A PATH SPACE FOR POSITIVE SEMIGROUPS

BY

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1. An existence theorem

Let $(E, \mathcal{E})$ be a locally compact space with countable basis and the σ-field of its Borel sets. We suppose given a real values kernel $m(t, x, A)$, $t \geqq 0$, $x \in E$, $A \in \mathcal{E}$ subject to the following hypotheses.

(a) For each $A \in \mathcal{E}$, m is measurable in (t, x) over the product σ-field $\mathcal{B}^+ \times \mathcal{E}$, where $\mathcal{B}^+$ denotes the Borel sets of $[0, \infty)$.

(b) For each (t, x), $m(t, x, A)$ is a non-negative finite measure on $\mathcal{E}$.

(c) $m(0, x, A) = I_A(x)$ where I_A is the indicator function of A.

(d) $m(t_1 + t_2, x, A) = \int_E m(t_1, x, dy)m(t_2, y, A)$ for all $t_1, t_2 \geqq 0$ and $A \in \mathcal{E}$ (the semigroup property.)

If, in addition, we assume $m(t, x, E) \leqq 1$, then m is "substochastic". There is in this case a great and well-known theory (the theory of Markov processes) of how to define and investigate measure spaces of "paths" determined by $m(t, x, E)$. In this paper we are concerned with showing that a certain part of this theory having to do with finite-valued stopping times can be directly extended to the more general case.

It should be remarked at the outset, however, that the substochastic case is much more inclusive than its definition might indicate. This fact arises from the existence of various methods of reduction of the general case to that case. For example, if for some $\Lambda > 0$ one has $m(t, x, E) < e^{\Lambda t}$, $t \geqq 0$, then all of our results can be easily reduced to known ones for the kernel $e^{-\Lambda t}m(t, x, A)$. More generally, if for some ε and $\Lambda > 0$, $m(t, x, E) < e^{(\Lambda - \varepsilon)t}$, for all $t > \varepsilon$, then under hypotheses necessary for the most part already in the substochastic case one can reduce our problem to that case by means of the substochastic kernels

$$\frac{e^{-\Lambda t}}{h(x)} m(t, x, dy)h(y)$$

where $h(x) = \int_0^\infty e^{-\Lambda s} m(s, x, E)\, ds$. It can be argued as an objection to this transformation that it introduces a distortion of the "process", but this is a matter of opinion. Hence to escape from the "sphere of influence" of the substochastic case, one must be ready at least to give up the exponential boundedness of the measures with t. There are, however, more general transformations which may yet effect the reduction by means of (other)

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multiplicative functionals. The limit of generality of these methods is not known, but it seems likely that beyond a certain point it becomes preferable to develop a direct approach, as is done here, rather than to reduce matters to the substochastic case. A further discussion of this point will be made in connection with the examples at the end of the paper.

To introduce our approach it is perhaps worthwhile to draw a connection with the theory of branching processes. In that theory, there are many representations of kernels $m(t, x, A)$ in the form

$$m(t, x, A) = E^x \sum_{j=1}^{k(t)} I_A(X_j(t)),$$

where $(X_1, \cdots, X_{k(t)})$ are the "branches" of the process at time t and E^x denotes integration over a probability space for the process starting at x (see [12] and [15]). Each of these branches, of which the number $k(t)$ is a random variable, has associated with it a branching time $\eta < \infty$ and termination time ζ, $\eta < \zeta \leqq \infty$, as well as a direct lineal ancestry up to time η and a path of values from time η to ζ. Let me imagine that the branches are now separated to become objects of a new refined space, except that the lineal ancestries are included along with each branch (and thus may have to be repeated, at least in part). There is induced on this space by the original measure a measure in which events along the branches are weighted according to the expected number of branches for which they occur. In this way, one arrives at a sample space whose objects have the form $(\eta, \zeta, w(t))$ where $0 \leqq \eta < \zeta \leqq \infty$ and $w(t) \in E$ for $0 \leqq t < \zeta$. On this sample space, there are induced measures μ^x such that for $0 \leqq t_1 < t_2 < \cdots t_n$ and $A_1, \cdots, A_n \in \mathcal{E}$ one has

$$\begin{aligned} &\int_{A_{n-1}} \cdots \int_{A_1} m(t_1, x, dy_1) m(t_2 - t_1, y_1, dy_2) \cdots \\ (1.1) \qquad &\qquad m(t_{n-1} - t_{n-2}, y_{n-2}, dy_{n-1}) m(t_n - t_{n-1}, y_{n-1}, A_n) \\ &\qquad = \mu^x\{(\eta, \zeta, w) : \eta \leqq t_n < \zeta \text{ and } w(t_1) \in A_1, \cdots, w(t_n) \in A_n\}. \end{aligned}$$

It is important to remark, however, that the measures μ^x on the σ-field generated by the sets having the form on the right of 1.1) are not thereby uniquely determined. Indeed, sets of this form do not form a ring. There are in general many such measures μ^x compatible with (1.1), and the reason for this variety can be expressed in the phrase that a greater rate of branching can be cancelled by a more rapid rate of termination. In other words, there may be more branches and larger total measures μ^x if the branches are also shorter.

We suppose, now, that the branching and termination rates both tend to infinity while (1.1) remains true. In the limit one would have $\eta \equiv \zeta$. Thus we arrive at the idea of a one-parameter family of measures μ_t^x, $t \geqq 0$, for each $x \in E$, such that for $0 \leqq t_1 < \cdots < t_n \leqq t$ and

$$A_1, \cdots, A_{n+1} \in \mathcal{E}, \qquad 1 \leqq n < \infty,$$

$$(1.2) \quad \int_{A_n} \cdots \int_{A_1} m(t_1, x, dy_1) m(t_2 - t_1, y_1, dy_2) \cdots m(t_n - t_{n-1}, y_{n-1}, dy_n) m(t - t_n, y_n, A_{n+1}) = \mu_t^x\{w(t_1) \in A_1, \cdots, w(t_n) \in A_n, w(t) \in A_{n+1}\}.$$

By strict analogy with (1.1), such measures for different t would be defined on disjoint sample spaces. However, it is clear from (1.2) that they can all be considered as defined on a single space consisting of paths in the ordinary stochastic sense if each μ_t^x is defined only on the σ-field $\mathfrak{F}(t)$ generated by the path up to time t. We thus have the following point of departure for our method.

DEFINITION 1.1. Let $\Omega = \times_{0 \leqq s < \infty} E_s$ and $\mathfrak{F} = \times_{0 \leqq s < \infty} \mathcal{E}_s$ where each $(E, \mathcal{E})_s$ is a replica of $(E, \mathcal{E})$ and "$\times$" denotes the Cartesian product, and let $\mathfrak{F}(t) = (\times_{0 \leqq s \leqq t} \mathcal{E}_s) \times (\times_{t < s} E_s), 0 \leqq t < \infty$.

THEOREM 1.1. *For each* $x \in E$ *and* $t \geqq 0$ *there is a unique measure* μ_t^x *on* $\mathfrak{F}(t)$ *for which* (1.2) *holds.*

Proof. If $0 \leqq t_1 < \cdots < t_n \leqq t$ are fixed and $A_{n+1} = E$, (1.2) is easily extended to define a measure on $\mathcal{E}^n$ considered as a sub-field of $\mathfrak{F}(t)$ associated with $\{t_1, \cdots, t_n\}$. Moreover, these measures are obviously consistent for overlapping sets of coordinates. The existence of the measure μ_t^x thus follows from Kolmogorov's theorem. The uniqueness is evident.

Remark. After completion of an initial version of this paper it was pointed out to me by Professor J. L. Doob that the measures $m^{-1}(t, x, E)\ \mu_t^x$ are actually Markovian on $\mathfrak{F}(t)$ with respect to the inhomogeneous transition function

$$p(s_1, x_1; s_2, dx_2) = \frac{m(s_2 - s_1, x_1, dx_2) m(t - s_2 x_2, E)}{m(t - s_1, x_1, E)}; \quad m(t - s_1, x_1, E) \neq 0$$

$$= 0; \qquad \text{otherwise.}$$

This makes Theorem 1.1 a standard application of Kolmogorov, and it is evidently the "underlying" reason for the existence of the martingales used below.

2. Regular processes and stopping times

We shall develop the regularity properties of the path functions by using the method of martingales. It is necessary to introduce three hypotheses.

Hypothesis 2.1. Let C denote the bounded, continuous functions, and

$C_c \subset C$ the elements with compact support. For $0 \leqq f$, f measurable over $\mathcal{E}$, and $t \geqq 0$, set

$$T_t f(x) = \int m(t, x, dy) f(y).$$

Then for $f = (f \wedge 0) + (f \vee 0) \in C_c$ we have

$$T_t f = T_t(f \vee 0) - T_t(-(f \wedge 0)) \in C,$$

where both terms are finite. Moreover, for $f \in C_c$, $\lim_{t \to 0} T_t f = f$ in the sense of pointwise convergence.

