A CHARACTERIZATION OF THE ALTERNATING GROUP OF DEGREE ELEVEN

BY

TAKESHI KONDO

Introduction

The main purpose of the present paper is to prove the following theorem.

THEOREM B. *Let G be a finite group satisfying the following conditions:*

(1) *G has no normal subgroup of index 2, and*

(2) *G contains an involution z_0 such that $C_G(z_0)$ is isomorphic to the centralizer of an involution in the center of an S_2-subgroup of A_{11}, the alternating group of degree eleven.*

Then G is isomorphic to A_{11}.

Let D be a 2-group of order 2^7 which is isomorphic to a wreath product of a dihedral group of order 8 by a group of order 2. An S_2-subgroup of A_{11} is isomorphic to D. Further, there are infinitely many simple groups with an S_2-subgroup isomorphic to D, namely $LF_4(q)$ ($q \equiv 3 \bmod 8$) and $U_4(q)$ ($q \equiv 5 \bmod 8$). So the present paper contains detailed discussions of a finite group with an S_2-subgroup isomorphic to D, which are more than necessary for the proof of Theorem B (cf. footnote 2)). The results are summarized in Theorem A of §4.

Notation:

$\mathrm{ccl}_X(x)$	a conjugate class in a group X containing x
$\langle \cdots \rangle$	a group generated by ...
X'	the commutator subgroup of a group X
$[x, y]$	$x^{-1}y^{-1}xy$
x^y	$y^{-1}xy$
$x \sim y$ in X	x is conjugate to y in a group X
$J(X)$	the Thompson subgroup of X (cf. [7])
$C_X^*(x)$	$\langle y \in X \mid y^{-1}xy = x^{\pm 1} \rangle$.
A_n (S_n)	the alternating (symmetric) group of degree n

The other notations are standard.

1. Preliminaries

(1.0) Let X be a finite group and S be an S_2-subgroup of X. Let K be a subset of S which is an intersection of S with a conjugate class of X. An element x of K is called an extreme element if $\mid C_S(x) \mid \geqq \mid C_S(y) \mid$ for any $y \in K$. The following is due to R. Brauer [1, p. 308].

(1.1) LEMMA. *If x is an extreme element of K, $C_S(x)$ is an S_2-subgroup of $C_X(x)$. Moreover, for every element y of K, there exists an isomorphism θ*

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of $C_S(y)$ *into* $C_S(x)$ *with the properties:*

(1) $\theta(y) = x$, *and*

(2) *if* $\theta(z) = z'$ *for* $z \in C_S(y)$ *and* $z' \in C_S(x)$, z' *is conjugate to* z *in* X.

(1.2) LEMMA. *Let* M *be a maximal subgroup of* S *and* x *be an involution of* S *outside* M. *If* x *is not conjugate in* X *to any element of* M, X *has a normal subgroup of index* 2.

This is due to J. G. Thompson. The proof is easily obtained by computing the transfer.

In the present paper, Lemmas (1.1) and (1.2) will be frequently used

(1.3) LEMMA.[1] *Let* S *be isomorphic to a* 2-*group generated by involutions* x_1, x_2, y_1, y_2 *and* w *subject to the relations*

$$[x_i, x_j] = [y_i, y_j] = [x_i, y_j] = 1 \qquad (i = 1, 2 \quad j = 1, 2)$$

$$x_1^w = x_2, \qquad y_1^w = y_2 .$$

Then X *has a normal subgroup of index* 2 *with* $\langle x_1, x_2, y_1, y_2\rangle$ *as an* S_2-*subgroup.*

