $\eta'_t : R^{n+k} \to R^{n+k}$ be given by

$$\eta'_t(x, y) = (1 - t(1 - s(x, y)))(x, y)$$

where $s(x, y) = \min(\| y \|, 1)$.

LEMMA 3.3 *Suppose that (α, β) is a numerable Euclidean bundle. Then there is a (fiberwise) deformation, η_t, of (α, β) such that*

(i) $\eta_0 = \mathrm{id}$

(ii) $\eta_1(E_\beta) \subset i_\alpha(B)$

(iii) *η_t is a homeomorphism, for $0 \leqq t < 1$ which is the identity on $i_\alpha(B)$.*

(iv) *η_t is a homeomorphism on $E_{\nu(\alpha,\beta)}$ for $0 \leqq t \leqq 1$.*

(v) *η_t agrees with η'_t, over b, with respect to the chosen coordinates.*

The proof of the following theorem, except for the modifications which are necessary to respect the additional structure, is due to Dold [1, Thm. 3.3]. It is, of course, possible to prove more general versions of the theorem, which is stated in its present form specifically for application in this paper.

THEOREM 3.4 (Dold). *Suppose that* (α, β) *and* (α', β') *are numerable* $C_0(X, Y)$*-bundles over* B *and* f *is a map such that*

$$\begin{array}{ccc} f: E_{(\alpha,\beta)} & \rightarrow & E_{(\alpha',\beta')} \\ p \Big\downarrow\Big\uparrow i & & p \Big\downarrow\Big\uparrow i' \\ \mathrm{id}: B & \longrightarrow & B \end{array}$$

commutes and is a fiber homotopy equivalence over each U_λ *of a numerable covering of* B. *Then,* f *is a fiber homotopy equivalence.*

A proof of Theorem 3.1 is provided by defining $f : E_\alpha \rightarrow E_{\beta \oplus \nu(\alpha,\beta)}$ by

$$f(e) = (\rho_1(e), \eta_1(e))$$

and applying Theorem 3.4. The U_λ may be taken to be a common numerable local trivialization cover of B. To see that f is a fiber homotopy equivalence over each U_λ it is sufficient (by arcwise connectivity) to verify that it is a fiber homotopy equivalence over b. This can be proven directly since f is representable over b, with respect to the chosen coordinates, by $(\rho_1'(e), \eta_1'(e))$.

4. Topological manifolds

A topological manifold (with boundary) of dimension n, M^n is a separable metric space such that, for each $x \in M^n$, there is an open set, U, containing x, which is homeomorphic to (either) R^n (or

$$R^n_+ = \{(x_1, \cdots, x_n) \in R^n \mid x_1 \geqq 0\}).$$

We may assume, without loss of generality, that the closure of U is homeomorphic to either D^n or D^n_+.

If M is a topological manifold with boundary, we define the boundary of M, bM, to consist of those points of M which do not have a neighborhood homeomorphic to R^n. We also define 2M to be $M \cup_{\mathrm{b}M} M$ and note that bM and $2M$ are both topological manifolds.

Suppose M^n and N^{n+k} are two topological manifolds with boundary.

DEFINITION 4.1 A one-to-one map $f : M \rightarrow N$ is said to be a (proper) locally flat embedding of M into N if, for every point $x \in M \backslash \mathrm{b}M$, there is an open set $U \subset N$, containing $f(x)$, and a homeomorphism

$$h : (R^{n+k}, R^n) \rightarrow (U, U \cap f(M))$$

and, for every point $x \in \mathrm{b}M$, there is an open set $U \subset N$, containing $f(x)$, and a homeomorphism

$$h : (R^{n+k}_+, R^n_+) \rightarrow (U, U \cap f(M)).$$

A map which is, locally, a locally flat embedding is a locally flat immersion.

We note that if $f : M \to N$ is a locally flat embedding (immersion) of topological manifolds with boundary, then

$$f \mid_{\mathrm{b}M} : \mathrm{b}M \to \mathrm{b}N \quad \text{and} \quad 2f : 2M \to 2N$$

are locally flat embeddings (immersions) of topological manifolds.

Suppose that M and N are topological manifolds and f is a locally flat immersion.

PROPOSITION 4.2. $\Delta : M \to M \times M \to M : \pi_1$ *has the structure of a numerable* R^n*-microbundle,* t_M, *called the tangent microbundle of* M.