Remark. The last condition rules out the possibility of "branching points" [13] but these would not be basic for our subject.

Hypothesis 2.2. $m(t, x, E)$ is continuous in (t, x) with respect to the product topology, and is nowhere 0.

Remark. The non-vanishing of m is assumed only for notational convenience. Moreover, it may be a consequence of Hypothesis 2.1.

Hypothesis 2.3. Unless E is compact, for each compact K the equation $\lim_{x \to \infty} m(t, x, K)/m(t, x, E) = 0$ holds uniformly in finite time intervals,

Remark. The role of this hypothesis is to prevent exits of the paths from E, which would necessitate extension of the kernels to an enlarged space.

We can now prove the theorems which lead up to regularity of the paths for μ_t^x.

THEOREM 2.1. *For $A \in \mathcal{E}$ and $0 \leqq t_1 \leqq t_2$, $m(t_1 - s, X(s), A)/m(t_2 - s, X(s), E)$ is a $\mu_{t_2}^x$-martingale, $0 \leqq s \leqq t_1$.*

Remark. The definition of martingale, like that of conditional probability, is not effected if the measure $\mu_{t_2}^x$ is normalized to have total measure 1.

Proof. For $0 \leqq s_1 < s_2 \leqq t_2$, let $\mu_{t_2}^x\,(X(s_2) \in dy \mid \mathcal{F}(s_1))$ denote the indicated conditional measure. It is not hard to convince oneself that in fact this is a function of $X(s_1)$ only, and that it is given by the expression

$$\frac{m(s_2 - s_1, X(s_1), dy) m(t_2 - s_2, y, E)}{m(t_2 - s_1, X(s_1), E)}.$$

Letting $E_{t_2}^x$ denote expectation (or integral) with respect to $\mu_{t_2}^x$ it follows that for $s_2 \leqq t_1$

$$E_{t_2}^x\left(\frac{m(t_1 - s_2, X(s_2), A)}{m(t_2 - s_2, X(s_2), E)} \,\middle|\, \mathcal{F}(s_1)\right) = \int \frac{m(t_1 - s_2, y, A) m(s_2 - s_1, X(s_1), dy)}{m(t_2 - s_1, X(s_1), E)}$$

$$= \frac{m(t_1 - s_1, X(s_1), A)}{m(t_2 - s_1, X(s_1), E)} \quad \text{a.e. } \mu_{t_2}^x.$$

This completes the proof.

To obtain the regularity of the paths it is useful to introduce separability for the process. Once the necessary properties are established we can discard the irregular paths and define the measures on a regular subset of Ω for which separability is automatic. Thus our use of separability is only in passing. For the moment, however, we assume that if E is not compact it has been compactified by a single point Δ, that for each x and t the fields $\mathfrak{F}(s)$, $s \leqq t$, are completed for μ_t^x whenever $X(s)$ is considered for this measure, and that $X(s)$ is then a standard modification separable for the closed sets (see Meyer [10, p. 57]—we can again assume in applying the definition that μ_t^x is normalized). It is to be noted that since, for $t_1 < t_2$, $u_{t_2}^x$ is absolutely continuous with respect to $\mu_{t_1}^x$ on $\mathfrak{F}(t_1)$ the necessary modifications can be chosen independent of t. Let Q_t^x denote a countable dense separating subset of $[0, t]$ for μ_t^x. We assume that Q_t^x increases with t for each x.

THEOREM 2.2. *For each (t, x), μ_t^x-almost-all paths have at most one right (left) limit value in E at each $s < t$ ($s \leqq t$) along the set Q_t^x.*

The following lemma will be used again later.

LEMMA 2.2. *For $f \in C_c$, $T_t f(x)$ is continuous in (t, x).*

Proof. Let $d(x, y)$ be a metric on E which generates the given topology (here we do not consider $\Delta \in E$), and let $S_\varepsilon(x)$ denote the sphere with center x and radius $\varepsilon > 0$. Suppose, to the contrary, that there is a sequence $(t_j, x_j) \to (t, x)$ such that $| T_{t_j} f(x_j) - T_t f(x) | > \varepsilon > 0$ for all j. We can assume without loss of generality that $f \geqq 0$. Two cases are distinguished.

Case 1. There is a decreasing subsequence of $\{t_j\}$. We can assume in this case that t_j is decreasing. From Hypothesis 2.1 it follows that for $\varepsilon > 0$ and $x \in E$

$$\lim_{t\to 0} m(t, x, S_\varepsilon(x)) = 1 \quad \text{and} \quad \lim_{t\to 0} m(t, x, K - S_\varepsilon(x)) = 0,$$

where K is any fixed compact set. Moreover, these convergences are uniform for x in compact sets for if not a contradiction would ensue at an x-limit point upon replacing ε by $\varepsilon/2$. Let K_f denote the support of f. It follows that

$$\liminf_{j\to\infty} T_{t_j} f(x_j)$$
$$= \liminf_{j\to\infty} \int_{K_f}\int_{K_f} m(t, x_j, dy) m(t_j - t, y, dx) f(x) \geqq T_t f(x).$$

On the other hand, if for $\delta > 0$ one had $\limsup_{j\to\infty} T_{t_j} f(x_j) > T_t f(x) + \delta$, then for any compact K it would follow from the boundedness of $m(t, x_j, E)$, $j = 1, \cdots$, that

$$\limsup_{j\to\infty} \int_{E-K} m(t, x_j, dy) m(t_j - t, y, K_f) \geqq (\delta)/(\max_x f).$$

By choosing K sufficiently large, however, this contradicts Hypothesis 2.3 in view of the boundedness of $m(t, x_j, E)$.

Case 2. There is an increasing subsequence of $\{t_j\}$. This case is treated similarly, using the boundedness of $m(t_j, x_j, E)$.

From this lemma and Hypothesis 2.1 there exists a sequence $\{f_n\}$ of non-negative elements of C_c such that if $x_1 \neq x_2$ there are $\varepsilon > 0$, n, and disjoint neighborhoods $N(x_1)$ and $N(x_2)$ such that for $t < \varepsilon$,

$$\left| \frac{T_t f_n(y_1)}{m(t, y_1, E)} - \frac{T_t f_n(y_2)}{m(t, y_2, E)} \right| > \varepsilon \qquad \text{for} \quad y_1 \,\varepsilon\, N(x_1), y_2 \,\varepsilon\, N(x_2).$$

It is easily seen by Theorem 2.1 that

$$\left\{ \frac{T_{t-s} f_n(X(s))}{m(t - s, X(s), E)}, \qquad 0 \leqq s \leqq t \right\}$$

is a countable family of μ_t^x-martingales, which accordingly have unique right limits along Q_t^x at all $s < t$, μ_t^x-a.e. Let $S(t)$ denote the set of paths where for some j, k, and $n > 0$, the martingale

$$\frac{T_{j2^{-k}t-s} f_n(X(s))}{m(j2^{-k}\, t - s, X(s), E)}$$

does not have right limits along Q_t^x at all $s < j2^{-k}\, t$. By the absolute continuity of $\mu_{t_2}^x$ with respect to $\mu_{t_1}^x$ for $t_1 < t_2$ it follows that $\mu_t^x(S(t)) = 0$. On the other hand, if x_1 and x_2 are distinct right (left) limit points in E along Q_t^x of a path at $s < t$ ($s \leqq t$) then choosing $0 < j2^{-k}\, t - s < \varepsilon$, for the ε and n (depending on x_1 and x_2) chosen above the corresponding martingale obviously could not have a right limit along Q_t^x at s. Hence such a path is in $S(t)$, and the proof is finished by an application of separability.

THEOREM 2.3. *For each (t, x) the process $X(s)$ is right continuous in measure with respect to μ_t^x, $0 \leqq s < t$.*

Proof. For $0 < h < t - s$, the conditional distribution of $X(s + h)$ given $\mathfrak{F}(s)$, or equivalently $X(s)$, is

$$\frac{m(h, X(s), dy) m(t - (s + h), y, E)}{m(t - s, X(s), E)}.$$

Since, for $\varepsilon > 0$, $\lim_{h \to 0} m(h, X(s), S_\varepsilon(X(s))) = 1$, while the ratio of the other two terms approaches 1 in small neighborhoods of $X(s)$ and the total measures are all 1 the result is now clear.