Proof. $C_S(w) = \langle w, x_1 x_2, y_1 y_2\rangle$ is a self-centralizing normal subgroup of order 8. Put $W = C_S(w)$. We know $C_X(W) = W \times U$ and $|U| =$ odd. Suppose that X has no normal subgroup of index 2. Then (1.2) implies that w must fuse to an element of $\langle x_1, x_2, y_1, y_2\rangle = V$. We claim that w must fuse to an element of $Z(S)$. Otherwise, $C_S(w)$ would be conjugate to a subgroup of V by (1.1), which contradicts the fact that $C_X(W) = W \times U$ and $|U| =$ odd. Take an S_2-subgroup S_1 of $C_X(w)$. Then we have $Z(S_1) \subset W$ and $Z(S_1) = \langle w, x_1 x_2\rangle, \langle w, y_1 y_2\rangle$ or $\langle w, x_1 x_2 y_1 y_2\rangle$. If $Z(S_1) = \langle w, x_1 x_2\rangle$, we see that x_1 is contained in $N_X(Z(S_1)) - C_X(Z(S_1))$, which is impossible because $N_X(Z(S_1))/C_X(Z(S_1))$ is of odd order Similarly, we get a contradiction also in other cases. Hence we have proved that $X > O^2(X)$ and w does not fuse to any element of V. Then any element of $S - V$ is not conjugate to an element of V, since an element of $S - V$ is of order 4 or is conjugate to w in S. Then the focal subgroup theorem yields our lemma.

2. Some properties of a 2-group D

Let D be a 2-group generated by involutions b_1, c_1, b_2, c_2 and u satisfying the following relations:

$$b_1^2 = c_1^2 = b_2^2 = c_2^2 = u^2 = 1, \qquad (b_i c_i)^4 = 1 \qquad (i = 1, 2),$$

$$[b_i, b_j] = [c_i, c_j] = [b_i, c_j] = 1 \qquad (i \neq j)$$

$$b_1^u = b_2, \qquad c_1^u = c_2 .$$

[1] This is due to K Harada (cf. [4, Lemma 5]). This lemma will be used only in the proof of Lemma (4.14).

Put $z_i = (b_i c_i)^2$, $a_i = b_i c_i$ $(i = 1, 2)$, $z = z_1 z_2$. Further we define four elementary abelian subgroups of order 16:

$$B = \langle b_1, z_1, b_2, z_2\rangle, \quad C = \langle c_1, z_1, c_2, z_2\rangle, \quad F_1 = \langle b_1, z_1, c_2, z_2\rangle$$

and
$$F_2 = \langle b_2, z_2, c_1, z_1\rangle,$$

D is a wreath product of a dihedral group of order 8 by a cyclic group of order 2. The Thompson subgroup $J(D)$ of D is $\langle b_1, c_1\rangle \times \langle b_2, c_2\rangle$, which is generated by all elementary abelian subgroups of order 16, that is, B, C, F_1 and F_2, and is isomorphic to a direct product of two copies of a dihedral group of order 8.

B, C, F_1 and F_2 are normal in $J(D)$ and selfcentralizing in D. B and C are normal in D. The conjugate classes of involutions of D are as follows:

elements	cardinality
z	1
z_1, z_2	2
$b_1, b_1 z_1, b_2, b_2 z_2$	4
$b_1 z, b_1 z_2, b_2 z_1, b_2 z$	4
$b_1 b_2, b_1 b_2 z_1, b_1 b_2 z_2, b_1 b_2 z$	4
$c_1, c_1 z_1, c_2, c_2 z_2$	4
$c_1 z, c_1 z_2, c_2 z, c_2 z_1$	4
$c_1 c_2, c_1 c_2 z_1, c_1 c_2 z_2, c_1 c_2 z$	4
$b_1 c_2 x$, $b_2 c_1 x$ for any $x \in \langle z_1, z_2\rangle$	8
$uxux^{-1}u$ for any $x \in \langle b_1, c_1\rangle$	8

See (3.7) for the conjugate classes of elements of order 4 which are used only in the proof of lemmas (3.8) and (3.9).

3. General properties of G and D

(3.0) Let G be a finite group with D as an S_2-subgroup. Put $H = C_G(z)$. Throughout the present paper, G, D and H will be used in this meaning.