PROPOSITION 4.3. $(\mathrm{id}, f) : M \to M \times N \to M : \pi_1$ *has the structure of a numerable* (R^{n+k}, R^n)*-microbundle,* $(f^* t_N, t_M)$, *called the induced microbundle of* f.

A proof of Proposition 4.2 was given by Milnor [11] which employed a special case of the following lemma, which we shall use to prove Proposition 4.3.

LEMMA 4.4. *Let* $\bar{H}(n + k; n)$ *denote the topological group of homeomorphisms of* D^{n+k} *onto itself which are the identity on* S^{n+k-1} *and invariant on* D^n. *There is a map*

$$\gamma : (\mathrm{Int}\, D^n) \times (\mathrm{Int}\, D^n) \to \bar{H}(n + k; n)$$

such that

(i) $\gamma(x, y)(x) = y$

(ii) $\gamma(x, x)(y) = y$.

In the proof of Proposition 4.3 it is helpful to keep in mind the case of a locally flat embedding since, in this case, it is possible to take $(E_{f^* t_N}, E_{t_M})$ to be the pair $(M \times N, M \times f(M))$. In the general case, however, the pair structure is slightly more complicated but follows, as well, from the following argument. For each $x \in M$, consider

$$h_x : (R^{n+k}, R^n) \to (U_{f(x)}, U_{f(x)} \cap f(U_x))$$

given by the local flatness structure of f at x, where $U_{f(x)}(U_x)$ is a neighborhood of $f(x)(x)$ in $N(M)$. The (R^{n+k}, R^n)-microbundle structure is, then, given by

$$g_x : U_x \times (R^{n+k}, R^n) \to M \times N$$

defined by

$$g_x(y, z) = (y, h_x(\gamma_x^{-1}(h_x^{-1}(f(x)), 0)(\gamma(0, \gamma(h_x^{-1}(f(x)), 0), h_x^{-1}(f(y)))(z)))).$$

In the case of locally flat immersions we can take E_{t_M} to be the union of the images of $R^n \times 0$ under the g_x.

Applying Theorem 2.4 to t_M and $(f^* t_N, t_M)$ we define the tangent R^n-bundle, τ_M, and the induced (R^{n+k}, R^n)-bundle, $(f^* \tau_N, \tau_M)$. The homotopy normal bunde of f is then defined by

$$\nu_f = \nu_{(f^* \tau_N, \tau_M)} .$$

If M and N are topological manifolds with boundary and f is a locally flat immersion we define

$$\tau_M \equiv \tau_{2M} \mid_M \quad \text{and} \quad \nu_f \equiv \nu_{2f} \mid_M .$$

As a corollary of Theorem 3.1 we have

THEOREM 4.5. *If $f : M \to N$ is a locally flat immersion $f^*\tau_N$ is a fiber homotopy equivalent to $\tau_M \oplus \nu_f$.*

We note that this weak Whitney duality theorem, together with the structure of the bundles, is sufficient to construct a Stiefel-Whitney characteristic class theory, see [2]. The following results are given to illustrate further the similarity of the homotopy normal bundle and the normal bundle in the smooth category.

PROPOSITION 4.6. *Suppose that $f : M \to N$ is a locally flat immersion; then $f \mid_{bM}$ is a locally flat immersion of bM into bN such that $\nu_{f \mid bM}$ is fiber homotopy equivalent to $\nu_f \mid_{bM}$.*

The proof follows from Definition 4.1 and Theorem 3.4 via the inclusion map.

THEOREM 4.7. *Suppose that $f : M \to N$ and $g : N \to 0$ are locally flat immersions; then gf is a locally flat immersion and ν_{gf} is fiber homotopy equivalent to $\nu_f \oplus f^*\nu_g$.*

It is easy to see that gf is a locally flat immersion. To prove the remainder of the theorem we let η_t denote the deformation defined by applying Lemma 3.3 to $(g^*\tau_0, \tau_N)$ and δ_t, the deformation defined by applying the lemma to $(f^*\tau_N, \tau_M)$. Finally, let ρ_t denote the deformation gotten from applying Lemma 3.2 to $(g^*\tau_0, \tau_N)$. The fiber map

$$h : E_{\nu_{gf}} \to E_{\nu_f} \oplus f^*(\nu_g)$$

is defined by $h(e) = (p(e), \delta_1(\rho_1(e)), \eta_1(e))$. The proof is concluded by applying Theorem 3.4 as before.