THEOREM 2.4. *For each (t, x), the process $X(s)$ is μ_t^x-a.e. bounded (i.e. contained in some compact set) $0 \leqq s \leqq t$.*

Proof. Let K_n, $n \geqq 1$, be compact sets with $d(K_n, E - K_{n+1}) > 0$ and $K_n \uparrow E$, and let g_n be continuous functions with $0 \leqq g_n \leqq 1$ and

$$\begin{aligned} g_n &= 1; \quad x \in K_n \\ &= 0; \quad x \notin K_{n+1}. \end{aligned}$$

Since the g_n increase pointwise it follows from Lemma 2.2 and Dini's theorem that $\lim_{n\to\infty} T_s g_n(x) = m(s, x, E)$ uniformly for $x \in K$ compact and $0 \leqq s \leqq t$. Hence for n sufficiently large, $T_s\, g_n$ is bounded away from 0 in the same domain. On the other hand, by Hypothesis 2.3,

$$\lim_{x\to\infty} \frac{T_s\, g_n(x)}{m(s, x, E)} = 0 \quad \text{uniformly in} \quad 0 \leqq s \leqq t.$$

Thus we see that if $S(t)$ denotes the set where $X(s)$ is unbounded in $0 \leqq s \leqq t$ then $S(t) = \bigcap_n S_n(t)$ where

$$S_n(t) = \left\{\inf_{0\leqq s\leqq t, s\in Q_t^x} \frac{T_{t-s}\, g_n(X(s))}{m(t - s, X(s), E)} = 0\right\}.$$

Setting

$$M_n(s) = \frac{T_{t-s}\, g_n(X(s))}{m(t - s, X(s), E)},$$

and applying the optional sampling theorem of Doob to the martingale $-M_n(s)$, using finite subsets of Q_t^x which increase to Q_t^x, we deduce that for $\varepsilon > 0$,

$$-\varepsilon\, \mu_t^x \{\sup_{0\leqq s\leqq t, s\in Q_t^x} -M_n(s) > -\varepsilon\} \leqq -\int_{\{\sup -M_n(s) > -\varepsilon\}} g_n(X(t))\, d\mu_t^x.$$

Letting $\varepsilon \to 0$ it follows (see Loeve [9, p. 636]) that $0 = \int_{S_n} g_n(X(t))\, d\mu_t^x$. However, since g_n is bounded away from 0 in compact K for large n, this implies that $\lim_{n\to\infty} \mu_t^x(S_n) = 0$, and hence that $\mu_t^x(S) = 0$. This completes the proof.

THEOREM 2.5. *For each (x, t) the set of paths which are bounded in finite time intervals, right continuous, and have left limits in $0 \leqq s \leqq t$, has μ_t^x-outer-measure $m(t, x, E)$.*

Proof. If we define $Y(s) = \lim_{t\downarrow s, t\in Q_\infty^x} X(t)$ if this limit exists for all $s < t$, and $Y(s) = \bar{x}$ otherwise where $\bar{x}$ is a fixed element of E, then by Theorems 2.2–2.4 it is clear that $Y(s)$ is a μ_t^x-standard modification of $X(s)$ with right continuous paths having left limits except in some μ_t^x-null set. Since any measurable set may be defined using only countably many coordinates, the result is now immediate.

DEFINITION 2.1. Let $\bar{\Omega}$ denote the subset of Ω consisting of the functions bounded in finite time intervals, right continuous, and having left limits. Let $\mathfrak{F}$, $\mathfrak{F}(t)$, and μ_t^x be the traces and restrictions of the corresponding σ-fields and measures of Definition 1.1 to $\bar{\Omega}$. Finally, let $X(s)$ denote the coordinate function for s on $\bar{\Omega}$.

We next turn to stopping times and the strong Markov property. Even though each μ_t^x concerns the paths only up to time t, it is not necessary to assume for this purpose that the stopping times are uniformly bounded.

The key to this difficulty consists in introducing measures μ_T^x corresponding to stopping times T in the same way that μ_t^x corresponds to t. However, since we have no measure corresponding to $t = \infty$ it seems necessary (unlike in the stochastic case) either to assume that the stopping times are finite valued, or else to restrict all assertions to the set where they are finite. The former approach will be followed in the present section.

DEFINITION 2.2. A non-negative measurable function $T < \infty$ on $(\bar{\Omega}, \mathfrak{F})$ is a "stopping time" if $\{T < t\} \in \mathfrak{F}(t)$ for all $t \geqq 0$. The σ-field $\mathfrak{F}(t)$ of the "past up to time $T+$" consists of the sets $S \in \mathfrak{F}$ for which $S \cap \{T < t\} \in \mathfrak{F}(T)$ for all $t \geqq 0$.

We can now introduce the measures μ_T^x, although the definition requires a subsequent justification.

DEFINITION 2.3. For each stopping time T we define μ_T^x on $\mathfrak{F}(T)$ as the unique measure for which the Radon-Nikodym derivative

$$d\mu_t^x/d\mu_T^x = m(t - T, X(T), E)$$

a.e.-μ_t^x on the set $\{T < t\}$ for all t.

Remark. It is not asserted, however, that $\mu_T^x(\bar{\Omega}) < \infty$.

To justify this definition we must establish both the existence of such a measure on $\{T < t\}$ for each t and the consistency of these definitions as t varies. The first point is easily seen because $m(t, x, E)$ is measurable in (t, x), hence we see that on $\{T < t\}$, $m^{-1}(t - T, X(T), E)$ is $\mathfrak{F}(T)$-measurable and we define for $S \in \mathfrak{F}(T)$

$$\mu_T^x(S \cap \{T < t\}) = \int_{S \cap \{T<t\}} (m(t - T, X(T), E))^{-1} \, du_t^x.$$

The consistency of this definition as t varies is not as easily obtained. To see what is required we note that for $t_1 < t_2$ the identity

$$d\mu_{t_2}^x/d\mu_{t_1}^x = m(t_2 - t_1, X(t_1), E)$$

on $\mathfrak{F}(t_1)$ is an obvious consequence of Theorem 1.1. Thus the consistency follows if we show that

$$\int_{S \cap \{T < t_1\}} (m(t_1 - T, X(T), E) m(t_2 - t_1, X(t_1), E))^{-1} \, d\mu_{t_2}^x$$
$$= \int_{S \cap \{T < t_1\}} (m(t_2 - T, X(T), E))^{-1} \, d\mu_{t_2}^x.$$

For this to hold, it is in turn sufficient that the following identity of conditional expectations be valid:

$$E_{t_2}^x((m(t_2 - t_1, X(t_1), E))^{-1} \mid \mathfrak{F}(T))$$
$$= \frac{m(t_1 - T, X(T), E)}{m(t_2 - T, X(T), E)} \quad \text{a.e.} \quad -\mu_{t_2}^x \quad \text{over} \quad \{T < t_1\}.$$

But this is a consequence of the optional sampling theorem of Doob, in view of Theorem 2.1, the right-continuity of $X(t)$, and the continuity in (t, x) of $m(t, x, E)$.

We are now ready to state and establish our strong Markov property. A yet more comprehensive form will be required in the next section.

DEFINITION 2.4. Let T be a stopping time and let $X_T(t)$, $t \geqq 0$ denote $X(T + t)$. Let $\mathfrak{F}_T(t)$ be the σ-field generated by $\mathfrak{F}(T)$ together with the sets $\{X_T(s) \in A\}$, $0 \leqq s \leqq t$, $A \in \mathcal{E}$. Then (see below) $\mathfrak{F}_T(t) \subset \mathfrak{F}$ and $X_T(t)$ is measurable over $\mathfrak{F}_T(t)$. An X_T-stopping time T_0 is a non-negative function on $\bar{\Omega}$ such that $\{T_0 \leqq t\} \in \mathfrak{F}_T(t)$ for all t, and the field $\mathfrak{F}_T(T_0)$ of the "past of X_T up to time T_0" is the class of all $S \in \mathfrak{F}$ such that

$$S \cap \{T_0 \leqq t\} \in \mathfrak{F}_T(t) \quad \text{for all } t.$$

We call the process $X(t)$ a "strong Markov process" if for each X_T-stopping time T_0 which is *measurable over* $\mathfrak{F}(T) = \mathfrak{F}_T(0)$ we have for each $S \in \mathfrak{F}(T)$

$$\mu^x_{T+T_0}(\{X(T + T_0) \in A\} \cap S) = \int_S m(T_0, X(T), A)\, d\mu^x_T. \tag{2.1}$$

Remark. Suppose (as will be shown) that $X(t)$ is a strong Markov process. If T is a stopping time and t is fixed, then $T \wedge t$ is a stopping time. Moreover $T_0 = t - (T \wedge t)$ is an $X_{T\wedge t}$-stopping time measurable over $\mathfrak{F}(T \wedge t)$. Since $T + T_0 = t$ we have for $S \in \mathfrak{F}(T \wedge t)$

$$\mu^x_t(\{X(t) \in A\} \cap S) = \int_S m(t - (T \wedge t), X(T \wedge t), A)\, d\mu^x_{T\wedge t}$$

$$= \int_{S\cap\{T<t\}} m(t - T, X(T), A)\, d\mu^x_T + \int_{S\cap\{T\geqq t\}} I_A(X(t))\, d\mu^x_t$$

where I_A is the indicator function of A, and we used the evident fact that on the set where two stopping times are equal the corresponding measures coincide. This identity may shed some light on the role of the measures μ^x_T and the strong Markov property.