(3.1) LEMMA. *z is not conjugate to z_1 in G.*
$N_G(J(D)) = D \cdot C_G(J(D))$ *and* $N_G(J(D)) \subseteq N_H(B) \cap N_H(C)$.

Proof. Since $J(D)$ is weakly closed in D with respect to G, any two elements of $Z(J(D)) = \langle z_1, z_2\rangle$ are conjugate in G if and only if they are conjugate in $N_G(J(D))$. On the other hand, from the structure of $J(D)$, the automorphism group of $J(D)$ is 2-group. Hence we have $N_G(J(D)) = D \cdot C_G(J(D))$. From this, our lemma follows.

(3.2) LEMMA. *$z_1 \sim b_1$ in G if any only if $z \sim b_1 z$ in G. Similarly, $z_1 \sim c_1$ in G if and only if $z \sim c_1 z$ in G.*

Proof. Suppose that $z_1 \sim b_1$ in G. Put $W = C_D(b_1) = B\langle c_2\rangle$. Then we have $W' = \langle z_2\rangle$. Denote by D_1 an S_2-subgroup of $C_G(b_1)$ with

$W \subset D_1 \subset C_G(b_1)$. By (3.1), we have $|D_1| = 64$. Furthermore $Z(D_1) \ni z_2$, since W' char W and $[D_1 : W] = 2$ and so $W' \lhd W$. Hence we have $Z(D_1) = \langle b_1, z_2 \rangle$. Since D_1 is conjugate in G to $J(D)$ by (3, 1), $Z(D_1)$ is conjugate to $Z(J(D)) = \langle z_1, z_2 \rangle$. Since $b_1 \sim z_1 \sim z_2$ in G, we have $b_1 z_2 \sim z$ in G by (3.1). Since $b_1 z_2 \sim b_1 z$ in D, we get $b_1 z \sim z$ in G. Suppose that $b_1 z \sim z$ in G. Then we have $W = C_D(b_1 z) = B\langle c_2 \rangle$. Since W is generated by B and F_1, W is contained in $J(D_2)$, where D_2 is an S_2-subgroup of $C_G(b_1 z)$ with $W \subset D_2 \subset C_G(b_1 z)$. Furthermore, $Z(J(D_2)) = \langle b_1 z, z_2 \rangle$. In the same way as above, we get $b_1 \sim z_1$ in G.

(3.3) Lemma. *We may and shall assume $b_1 \not\sim z$ in G and $c_1 \not\sim z$ in G.*

Proof. This follows from (3.1) and (3.2), by interchanging a pair $b_1 b_2$ (resp. $c_1 c_2$) by $b_1 z b_2 z$ (resp. $c_1 z c_2 z$) if necessary.

(3.4) Lemma. *B and C are weakly closed in D with respect to G.*

Proof. Suppose that $B^x \subset D$ for $x \in G$. Then we have $B^x \lhd J(D)$. Since $N_G(B^x) \supset D^x, J(D)$, there exists an element y of $N_G(B^x)$ such that $D^{xy} \supset J(D) \supset B^x$. Then we have $D^{xy} \subset N_G(J(D)) \subset N_H(B)$ by (3.1). Hence there exists an element w of $N_H(B)$ such that $D^{xy} = D^w$. Since

$$xyw^{-1} \in N_G(D) \subset N_G(J(D)) \subset N_H(B),$$

we have $B^{xy} = B^w = B$, and so $B^x = B^{xy} = B$. Thus we have proved that B is weakly closed in D. Similarly, C is weakly closed in D with respect to G.

(3.5) Lemma. *If X is a 2-subgroup of G containing B (resp. C), X normalizes B (resp. C). Furthermore, any two elements of B (resp. C) are conjugate in G if and only if they are conjugate in $N_G(B)$ (resp. $N_G(C)$).*

Proof. This is an immediate consequence of (3.4). This lemma is very useful for the discussions in §4.