In considering the question of invariance of homotopy normal bundles under isotopies one is lead to consider the concept of a concordance between two locally flat immersions. It is a generalization of what one might call a locally flat isotopy of locally flat immersions and for which it is possible to prove a slightly stronger invariance theorem.

DEFINITION 4.8. Two locally flat immersions, f and g, of M into N are said to be concordant if there is a locally flat immersion

$$F : M \times I \to N \times I$$

which extends

$$f \times (0) : M \times (0) \to N \times (0) \quad \text{and} \quad g \times (1) : M \times (1) \to N \times (1).$$

THEOREM 4.9. *If f and g are concordant locally flat immersions then ν_f and ν_g are fiber homotopy equivalent.*

This can be proved by considering the homotopy normal bundle of the concordance, applying the numerable "homotopy" theorem [5, pp. 48–51], and Proposition 4.6.

In considering the equivalence of the homotopy normal bundle to other normal structures it should be noted that Milnor [11] has shown that the tangent microbundle of a smooth manifold is microbundle equivalent to the tangent vector bundle so that the tangent Euclidean bundle is equivalent, as a Euclidean bundle, to the tangent vector bundle. The situation for normal structures is more complicated, making the following definition necessary.

DEFINITION 4.10. Suppose that $f : M^n \to N^{n+k}$ is an embedding (immersion) of topological manifolds. f is said to have a tubular neighborhood if there is an R^k-bundle, $\nu = (E_\nu, p_\nu, M)$ and an embedding (immersion), F such that

(i) the diagram commutes

$$\begin{array}{ccc} F: E_\nu & \longrightarrow & N \\ & {\nwarrow}_{i} \quad {\nearrow}_{f} & \\ & M & \end{array}$$

(ii) there is a cover, $\{U_\alpha\}_{\alpha \in A}$, such that $F \mid_{p_\nu^{-1}(U_\alpha)}$ is a homeomorphism.

THEOREM 4.11. *If $f : M \to N$ has a tubular neighborhood then $(f^* \tau_N, \tau_M)$ is (R^{n+k}, R^n)-bundle equivalent to $(\tau_M \oplus \nu, \tau_M)$.*

A proof can be given following Milnor [11, Thm. 5.9]

COROLLARY 4.12. *Tubular neighborhoods are unique up to fiber homotopy equivalence.*

COROLLARY 4.13. *If f is a smooth embedding or immersion of smooth manifolds the normal vector bundle, ν, is fiber homotopy equivalent to ν_f.*

The first corollary follows from Theorem 4.11 since every tubular neighborhood is fiber homotopy equivalent to ν_f. The second is proven by showing that, following Lang [8, p. 73], smooth embeddings and immersions have tubular neighborhoods.

In the category of topological manifolds and locally flat immersions or embeddings the situation is more complicated since Fadell has given different constructions of the normal fiber space in each case. First, suppose that $f : M \to N$ is a locally flat embedding. Let N_0 denote those paths in N^I, ω, such that $\omega(t) \in f(M)$ if and only if $t = 0$. Let $N = N_0 \cup C$, where C denotes the constant paths at points of $f(M)$, and define $p : N \to M$ by $p(\omega) = f^{-1}(\omega(0))$. This is Fadell's normal fiber space, ν.

THEOREM 4.14. *ν is a fiber homotopy equivalent to ν_f.*

This is proved by applying Theorem 3.4 to the map $g : E_{\nu_f} \to E_\nu$ given by $g(x)(t) = \rho_{1-t}(x)$, where ρ_t is defined in Lemma 3.2.

If f is a locally flat immersion the definition of Fadell's normal fiber space is much more difficult. Thus we shall not give the details of the following weak equivalence theorem.

THEOREM 4.15. *$g : E_{\nu_f} \to E_\nu$ defined as above, has the property that*

$$g_* : H_*(p_{\nu_f}^{-1}(x), p_{\nu_f}^{-1}(x) \setminus f(x)) \to H_*(p_\nu^{-1}(x), p_\nu^{-1}(x) \setminus \tilde{f}(x))$$

is an isomorphism.

Here $p_\nu^{-1}(x)$ is a function space similar to that used for embeddings and $\tilde{f}(x)$ denotes the constant path at $f(x)$. This seems to be the strongest statement that can be made since the homotopy type of Fadell's fiber is not known. The proof of this theorem parallels that of the previous except that the argument showing that g_* is an isomorphism is precisely that given by Fadell [3] to identify the weak homotopy type of the fiber.

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