THEOREM 2.6. *Under Hypotheses* 2.1–2.3, *$X(t)$ is a strong Markov process.*

Proof. We introduce the X-stopping times

$$T_n = (k + 1)2^{-n} \quad \text{on} \quad \{k2^{-n} \leqq T < (k + 1)2^{-n}\}, \qquad 0 \leqq k, n > 0,$$

and also the X_T-stopping times

$$T_{0,m} = (k + 1)2^{-m} \quad \text{on} \quad \{k2^{-m} \leqq T_0 < (k + 1)2^{-m}\},$$

which are measurable over $\mathfrak{F}(T)$. Since clearly $X(T_n + t)$ is measurable over $\mathfrak{F}$ and $\lim_{n\to\infty} X(T_n + t) = X(T + t)$ it follows that $X(T + t)$ is measurable over $\mathfrak{F}$ as asserted in Definition 2.4. Also, it is clear that $m(T_0, X(T), A)$, $A \in \mathcal{E}$, is measurable over $\mathfrak{F}(T)$. To prove the theorem it suffices to show that

$$(2.2)\qquad E^x_{T+T_0}(f(X(T+T_0));S) = \int_S\int_E m(T_0, X(T), dy)f(y)\,d\mu^x_T$$

for each $S \in \mathfrak{F}(T)$ and non-negative function $f \in C_c$, where both sides of (2.2) may equal $+\infty$. We shall first prove a form of (2.2) with S replaced by $S \cap \{T_N + T_{0,M} < t\}$ where t, N, and M are fixed, and T replaced by T_n and T_0 by $T_{0,m}$ for $n \geqq N$ and $m \geqq M$, namely

$$(2.3)\qquad \begin{aligned} &E^x_t[f(X(T_n + T_{0,m}))m^{-1}(t - T_n - T_{0,m}, X(T_n + T_{0,m}), E);\\ &\qquad S \cap \{T_N + T_0,M < t\}]\\ &\quad = \int_{S\cap\{T_N+T_{0,M}<t\}}\int_E m(T_{0,m}, X(T_n), dy)f(y)\,d\mu^x_{T_n} \end{aligned}$$

which becomes (2.2) with the stated replacements if we apply Definition 2.3 using the fixed t. To prove (2.3) let us consider the set where $T_n = k2^{-n}$ and $T_{0,m} = k_0 2^{-m}$ for fixed k and k_0. Evidently on this set T_N and $T_{0,M}$ likewise have prescribed values, hence $T_N + T_{0,M}$ is a constant which we take without loss of purpose to be less than t. Denoting this set by $S(k, k_0)$, replacing S by $S \cap S(k, k_0)$, and setting $t_{k,k_0} = k2^{-n} + k_0 2^{-m}$, (2.3) may be written in the form

$$(2.4)\qquad \begin{aligned} &E^x_t[f(X(t_{k,k_0}))m^{-1}(t - t_{k,k_0}, X(t_{k,k_0}), E); S \cap S(k, k_0)]\\ &\quad = \int_{S\cap S(k,k_0)}\int_E m(k_0 2^{-m}, X(k\,2^{-n}), dy)f(y)\,d\mu^x_{k2^{-n}}. \end{aligned}$$

If on the right we multiply and divide the integrand by the factor $m(t - t_{k,k_0}, y, E)$ and then use the facts that $S \cap S(k, k_0) \in \mathfrak{F}(T) \subset \mathfrak{F}(T_n)$ while $\mu^x_{T_n}$ and $\mu^x_{k2^{-n}}$ coincide on $S \cap S(k, k_0)$, then it becomes obvious that (2.4) holds. We now sum this equation over k and k_0 for which $T_N - T_{0,M} < t$ to obtain (2.3). The next step is to let first $n \to \infty$ and then $m \to \infty$. Still confining attention to the set $\{T_N + T_{0,M} < t\}$ we show that

$$\lim_{n\to\infty} d\mu^x_{T_n} = d\mu^x_T$$

in the sense of pointwise convergence of the Radon-Nikodym derivative of these measures to 1 over this set. In fact, it follows from the consistency of Definition 2.3 and the fact that $\{T_n = k2^{-n}\} \in \mathfrak{F}(T)$ that

$$d\mu^x_{T_n}/d\mu^x_T = m(T_n - T, X(T), E), \quad \mu^x_T\text{-a.e.}$$

Letting $K_T = \{X(T) \in K\}$ for compact K, we have $K_T \in \mathfrak{F}(T)$, and

$$\lim_{n\to\infty} m(T_n - T, X(T), E) = 1$$

in the sense of bounded pointwise convergence on K_T. We can now let $n \to \infty$ and then $m \to \infty$ on the right of (2.3), and apply Fatou's lemma on the one hand, and restriction to K_T followed by an application of bounded convergence (note that $\int_E m(T_{0,m}, X(T_n), dy)f(y)$ is bounded for fixed m over the given set) and then monotone convergence as $K \uparrow E$ on the other, to obtain the limit

$$\int_{S\cap\{T_N+T_{0,M}<t\}} \int_E m(T_0, X(T), dy)f(y)\, d\mu_T^x .$$

As for the left side of (2.3) as $n \to \infty$ and $m \to \infty$, we can proceed in the same way using the expression

$$d\mu^x_{T_n+T_{0,m}}/d\mu^x_{T+T_0} = m(T_n + T_{0,m} - (T + T_0), X(T + T_0), E),$$

and introducing the sets K_{T+T_0} over which the measures $d\mu^x_{T+T_0}$ are finite and the convergence of the above expression to 1 is bounded, to obtain the limit

$$E^x_{T+T_0}[f(X(T + T_0));\, S \cap \{T_N + T_{0,M} < t\}].$$

To complete the proof of the theorem, it now remains only to let $N \to \infty$, $M \to \infty$, and $t \to \infty$.

3. Terminated processes

In the study of Markov processes the concepts of quasi-left-continuity and terminal times are of importance [2]. It seems likely that they can be extended to the present case, but this would take us too far afield. Among the terminal times, however, the times of first reaching a given set are important in various analytical problems, such as solving elliptic and parabolic partial differential equations in a given domain, and will be treated along the same general lines as in [1 Theorem 5.1]. It is necessary to introduce certain extensions and completions of the σ-fields.

DEFINITION 3.1. Let $\mathfrak{F}(s+) = \bigcap_{t>s} \mathfrak{F}(t)$, $0 \leqq s < \infty$, and, noting that Definition 2.3 may be applied (with s in place of T) to define a measure $d\mu_s^x$ on $\mathfrak{F}(s+)$ agreeing with its previous definition on $\mathfrak{F}(s)$, let $\mathfrak{F}^*(s+)$ denote the intersection over $x \in E$ of the completions of $\mathfrak{F}(s+)$ with respect to $d\mu_s^x$. Similarly, let $\mathfrak{F}^*(s)$ denote the analogous completion of $\mathfrak{F}(s)$.

THEOREM 3.1. *$F^*(s+) = \mathfrak{F}^*(s)$ for $0 \leqq s < \infty$.*

Proof. It suffices to show that for each $x \in E$ the two fields $\mathfrak{F}(s)$ and $\mathfrak{F}(s+)$ differ at most by sets of μ_s^x-measure 0. Let $\delta > 0$ be fixed, and let $0 < t_1 < \cdots < t_k \leqq t < t_{k+1} < \cdots < t_n \leqq t + \delta$, and $f_1, \cdots, f_n \in C_c$. Then we have

$$\begin{aligned} E^x_{t+\delta}\,[\textstyle\prod_{j=1}^n f_j(X(t_j)) \mid \mathfrak{F}(t+)] &= \lim_{\varepsilon\to 0+} E^x_{t+\delta}[\textstyle\prod_{j=1}^n f_j(X(t_j)) \mid \mathfrak{F}(t+\varepsilon)] \\ &= \lim_{\varepsilon\to 0+} \textstyle\prod_{j=1}^k f_j(X(t_j)) \\ &\quad \int\cdots\int\Big[m(t_{k+1} - t - \varepsilon, X(t+\varepsilon), dy_1)\cdots \\ &\quad m(t_n - t_{n-1}, y_{n-k-1}, dy_{n-k}) \\ &\quad \cdot\frac{m(t+\delta-t_n, y_{n-k}, E)}{m(\delta-\varepsilon, X(t+\varepsilon), E)} \prod_{j=k+1}^n f_j(y_{j-k})\Big] \\ &= E^x_{t+\delta}\,[\textstyle\prod_{j=1}^n f_j(X(t_j)) \mid \mathfrak{F}(t)] \qquad \mu^x_{t+\delta} \text{ a.e.,} \end{aligned}$$

i.e. the limit is obtained by setting $\varepsilon = 0$, in view of Lemma 2.2. It follows easily from this that for $S \in \mathfrak{F}(t + \delta)$, and hence in particular for $S \in \mathfrak{F}(t+)$, one has

$$\mu_{t+\delta}^x(S \mid \mathfrak{F}(t+)) = \mu_{t+\delta}^x(S \mid \mathfrak{F}(t)), \quad \mu_{t+\delta}^x\text{-a.e.}$$

It is now only routine to see that for $S \in \mathfrak{F}(t+)$,

$$\mu_t^x(S \mid \mathfrak{F}(t+)) = \mu_t^x(S \mid \mathfrak{F}(t)), \quad \mu_t^x\text{-a.e.,}$$

since $d\mu_{t+\delta}^x / d\mu_t^x = m(\delta, X(t), E)$ which is measurable over $\mathfrak{F}(t)$. This implies that $\mu_t^x(S \mid \mathfrak{F}(t)) = 0$ or 1, μ_t^x-a.e., and setting $S_0 = \{\mu_t^x(S \mid \mathfrak{F}(t)) = 1\}$ for a particular choice of the conditional measure we see that S_0 is $\mathfrak{F}(t)$-measurable and $\mu_t^x(S \cap S_0) = \mu_t^x(S)$, $\mu_t^x(S \cap (E - S_0)) = 0$, which completes the proof.