(3.6) Lemma. *If z is not conjugate in G to any element of D distinct from z, G has a normal subgroup of index 2.*

Proof. Assume by way of contradiction that $G = O^2(G)$. Put $W = B\langle u, c_1 c_2 \rangle$. Then by (1.2), an involution c_1 of $D - W$ must be conjugate to one of

$$\{z_1, b_1, b_1 z, b_1 b_2, c_1 c_2, u\}$$

which are involutions of W. If $c_1 \sim z_1$ in G, we have $c_1 z \sim z$ because of (3.2). This contradicts the assumption of our lemma. If $c_1 \sim b_1$ in G, by (1.1), $C_D(c_1)$ and $C_D(b_1)$ are S_2-subgroups of $C_G(c_1)$ and $C_G(b_1)$ respectively. Then C is conjugate to B or F_1 in G, since B and F_1 are just two elementary subgroups of order 16 contained in $C_D(b_1) = B\langle c_2 \rangle$ and $C \subset C_D(c_1)$. This is impossible because of (3.4). Similarly we get $c_1 \not\sim b_1 z$ in G. If $c_1 \sim b_1 b_2$ in G, by (1.1), $C_D(c_1)$ and $C_D(b_1 b_2)$ are S_2-subgroups of $C_G(c_1)$ and $C_G(b_1 b_2)$

respectively. Hence $C_D(c_1)$ and $C_D(b_1 b_2)$ are conjugate in G, which is impossible since $C_D(c_1)$ and $C_D(b_1 b_2)$ are not isomorphic. Similarly we get $c_1 \not\sim c_1 c_2$ in G. If $c_1 \sim u$ in G. $C_D(u)$ is conjugate in G to a subgroup of $C_D(c_1)$ by (1 1). Since z_1 (resp. z) is only one square element of $C_D(c_1)$ (resp. $C_D(u)$), we must have $z_1 \sim z$ in G, which is impossible because of (3.1). Thus we get a contradiction.

(3.7) In the proof of Lemmas (3.8) and (3.9), some properties of elements of D with order 4 will be used. The conjugacy classes of D with order 4 are as follows:

representatives	a_1	$a_1 z_2$	$a_1 a_2$	$b_1 a_2$	$c_1 a_2$	uz_1	ub_1	uc_1
cardinality	4	4	4	8	8	8	16	16
squares of representatives	z_1	z_1	z	z_2	z_2	z	$b_1 b_2$	$c_1 c_2$

We have

$$C_D(a_1 a_2) = \langle a_1, a_2, u\rangle \text{ and } C_D^*(a_1 a_2) = \langle b_1 c_2\rangle \cdot C_D(a_1 a_2).$$

There are sixteen involutions which invert $a_1 a_2$, and they are contained in $\langle z_1, a_1 a_2, b_1 b_2\rangle$ or $\langle z_1, a_1 a_2, b_1 b_2\rangle$. We have

$$C_D(ub_1) = \langle z, ub_1\rangle \quad\text{and}\quad C_D^*(ub_1) = \langle b_1\rangle \cdot C_D(ub_1).$$

The set of all involutions which invert ub_1 is $C_D^*(ub_1) - C_D^*(ub_1)$ and so, $C_D^*(ub_1)$ is a "generalized dihedral group" of order 16. Similarly, the set of all involutions which invert uc_1 are $C_D^*(uc_1) - C_D(uc_1)$.

(3.8) LEMMA. *If $u \sim z$ in G, we have $z \sim b_1 b_2$ or $z \sim c_1 c_2$ in G and $b_1 b_2 \not\sim c_1 c_2$ in G.*

Proof. Consider an isomorphism of $C_D(u) = \langle u\rangle \times \langle b_1 b_2, a_1 a_2\rangle$ into $D = C_D(z)$ defined in (1.1). Since $z \not\sim z_1$ in G and $\theta(u) = z = (a_1 a_2)^2$, $\theta(a_1 a_2)$ must be contained in c $\mathrm{cl}_D(ub_1)$ or c $\mathrm{cl}_D(uc_1)$ (cf. the table of (3.7)). Hence we may assume

$$\theta(a_1 a_2) = ub_1 \text{ or } uc_1, \quad \text{an} \quad \theta(C_D(u)) = \langle z\rangle \times \langle b_1, ub_1\rangle \text{ or } \langle z\rangle \times \langle c_1, uc_1\rangle.$$