We next introduce the stopped processes.

Definition 3.2. The first passage time T to a set $A \in \mathcal{E}$ is given by $T = \inf t \geqq 0: X(t) \in A; T \leqq \infty$. The process $Y(t)$ derived by stopping $X(t)$ upon reaching A is defined by

$$\begin{aligned} Y(t) &= X(t); \quad t \leqq T \\ &= X(T); \quad t > T. \end{aligned}$$

It is known that T is an $\mathfrak{F}^*(t+)$-stopping time in the sense that $\{T \leqq t\} \in \mathfrak{F}^*(t+)$, $0 \leqq t < \infty$, and hence it is an $\mathfrak{F}^*(t)$-stopping time (see [11, pp. 73 and 104] (on the latter, the necessary completion was not mentioned)). We define the past $\mathfrak{F}^*(T_1)$, for any $\mathfrak{F}^*(t)$-stopping time T_1, as the class of all sets S: $S \cap \{T_1 \leqq t\} \in \mathfrak{F}^*(t)$, $0 \leqq t \leqq \infty$, where we define $\mathfrak{F}^*(\infty)$ as the σ-field generated by $\bigcup_{t \geqq 0} \mathfrak{F}^*(t)$, so that $\{T_1 = \infty\} \in \mathfrak{F}^*(\infty)$. Since $Y(t)$ has right continuous path functions it follows by standard argument that it is measurable over $\mathfrak{F}^*(t) \cap \mathfrak{F}^*(T)$, $0 \leqq t < \infty$. We wish to show that $Y(t)$ is a strong "Markov" process with respect to these σ-fields in the same sense as Definition 2.4. From this point on, we assume that measures μ_t^x and $\mu_{T_1}^x$ for *finite* $\mathfrak{F}^*(t)$-stopping times T_1, are extended in the obvious way to $\mathfrak{F}^*(t)$ and $\mathfrak{F}^*(T_1)$. For such T_1 we shall say that a set S_0 is in the "past of $X(T_1 + s)$ up to time t" if it is in the (joint) σ-field generated by $X(T_1 + s)$, $0 \leqq s \leqq t$. Note that $X(T_1 + s)$ is measurable over $\mathfrak{F}^*(T_1 + s)$. For S_0 in the past of $X(T_1 + s)$ up to time t we define the shift θ_{T_1} of S_0 by

$$\theta_{T_1}(S_0) = \{\theta_{T_1} w: w \in S_0\},$$

where $(\theta_{T_1} w)(t) = w(T_1 + t)$, $0 \leqq t < \infty$. Clearly, $\theta_{T_1}(S_0) \in \mathfrak{F}(t)$.

Theorem 3.2. *Let T_1 be a finite $\mathfrak{F}^*(t)$-stopping time, $S \in \mathfrak{F}^*(T_1)$, and S_0 be in the past of $X(T_1 + s)$ up to time $t < \infty$. Then*

$$\mu_{T_1+t}^x(S_0 \cap S) = E_{T_1}^x(\mu_t^{X(T_1)}(\theta_{T_1}(S_0)); S).$$

Remark. It is evident that $S_0 \cap S \in \mathfrak{F}^*(T_1 + t)$, and that $\mu_t^{X(T_1)}(\theta_{T_1}(S_0))$ is $\mathfrak{F}^*(T_1)$-measurable, so that the last equation is well defined.

Proof. First of all, it is easy to check that the strong Markov property (2.1) remains valid for $S \in \mathfrak{F}^*(T)$, since then $S \in \mathfrak{F}^*(T + T_0)$ and it may be replaced by a set in $\mathfrak{F}(T)$ and in $\mathfrak{F}(T + T_0)$ differing from it by μ_T^x and $\mu_{T+T_0}^x$-measure 0. Next, it is clear that (2.1) also remains valid if

$$\mu_{T+T_0}^x(\{X(T+T_0) \in A\} \cap S)$$

is extended to $E_{T+T_0}^x[f(X(T+T_0)); S]$ for $f \in b^+(\mathcal{E})$ (the non-negative, bounded, $\mathcal{E}$-measurable functions) with an analogous extension of the right side. Now let $0 \leqq s_1 < \cdots < s_n = t$; $A_1, \cdots, A_n \in \mathcal{E}$; and set

$$S_k = \bigcap_{j=1}^k \{X(T_1 + s_j) \in A_j\}, \quad 1 \leqq k \leqq n.$$

Applying the extended form of (2.1) successively with the stopping times $T_1 + s_{n-1}, T_1 + s_{n-2}, \cdots, T_1$ in place of T one obtains

$$\begin{aligned}
\mu_{T_1+t}^x(S_n \cap S) &= \mu_{T_1+t}^x[\{X(T_1 + t) \in A_n\} \cap S_{n-1} \cap S] \\
&= \int_{S_{n-1}\cap S} m(t - s_{n-1}, X(T_1 + s_{n-1}), A_n)\, d\mu_{T_1+s_{n-1}}^x \\
&= \\
&\vdots \\
&= \int_S \int_{A_{n-1}} \cdots \int_{A_1} m(s_1, X(T_1), dy_1) m(s_2 - s_1, y_1, dy_2) \\
&\quad \cdots m(s_{n-1} - s_{n-2}, y_{n-2}, dy_{n-1}) m(t - s_{n-1}, y_{n-1}, A_n)\, d\mu_{T_1}^x .
\end{aligned}$$

Hence the theorem is proved for sets S_0 of the form S_n, and since these generate the past of $X(T_1 + s)$ up to time t the proof is now evident.

We come now to the basic theorem concerning the stopped process $Y(t)$ of Definition 3.2.

THEOREM 3.3. *For each finite $\mathfrak{F}^*(t)$-stopping time T_1 and $S \in \mathfrak{F}^*(T_1) \cap \mathfrak{F}^*(T)$ we have*

$$\mu_{(T_1+t)\wedge T}^y [\{Y(T_1 + t) \in D\} \cap S] = \int_S \mu_{(t\wedge T)}^{Y(T_1\wedge T)} \{Y(t) \in D\}\, d\mu_{T_1\wedge T}^y ;$$

$$y \in E, D \in \mathcal{E}, 0 \leqq t < \infty .$$

Remarks. To see that these expressions are well-defined it is necessary to recall that $Y(t) = Y(t \wedge T)$ for all t. Applied with T_1 constant, this theorem states that $Y(t)$ is "Markovian" with respect to the "transition function" $\mu_{t\wedge T}^y\{Y(t) \in D\}$ in the same sense as $X(t)$ with respect to $m(t, x, D)$. The proof can actually be applied with $T_1 \leqq \infty$ if S is replaced by $S \cap \{T_1 < \infty\}$ and T_1 is assumed $\mathfrak{F}^*(T)$-measurable. The restrictions placed on T_1 are a little less restrictive than their analogues of Theorem 2.6 in that $\mathfrak{F}^*(T)$-measurability of T_1 is not required. On the other hand, the T_0 of Theorem 2.6 has been replaced by constants. This is only for convenience, however, as the general analogue will be easily obtained below as a corollary. Finally, if $\mathcal{E}^*$ denotes the intersection over all finite measures μ on $\mathcal{E}$ of the μ-completions of $\mathcal{E}$, then the "transition function" $\mu^y_{t\wedge T}\{Y(t) \in D\}$ is seen, by a standard reasoning, to be $\mathcal{B}^+ \times \mathcal{E}^*$-measurable in (t, y) for fixed D. In short, $\{Y(t) \in D\}$ is in the completion of $\mathfrak{F}(t+)$ with respect to any σ-finite measure, as in [11, loc. sit], which implies $\mathcal{E}^*$-measurability of $\mu^y_{t\wedge T}\{Y(t) \in D\}$, and joint measurability then follows from the right continuity of the path functions and the continuity of $m(t, x, E)$.

Before proving the theorem, it is necessary to introduce the concept of a "regular point" and to prove two lemmas concerning it.

Definition 3.3. A point $x \in E$ is regular for $A \in \mathcal{E}$ if

$$\mu^x_t\{(\inf s > 0: X(s) \in A) \neq 0\} = 0$$

for some (and hence for all) $t \geqq 0$.

Remark. It follows again as in [11] that

$$\{(\inf s > 0: X(s) \in A) \neq 0\} \in \mathfrak{F}^*(0+) = \mathfrak{F}^*(0)$$

which provides the meaning of the definition.

ϕ or Ω Lemma 3.1. *For each $x \in E$, either x is regular for A or else, for all $t \geqq 0$, $\mu^x_t\{(\inf s > 0: X(s) \in A) = 0\} = 0$.*

Proof. Let $\tilde{S} \in \mathfrak{F}(0)$ differ from $\{(\inf s > 0: X(s) \in A) \neq 0\}$ by a μ^x_0-null set. We apply Theorem 3.2 with $T_1 = 0$ and $S = S_0 = \tilde{S}$. It follows that

$$\mu^x_t(\tilde{S}) = E^x_0[\mu^x_t(\tilde{S}); \tilde{S}] = \mu^x_0(\tilde{S})\mu^x_t(\tilde{S}).$$

Thus either $\mu^x_t(\tilde{S}) = 0$ or else $\mu^x_0(\tilde{S}) = 1$. In the former case, x is regular in view of the absolute continuity of μ^x_t with respect to μ^x_0 on $\mathfrak{F}^*(0)$. In the latter case, since clearly $\mu^x_0(\Omega) = 1$, we have $\mu^x_t(\tilde{S}) = \mu^x_t(\Omega)$ and thus $\mu^x_t(\Omega - \tilde{S}) = 0$, completing the proof.

Let A_r denote $\{x \in E: x \text{ is a regular point of } A\}$. Then $A_r \in \mathcal{E}^*$, as in the remarks to Theorem 3.3, and accordingly expressions such as $\mu^x_t\{X(t) \in A_r\}$ are well defined. To apply the ϕ or Ω lemma in proving the theorem, we require

Lemma 3.2. *For x, T, and T_1 as before,*

$$\{T \leqq T_1\} \quad and \quad \{X(T_1 \wedge T) \in A \cup A_r\}$$

differ by a set of $\mu^x_{(T_1\wedge T)}$-measure 0.

Proof. Let us define $T_{T_1} = \inf t \geqq T_1 : X(t) \in A;\ T_{T_1} \leqq \infty$. Then clearly

$$\begin{aligned}\mu^x_{T_1 \wedge T}\{T_1 < T\} \cap \{X(T_1 \wedge T) \in A \cap A_r\} \\ &\leqq \mu^x_{T_1\wedge T}\{T_{T_1} > 0\} \cap \{X(T_1 \wedge T) \in A \cup A_r\} \\ &= E^x_{T_1\wedge T}[\mu_0^{X(T_1\wedge T)}\{T > 0\}; \{X(T_1 \wedge T) \in A \cup A_r\}] \\ &= 0,\end{aligned}$$

where for the first equality we use Theorem 3.2 after replacing $\{T_{T_1} > 0\}$ by a set S_0 in the "past of $X(T_1 \wedge T + s)$ up to time $s = 0$" which differs from it by at most a set of $\mu^x_{T_1\wedge T}$-measure 0, and such that $\theta_{T_1\wedge T}(S_0)$ differs from $\{T > 0\}$ by at most a set of $\mu_0^{X(T_1\wedge T)}$-measure 0 for $\mu^x_{T_1\wedge T}$-almost-all values of $X(T_1 \wedge T)$. That such a set S_0 exists is clear by considering the measure on the past of $X(T_1 \wedge T) + s$ up to time $0+$ which is generated as the sum of

$$\mu^x_{T_1\wedge T}(\cdot) \quad \text{and} \quad E^x_{T_1\wedge T}\,\mu_0^{X(T_1\wedge T)}\theta_{T_1\wedge T}(\cdot).$$

On the other hand, we have similarly

$$\begin{aligned}0 &= \mu^x_{T_1\wedge T}\{T \leqq T_1\} \cap \{T > T_1\} \\ &\geqq E^x_{T_1\wedge T}[\mu_0^{X(T_1\wedge T)}\{T > 0\}; \{T \leqq T_1\} \cap \{X(T) \notin A \cup A_r\}] \\ &= \mu^x_{T_1\wedge T}\{T \leqq T_1\} \cap \{X(T) \in A \cup A_r\}.\end{aligned}$$

This completes the proof.

We are now in a position to prove the theorem. Separating the two sides of the equality into two terms each, we must show that

$$\begin{aligned}\mu^x_{(T_1+t)\wedge T}\{X((T_1 + t) \wedge T) \in D\} \cap \{T \leqq T_1\} \cap S \\ + \mu^x_{(T_1+t)\wedge T}\{X((T_1 + t) \wedge T) \in D\} \cap \{T > T_1\} \cap S \\ = E^x_{T_1\wedge T}[\mu_{t\wedge T}^{X(T_1\wedge T)}\{X(t \wedge T) \in D\}; \{T \leqq T_1\} \cap S] \\ + E^x_{T_1\wedge T}[\mu_{t\wedge T}^{X(T_1\wedge T)}\{X(t \wedge T) \in D\}; \{T > T_1\} \cap S].\end{aligned}$$

In the left terms on both sides we may replace $\{T \leqq T_1\}$ by

$$\{X(T_1 \wedge T) \in A \cup A_r\},$$

in view of Lemma 3.2 and the absolute continuity of $\mu^x_{(T_1+t)\wedge T}$ with respect to $\mu^x_{T_1\wedge T}$. Both of these terms then reduce to

$$\mu^x_T\{X(T) \in D\} \cap \{X(T_1 \wedge T) \in A \cup A_r\} \cap S.$$

As for the two second terms, we can write $(T_1 + t) \wedge T = T_1 + (t \wedge T_{T_1})$ on $\{T > T_1\}$. Let me suppose for a moment that $t \wedge T_{T_1}$ is actually measurable with respect to the past of $X(T_1 + s)$ up to time t, instead of only with respect to its completion. Then we could apply Theorem 3.2 to obtain

$$\mu^x_{(T_1+t)\wedge T}\{X(T_1 + (t \wedge T_{T_1})) \in D\} \cap \{T > T_1\} \cap S$$
$$= E^x_{T_1+t}[m^{-1}(t - (t \wedge T_{T_1}), X(T_1 + (t \wedge T_{T_1})), E)I_{\{X(T_1+t\wedge T_{T_1})\in D\}} ; \{T > T_1\} \cap S]$$
$$= E^x_{T_1}[E^{X(T_1)}_t m^{-1}(t - t \wedge T, X(t \wedge T), E)I_{\{X(t\wedge T)\in D\}} ; \{T > T_1\} \cap S]$$
$$= E^x_{T_1\wedge T}[\mu^{X(T_1\wedge T)}_{t\wedge T}\{X(t \wedge T) \in D\}; \{T > T_1\} \cap S],$$

where we use the fact that in Theorem 3.2 we may replace $\mu^x_{T_1+t}(S_0 \cap S)$ by $E^x_{T_1+t}[f(w); S]$ if $f(w)$ is non-negative and measurable over the past of $X(T_1 + s)$ up to time t and if we also introduce $f(\theta_{T_1}(w))$ on the right side. To justify this step rigorously, we must replace $t \wedge T_{T_1}$ by another non-negative function which is in the past of $X(T_1 + s)$ up to time t and which agrees with $t \wedge T_{T_1}$ except on sets of the appropriate measure 0, just as in the proof of Lemma 3.2. Specifically, the new function must equal $t \wedge T_{T_1}$, $\mu^x_{T_1+t}$-a.e., and also its shift by T_1 must equal $t \wedge T$, $\mu^{X(T_1)}_t$-a.e. for $\mu^x_{T_1}$-almost-all values of $X(T_1)$. The existence of such a function follows without difficulty, as in Lemma 3.2, and using it in an intermediary role to justify the second equality above, the proof is complete.