Hence we get $\theta(z) = \theta((a_1 a_2)^2) = b_1 b_2$ or $c_1 c_2$, namely, $z \sim b_1 b_2$ or $c_1 c_2$. Assume that

$$\theta(C_D(u)) = \langle z\rangle \times \langle b_1, ub_1\rangle,$$

because, also in case where $\theta(C_D(u)) = \langle z\rangle \times \langle c_1, uc_1\rangle$, the argument is similar. Put

$$C_1 = C_D(u) - \langle u, a_1 a_2\rangle \quad\text{and}\quad C_2 = \langle u, ub_1, b_1\rangle - \langle u, ub_1\rangle,$$

which are the sets of all involutions of $C_D(u)$ and $\theta(C_D(u))$ inverting $a_1 a_2$ and ub_1 respectively. θ must map C_1 onto C_2. If $b_1 b_2 \sim c_1 c_2$ in G, we have $z \sim b_1 b_2 \sim c_1 c_2$ in G, and so, any element of C_1 is conjugate to z in G. Since

$C_1 \ni b_1$, this forces to be $b_1 \sim z$ in G, which is impossible because of (3.3). Thus we get $b_1 b_2 \nsim c_1 c_2$ in G.

(3.9) LEMMA. *If $u \sim z_1$ in G, we have $z_1 \sim b_1 b_2 \sim c_1 c_2$ in G.*

Proof. Consider an isomorphism θ from $C_D(u)$ into $C_D(z_1) = J(D)$ defined in (1.1). Then by (3.7), we may assume

$$\theta(a_1 a_2) = a_1 a_2 \quad \text{and} \quad \theta(C_D(u)) = \langle z_1 , a_1 a_2 , b_1 b_2\rangle \text{ or } \langle z_1 , a_1 a_2 , b_1 c_2\rangle.$$

Put $C_1 = C_D(u) - \langle u, a_1 a_2\rangle$ and $C_2 = \theta(C_D(u)) - \langle z_1 , a_1 a_2\rangle$. We have

$$C_1 = \langle b_1 b_2 , b_1 b_2 z, c_1 c_2 , c_1 c_2 z, ub_1 b_2 , ub_1 b_2 z, uc_1 c_2 , uc_1 c_2 z\rangle,$$

and

$$C_2 = \mathrm{ccl}_D(b_1 b_2) \cup \mathrm{c\,cl}_D(c_1 c_2) \text{ or } \mathrm{ccl}_D(b_1 c_2)$$

according to whether $\theta(C_D(u)) = \langle z_1 , a_1 a_2 ,b_1 b_2\rangle$ or $\langle z_1 , a_1 a_2 , b_1 c_2\rangle$. Then the assumption that $u \sim z_1$ in G forces to be $z_1 \sim b_1 b_2 \sim c_1 c_2$ in G, since θ maps C_1 onto C_2. This completes the proof.

4. Conjugacy classes of involutions of G, where G = $O^2(G)$

(4.0) Throughout this section, we shall assume that G has no normal subgroup of index 2. The results of this section can be summarized as follows.

THEOREM A. (i) *G has normal subgroups G_1 and G_2 such that $|G/G_1|$ $|G_2|$ are odd and G_1/G_2 is a non-abelian simple group with an S_2-subgroup isomorphic to D.*

(ii) *G has two classes of involutions. If notation is chosen suitably, the possibilities for the fusion of involutions of G are*

Case I. $z \sim b_1 z \sim c_1 z \sim b_1 c_2 \sim b_1 b_2 \sim u \mid z_1 \sim b_1 \sim c_1 \sim c_1 c_2$, *or*