COROLLARY 3.1. *The statement of Theorem 3.3 remains valid if we replace t by any non-negative, real function T_0 measurable over $\mathfrak{F}^*(T_1) \cap \mathfrak{F}^*(T)$.*

Proof. It suffices to prove the result with S replaced by $S \cap \{T_0 \leqq t\}$, since the desired statement then follows by letting $t \to \infty$. Thus we can assume that T_0 is bounded by t over S. Next, we remark that since $Y(T_1 + s)$, $0 \leqq s \leqq t$, has right continuous path functions, it is jointly measurable in (s, w) over $\mathfrak{B}^+ \times \mathfrak{F}^Y_{T_1}(t)$ where $\mathfrak{F}^Y_{T_1}(t)$ denotes the past of $Y(T_1 + s)$ up to time t. It follows from this that $S \cap \{Y(T_1 + T_0) \in D\}$ is in the joint σ-field of $\mathfrak{F}^*(T_1) \cap \mathfrak{F}^*(T)$ and $\mathfrak{F}^Y_{T_1}(t)$. Let us introduce in place of this joint σ-field a product σ-field $(\mathfrak{F}^*(T_1) \cap \mathfrak{F}^*(T)) \times \mathfrak{F}^Y_{T_1}(t)$ over $\bar{\Omega} \times \bar{\Omega}$, and measures $\bar{\mu}^y_{(T_1+t)\wedge T}$ on the product which agree with those on the joint σ-field for corresponding sets. More precisely, we proceed as in Theorem 3.2 to extend Theorem 3.3 to intersections $S_0 \cap S$ for $S_0 \in \mathfrak{F}^Y_{T_1}(t)$ and $S \in \mathfrak{F}^*(T_1) \cap \mathfrak{F}^*(T)$. Since $\{Y(T_1 + s) \in D\}$ is in $\mathfrak{F}^*(T_1 + s) \cap \mathfrak{F}^*(T)$, $0 \leqq s \leqq t$, this extension is carried out just as before. We use the same values for $\bar{\mu}^y_{(T_1+t)\wedge T}$ on $S_0 \times S$ as are thus obtained for $S_0 \cap S$. These definitions now determine their extension to the product σ-field uniquely in accordance with the right side of Theorem 3.3, where $\{Y(t) \in D\}$ is replaced by the section in $\mathfrak{F}^Y_{T_1}(t)$ (of a given set in the product σ-field at the given point in the first space $\bar{\Omega}$) shifted by T_1. The mapping of any product-measurable set $\bar{S}$ onto its diagonal $\{w: (w, w) \in \bar{S}\}$ is then measure-preserving from $\bar{\mu}^y_{(T_1+t)\wedge T}$ to $\mu^y_{(T_1+t)\wedge T}$, and it is onto the joint σ-field. Since this method is not new, the details may be left to the reader. It is obvious that the section at w corresponding to $S \cap \{Y(T_1 + T_0) \in D\}$ is simply $\{Y(T_1 + T_0) \in D\}$ or the null set, depending on whether the point w in the first space is in S or not, where $T_1 + T_0$ is a function of this w. The corollary now follows from the definition of $\bar{\mu}^y_{(T_1+t)\wedge T}$, and the fact that the diagonal mapping is measure preserving.

DEFINITION 3.4. Let $T^Y_t f(x) = E^x_{t\wedge T} f(X(t \wedge T))$, for $f \in b^+(\mathcal{E})$.

In other words, T^Y_t is the semigroup of the process $X(t)$ stopped on reaching A, except that no function space is assumed to remain stationary under its application. Since the measures $\mu^x_{t\wedge T}$ need not be finite (as can be seen by considering in R^2 a process of outward radial translation along a sequence of radii chosen from $x = 0$ with different probabilities, and having different translation rates and different variations of mass, each radial mass reaching its maximum upon arrival at $A = \{x_1^2 + x_2^2 \geqq 1\}$ and then decreasing rapidly), this is only to be expected. $T^Y_t f(x)$ can nevertheless be regarded as the solution with the same generating mechanism as $T_t f(x)$ outside of A and the boundary values prescribed by f on A.

The final matters which we wish to consider involve the "harmonic" and "excessive" functions for T^Y_t.

THEOREM 3.4. *Let $0 \leqq f$ be $\mathcal{E}$-measurable, and suppose that either*
(a) $T_t f = f < \infty, 0 \leqq t$ *or*
(b) $T_t f \leqq f \leqq \infty$, $T_t f < \infty$ *for* $t > 0$, *and* $\lim_{t\to 0} T_t f = f$.
Then if (a) *holds* $T^Y_t f = f$, *while if* (b) *holds then* $T^Y_t f \leqq f \leqq \infty$, $T^Y_t f < \infty$ *for* $t > 0$, *and* $\lim_{t\to 0} T^Y_t f = f$.

Remark. It is easy to check that under (a) or (b) $f(X(s))/m(t - s, X(s), E)$ is a μ^x_t-martingale or supermartingale, respectively. It is thus plausible that the theorem should follow by optional stopping of these processes at A. However, such an approach involves knowing that the processes are right continuous in s, and this is not obvious. The proof in [7] involves quasi-left-continuity, and thus is not presently available here. It is therefore necessary to use a direct method, based on the following lemma and its proof.

LEMMA 3.3. *Under* (a) *or* (b) *above,* $E^x_T f(X(T)) \leqq f(x)$ *for all* $x \in E$, *where* E^x_T *is computed over* $\{T < \infty\}$ *as in Definition* 2.3.

Proof. It obviously suffices to prove the lemma for $e^{-\alpha t}T_t$ in place of T_t if α is permitted to be an arbitrary positive number. Next, if there is an increasing sequence f_n of non-negative functions with limit f for each of which the lemma holds, then it also holds for f by Fatou's lemma. We show that f is such a limit with $f_n = \int_0^\infty e^{-\alpha t} T_t g_n \, dt$, $0 \leqq g_n$, and that the lemma holds for these f_n. Indeed, for $\varepsilon_n \downarrow 0$ let

$$g_n = \varepsilon_n^{-1}(f - e^{-\alpha\varepsilon_n}T_{\varepsilon_n} f).$$

Then it is easy to compute that $f_n = \varepsilon_n^{-1} \int_0^{\varepsilon_n} e^{-\alpha t} T_t f \, dt$ which does increase to

f as required (it may be $+\infty$ at certain x under b)). But for f_n we have the decomposition

$$\begin{aligned}
f_n(y) &= \int_0^\infty e^{-\alpha t} T_t\, g_n(x)\, dt \\
&= \int_0^\infty e^{-\alpha t} E_t^x \,[g_n(X(t)) I_{\{t<T\}}]\, dt + \int_0^\infty e^{-\alpha t} E_t^x \,[g_n(X(t)) I_{\{t \geqq T\}}]\, dt \\
&= \int_0^\infty e^{-\alpha t} E_t^x \,[g_n(X(t)) I_{\{t<T\}}]\, dt + \int_0^\infty e^{-\alpha t} E_T^x \,[E_{t-T}^{X(T)} g_n(X(t-T)); T \leqq t]\, dt \\
&= \int_0^\infty e^{-\alpha t} E_t^x \,[g_n(X(t)) I_{\{t<T\}}]\, dt \\
&\qquad + E_T^x \left[\int_T^\infty e^{-\alpha t} E_{t-T}^{X(T)} g_n(X(t-T))\, dt;\, T < \infty\right] \\
&= \int_0^\infty e^{-\alpha t} E_t^x \,[g_n(X(t)) I_{\{t<T\}}]\, dt \\
&\qquad + E_T^x \left[e^{-\alpha T} \int_0^\infty e^{-\alpha t} E_t^{X(T)} g_n(X(t))\, dt;\, T < \infty\right].
\end{aligned}$$

The last term is simply $E_T^x\, f_n(X(T))$ for the semigroup $e^{-\alpha t}\, T_t$, and since the next to last term is non-negative the proof is complete.

Returning to the proof of the theorem, if we use $T \wedge t$ in the lemma in place of T we obtain $T_t^Y f \leqq f$. It remains to prove the reverse inequality under (a), and the convergence to f as $t \to 0$ under (b). For the former, we observe that the next to last term in then bounded by

$$\int_0^t e^{-\alpha s} T_s\, g_n(x)\, ds = \varepsilon_n^{-1}(1 - e^{-\alpha \varepsilon_n}) \int_0^t e^{-\alpha s}\, ds\, f(x) \uparrow (1 - e^{-\alpha t}) f(x) \quad \text{as } n \to \infty.$$

Since $f_n \uparrow f$ it follows that

$$f(x) - T_t^Y f(x) \leqq \lim_{\alpha \to 0} (1 - e^{-\alpha t}) f(x) = 0,$$

as required. As for the later, if $f(x) < \infty$ then it suffices to show that the same term is small for small t, uniformly in α and n: $\varepsilon_n < t$. Since

$$\int_0^t e^{-\alpha s} T_s\, g_n(x)\, ds = \varepsilon_n^{-1} \left(\int_0^{\varepsilon_n} e^{-\alpha s} T_s f\, ds - \int_t^{t+\varepsilon_n} e^{-\alpha s} T_s f\, ds \right)$$

the result follows. If $f(x) = \infty$, however, we must prove that the last term above tends to ∞ as $t \downarrow 0$ and $n \to \infty$, with $T \wedge t$ in place of T. This term exceeds

$$E_t^x \left[e^{-\alpha t} \int_0^\infty e^{-\alpha s} E_s^{X(t)} g_n(X(s));\, t < T \right]$$

which increases to $\int_0^\infty e^{-\alpha s}\, E_s^x\, g_n(X(s))\, ds$ as $t \to 0+$ unless x is a point of

$A \cup A_r$. The last integral is f_n, whence the result in this case. If $x \in A \cup A_r$, then we have f_n immediately, and the proof is complete.