Case II. $z \sim b_1 z \sim c_1 z \sim b_1 c_2 \mid z_1 \sim b_1 \sim c_1 \sim b_1 b \sim c_1 c_2 \sim u.$

(iii) *$C_G(z)$ has a normal subgroup K of index 4 with the following properties;*

(α) *an S_2-subgroup of K is $\langle z_1 , z_2 , b_1 b_2 , c_1 c_2 , u\rangle$, which is an extraspecial 2-group of order 32.*

(β) *K has no normal subgroup of index 2.*

(iv) *$C_G(z_1)$ has a normal subgroup K_1 of index 8 with the following properties:*

(α) *an S_2-subgroup of K_1 is $\langle b_2 , c_2\rangle$, which is a dihedral group of order 8, and*

(β) *K_1 has no normal subgroup of index 2.*

(v) *$N_G(C)/C_G(C)$ (resp. $N_H(C)/C_H(C)$) is isomorphic to S_5 (resp. S_4). In Case I, $N_G(B)/C_G(B)$ (resp. $N_H(B)/C_H(B)$) is isomorphic to a 3-Sylow-normalizer of A_8 (resp. a dihedral group of order 8), while in Case II, $N_G(B)/C_G(B)$ (resp. $N_H(B)/C_H(B)$) is isomorphic to S_5 (resp. S_4).*

Remark. We know only two examples of finite simple groups with Case I for the fusion of involutions, namely A_{10} or A_{11}, while there exist infinitely many simple groups with Case II, namely $LF_4(q)$ ($q \equiv 3 \bmod 8$) or $U_4(q)$ ($q \equiv 5 \bmod 8$).

(4.1) LEMMA. *z is conjugate in G to an element which is distinct from z and is contained in B or C.*

Proof. Suppose false. Then the assumption $G = O^2(G)$ and (3.6) yields that $z \sim u$ or $b_1 c_2$ in G. If $z \sim u$, we have $z \sim b_1 b_2$ or $c_1 c_2$ in G from (3.8). Hence we may assume that $z \sim b_1 c_2$ in G and z is not conjugate to any element of B or C. Denote by D_1 an S_2-subgroup of $C_G(b_1 c_2)$ with $C_D(b_1 c_2) = F_1 \subset D_1 \subset C_G(b_1 c_2)$. Put $W = \langle J(D), J(D_1)\rangle$. Then we have $W \triangleright F_1$ and $z \sim b_1 c_2$ in W. Hence we get $[N_G(F_1) : H \cap N_G(F_1)] = 5$. If x is an element of $N_G(F_1) - C_G(F_1)$ with $x^5 \in C_G(F_1)$, x acts fixed-point-free on F_1. This forces to be $z_1 \sim b_1 \sim b_1 z_2$ in G, which is impossible because of (3.2).

(4.2) We may and shall assume that z is conjugate to an element in B distinct from z, since B and C play symmetric role.

(4.3) LEMMA. *$z \sim b_1 z$ and $z_1 \sim b_1$ in G. More precisely, there exists an element β in $N_G(B) \cap NG(F_2)$ of odd order such that $\beta^3 \in C_G(B)$, $z_2^\beta = b_2$, $b_2^\beta = b_2 z_2$, $z^\beta = b_2 z_1$, $(b_2 z_1)^\beta = b_2 z$, and $[\beta \;\; z_1] = 1$. We have $[\beta \;\; b_1] = 1$ or $[\beta \;\; b_1 z] = 1$, and $b_1 b_2 \sim z$ or z_1 in G according to whether $b_1^\beta = b_1$ or $(b_1 z)^\beta = b_1 z$. Moreover, we have $b_2 c_1 \sim c_1$ or $c_1 z$ in G.*

Proof. First we shall show that $z \sim b_1 z$ in G. Suppose false. Then, from (3.1), (3.3) and (4.2), we must have $z \sim b_1 b_2$ in G and so in $N_G(B)$ because of (3.5). Hence we get $[N_G(B) : H \cap N_G(B)] = 5$. If x is an element of $N_G(B) - C_G(B)$ with $x^5 \in C_G(B)$, x acts fixed-point-free on B. This forces $b_1 \sim z_1$ in G and so $z \sim b_1 z$ in G because of (3.2). This is a contradiction. Thus we have proved that $z \sim b_1 z$ in G and so $b_1 \sim z_1$ in G because of (3.2). Denote by D_1 an S_2-subgroup of $C_G(b_2 z)$ with $C_D(b_2 z) = B\langle c_1\rangle \subset D_1 \subset C_G(b_2 z)$. Put $U = C_D(b_2 z)$. Then we have $U' = \langle z_1\rangle$ and $Z(U) = \langle b_2, z_1, z_2\rangle$. Since $[J(D_1) : U] = 2$ we have $Z(J(D_1)) = \langle b_2 z, z_1\rangle$ and $J(D_1)$ normalizes $Z(U) = \langle b_2, z_1, z_2\rangle$. Put $W = \langle J(D), J(D_1)\rangle$. Then we have $Z(W) = \langle z_1\rangle$ and W normalizes $B\langle c_1\rangle = U$, $\langle b_2, z_1, z_2\rangle$, B and F_2 by (3.5). (Note that B and F_2 are exactly two elementary subgroups of order 16 contained in $B\langle c_1\rangle$.) Furthermore, z is conjugate to $b_2 z$ in W. $W/C_W(B)$ is not 2-group. Otherwise W would be $W = \langle c_1, c_2\rangle C_W(B)$ because of $Z(W) = \langle z_1\rangle$, against the fact that $z \sim b_2 z$ in W. Hence we can find an element β of W such that $[\beta, B] \neq 1$. Since $C_G(B) = B \times Y$ and $|Y| =$ odd, we may assume that β is of odd order. Then we have $[\beta, \langle b_2, z_1, z_2\rangle] \neq 1$. From $[\beta, z_1] = 1$, $z \sim b_1 z$ and $z_1 \sim b_1$ in G, we obtain $z_2^\beta = b_2$, $b_2^\beta = b_2 z_2$, $z^\beta = b_2 z_1$ and $(b_2 z_1)^\beta = b_2 z$ by interchanging β by β^{-1} if necessary. This implies that β^3 centralizes $\langle b_2, z_1, z_2\rangle$ and so B. Since $[\beta, \;\; z_1] = 1$ and β normalizes B, β must fix an element of $B - \langle b_2, z_1, z_2\rangle$. If β fixes an element of $\mathrm{ccl}_D(b_1 b_2) \subset B - \langle b_2, z_1, z_2\rangle$, we have $(b_1 b_2)^\beta = b_1 b_2$

or $(b_1 b_2 z)^\beta = b_1 b_2 z$. In the former case, $b_1 b_2 = b_1^\beta b_2 z_2$ and so $b_1^\beta = b_1 z_2$ which is impossible since $b_1 \sim z_1$ and $b_1 z_2 \sim z$ in G. In the second case, $b_1 b_2 z = (b_1 b_2 z)^{\beta^2} = b_1^{\beta^2} x_2 b_2 z$ and so $b_1^{\beta^2} = b_1 z_2$, which is impossible. Hence we have $b_1^\beta = b_1$ or $(b_1 z)^\beta = b_1 z$. If $b_1^\beta = b_1$, we have $(b_1 b_2)^{\beta^2} = b_1 z_2 \sim z$ in G. If $(b_1 z)^\beta = b_1 z$ we have $b_1^\beta = b_1 b_2 z$ and so $z_1 \sim b_1 b_2$ in G. Finally we shall show that $b_1 c_2 \sim c_1$ or $c_1 z$ in G. The involutions of $F_2 - \langle b_2, z_2, z_1\rangle$ are $c_1 \sim c_1 z_1$, $c_1 z_2 \sim c_1 z$ and $b_2 c_1 \sim b_2 c_1 z_1 \sim b_2 c_1 z_2 \sim b_2 c_1 z$. If $c_1^\beta = c_1$, we have $(c_1 z_2)^\beta = b_2 c_1$. If $c_1^\beta = c_1 z_1$, $(c_1 z_2)^\beta = b_2 c_1 z$. If $c_1^\beta = c_1 z_2$, $(c_1 z_2)^\beta = b_2 c_1 z_2$. If $c_1^\beta = c_1 z$, $(c_1 z_2)^\beta = b_2 c_1 z$. If $c_1^\beta \neq c_1$, $c_1 z_1$, $c_1 z_2$ and $c_1 z$, we have $c_1^\beta = b_2 c_1$, $b_2 c_1 z_1$, $b_2 c_1 z_2$ or $b_2 c_1 z$ since β normalizes F_2 and $\langle b_2, z_2, z_1\rangle$. Thus, in any case, we get $b_1 c_2 \sim c_1$ or $c_1 z$ in G.