The next, and last, theorem establishes the existence and uniqueness of the solutions of $T_t^Y f = f$ with non-negative boundary values on A.

THEOREM 3.5. (i) *If* $0 \leqq f$ *is* $\mathcal{E}$*-measurable, and if* $T_t^Y f(x)$ *satisfies* (a) *or* (b) *of Theorem* 3.4, *then* $f(x) =$ *or* $\geqq E_T^x[f(X(T)); T < \infty]$ *respectively at any* $x \in E$ *for which* $\lim_{t\to\infty} E_t^x [f(X(t)); t < T] = 0$.

(ii) *Conversely, let* $0 \leqq f$ *be defined and* $\mathcal{E}^*$*-measurable over* $A \cup A_r$. *Then* $f(x) = E_T^x [f(X(T)); T < \infty]$ *is well defined and we have, for all* $x \in E$, $T_t^Y f(x) = f(x)$.

Proof. The proof of (i) is immediate, since

$$f(x) = \text{or} \geqq T_t^Y f(x) = E_T^x[f(X(T)); T \leqq t] + E_t^x [f(X(t)); T > t]$$
$$\to E_T^x[f(X(T)); T < \infty].$$

To prove (ii) we must use Theorem 3.3 when $D \in \mathcal{E}^*$ instead of $D \in \mathcal{E}$, or equivalently, in its integral form with f measurable over $\mathcal{E}^*$ instead of over $\mathcal{E}$. Such a D may be replaced by $\tilde{D} \in \mathcal{E}$ for which $\{Y(T_1 + t) \in D\}$ and $\{Y(T_1 + t) \in \tilde{D}\}$ differ by $\mu^y_{(T_1+t)\wedge T}$-measure 0, and for which $\{Y(t) \in D\}$ and $\{Y(t) \in \tilde{D}\}$ differ by $\mu_{t\wedge T}^{Y(T_1\wedge T)}$-measure 0 for $\mu^y_{T_1\wedge T}$-a.e. value of $Y(T_1 \wedge T)$, justifying the generalization.

Set

$$g(x) = f(x); \quad x \in A \cup A_r.$$
$$= 0; \qquad \text{otherwise}$$

Then for $t_1 < t_2$ we have by Theorem 3.3 and Lemma 3.2,

$$E^y_{t_2\wedge T}\, g(Y(t_2)) = E^x_{t_1\wedge T}\, E^{X(t_1\wedge T)}_{(t_2-t_1)\wedge T}[f(X(T)); T \leqq t_2 - t_1].$$

As $t_2 \to \infty$ the left side increase to $f(x)$, while by the monotone convergence theorem the right side increases to $T_{t_1}^Y f(x)$, Q.E.D.

4. Examples and remarks

The foregoing theorems have indicated that part of the theory of Markov processes can be extended directly to more general positive semigroups. At the same time, they have revealed a certain limitation—namely, they do not provide any way to extend results involving $t \to \infty$. Such problems may be treated in a sense, however, if the kernels can be reduced to stochastic ones by a suitable transformation. Even if the result is only substochastic, as in the case of the h-path transformation when $T_t h < h$, it affords a simpler method than that above. Also, whether or not such an h is available, there may be a Markov process $Z(t)$ and a multiplicative functional $\mathfrak{M}(t)$ such that the "semigroup" may be represented in the form $T_t f(x) = E^x(\mathfrak{M}(t) f(Z(t)))$.

Example 1. Let $Z(t)$ be Brownian motion in R^1, let

$$V(x) = 0; \qquad |x| \leqq 1$$
$$= \log|x|; \quad |x| > 1,$$

and define, for $f \in b^+(R)$,

$$T_t f(x) = E^x\left[\left(\exp\int_0^t V(Z(s))\,ds\right)f(Z(t))\right].$$

We can show that this semigroup is defined from a kernel $m(t, x, A)$ satisfying all of our hypotheses, but which is not exponentially bounded and thus cannot be reduced by the h-path method. In fact, all of the properties are evident [see 14] except two: that $T_t 1(x) < \infty$ for all t, and that for all $\Lambda > 0$, $T_t 1(x) > e^{\Lambda t}$ for t sufficiently large. To show the former, note that by Jensen's inequality

$$E^x \exp\int_0^t V(Z(s))\,ds \leqq E_x(1/t)\int_0^t \exp tV(Z(s))\,ds$$
$$\leqq E^x(1/t)\int_0^t 1 + |Z(s)|^t\,ds < \infty.$$

On the other hand, if we replace $V(x)$ by $V(x) \wedge (\Lambda + \varepsilon)$ for $\varepsilon > 0$, which will not increase $T_t 1(x)$, and then consider $e^{-(\Lambda+\varepsilon)}\, T_t 1(x)$, for which the corresponding V is now $V(x) \wedge (\Lambda + \varepsilon) - (\Lambda + \varepsilon)$, then since this function vanishes for all large x it is easy to see that the corresponding $e^{-(\Lambda+\varepsilon)}T_t\, 1(x)$ will not approach 0 exponentially fast, but only at a rate of the order of

$$1 - \frac{2\delta}{\sqrt{2\pi t}}\int_c^\infty \frac{-x^2}{2t}\,dx = \frac{\delta}{\sqrt{2\pi t}}\int_{-c}^c \exp\frac{-x^2}{2t}\,dx = O\,(2\pi t)^{-1/2},$$

which represents the mass remaining for all $t > \varepsilon' > 0$ in the region where $V \equiv 0$. Thus $T_t\, l(x) > e^{\Lambda t}$ for large t.

It is natural to inquire as to the generality of such representations of semigroups by Markov processes and multiplicative functionals. Let us conclude with three examples of the following variety. In the first, such a representation is evident, but $\mathfrak{M}(t)$ is of unbounded variation. In the second, a representation is again clear, but $\mathfrak{M}(t)$ is a (random) step function. In the third, although it is evident what should be taken as $Z(t)$, it does not seem possible to define $\mathfrak{M}(t)$ to depend only upon the paths of $Z(t)$.

Example 2. Let $T_t f(x) = h(x)f(x + t)h^{-1}(x + t)$ where $x \in R^1$ and $h(x)$ is a fixed, strictly positive, continuous function, which may be of unbounded local variation. The process $Z(t)$ is evidently right translation with unit speed, and $\mathfrak{M}(t) = h(x)h^{-1}(x + t)$ for the path starting at x.

Example 3. Let $m(x, A)$ be a measure on $\mathcal{E}$ for each x, measurable in x for each A, with $m(x, E)$ bounded, positive, and $\int m(x, dy)f(y)$ continuous for bounded continuous f. Consider the semigroup

$$T_t f(x) = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} \int \cdots \int m(x, dy_1) \cdots m(y_{k-1}, dy_k) f(y_k)$$

where $\lambda > 0$ is fixed. The hypotheses are clearly satisfied. An underlying sub-Markovian process may be obtained by replacing each $m(x, A)$ with $m(x, E) > 1$ by $m(x, A)/m(x, E)$. In this case the multiplicative functional would be constant except at the "jumps" of the process, and there it would be multiplied by $m(X(t-), E)$ when $m(X(t-), E) > 1$.

Example 4. Here we describe a situation in which there is an underlying Markov process but the definition of a suitable multiplicative functional to define the "larger" semigroup may pose insuperable difficulties. For a given Markov process $X(t)$ and additive functional $A(t)$, (not assumed positive), let $0 \leqq m(x)$ be a bounded $\mathcal{E}$-measurable function, and let $\rho_1, \rho_2, \cdots$ be a sequence of independent, exponentially distributed random variables independent of $X(t)$. A particle moves along the path of $X(t)$, and at time $T_1 = \inf t > 0: A(t) \geqq \rho_1$ its "mass" is multiplied by $m(X(T_1))$. It then proceeds as before until at time $T_2 = \inf t > T_1 : A(t) - A(T_1) \geqq \rho_2$ its "mass" is multiplied by $m(X(T_2)))$, and it continues in this manner indefinitely. The semigroup is defined by integration with respect to the distribution of total "mass" of the particle, the semigroup property being evident from the description. Evidently there is no intrinsic difficulty in satisfying the hypotheses on $m(t, x, A)$. The attempt to introduce a multiplicative functional to represent the semigroup by means of $X(t)$, however, leads to the expression $\exp \int_0^t (m(X(s)) - 1)\, dA(s)$. Here the integral will not be well defined without further restrictions on $A(t)$, and otherwise there would appear to be no multiplicative functional available to define the semigroup.

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