(4.4) LEMMA. *z is conjugate in G to an element which is distinct from z and is contained in C.*

Proof. Suppose false. By Lemma (1.2) of Thompson and the assumption $G = O^2(G)$, c_1 must be conjugate in G to an element of $B\langle u, c_1 c_2\rangle$. Since every element of B fuses to z or z_1 by (4.3) and $c_1 \sim z_1$ in G implies $c_1 z \sim z$ in G by (3.2), we have $c_1 \sim c_1 c_2$ or $c_1 \sim u$ in G. If $c_1 \sim c_1 c_2$ in G, by (1.1), $C_D(c_1)$ and $C_D(c_1 c_2)$ are S_2-subgroups of $C_G(c_1)$ and $C_D(c_1 c_2)$ respectively. Hence $C_D(c_1)$ is conjugate to $C_D(c_1 c_2)$, which is impossible since they are not isomorphic. If $c_1 \sim u$, $C_D(u)$ must be conjugate in G to a subgroup of $C_D(c_1)$ by (1.1). Since z and z_2 are only one square elements of $C_D(u)$ and $C_D(z_2)$ respectively, we must have $z \sim z_2$ in G. This contradicts (3.1). This completes the proof of the lemma.

(4.5) LEMMA. *There exists an element γ in $N_G(C) \cap N_G(F_1)$ of odd order such that $\gamma^3 \in C_G(C)$, $z_2^\gamma = c_2$, $c_2^\gamma = c_2 z_2$, $z^\gamma = c_2 z_1$, $(c_2 z_1)^\gamma = c_2 z$ and $[\gamma \;\; z_1] = 1$. We have $[\gamma \;\; c_1] = 1$ or $[\gamma \;\; c_1 z] = 1$ and $c_1 c_2 \sim z$ or z_1 according to whether $c_1^\gamma = c_1$ or $(c_1 z)^\gamma = c_1 z$.*

Proof. This can be proved in the same way as in (4.3).

(4.6) LEMMA. *G has two classes of involutions If notation is chosen suitably, the possibilities for the fusion of involutions of G are*

Case I. $z \sim b_1 z \sim c_1 z \sim b_1 b_2 \sim u \mid z_1 \sim b_1 \sim c_1 \sim c_1 c_2$

Case II. $z \sim b_1 z \sim c_1 z \mid z_1 \sim b_1 \sim c_1 \sim b_1 b_2 \sim c_1 c_2 \sim u$.

Moreover, we have $b_1^\beta = b_1$ and $(c_1 z)^\gamma = c_1 z$ in Case I, *while we have $(b_1 z)^\beta = b_1 z$ and $(c_1 z)^\gamma = c_1 z$ in Case* II.

(Remark that whether $z \sim b_1 c_2$ or $z_1 \sim b_1 c_2$ has not been determined yet. In (4.9) we shall show that $b_1 c_2 \sim z$ in G in both cases.)

Proof. By (4.3) and (4.5), any involution of $J(D)$ must be conjugate to z or z_1 in G. By (1.2) and the assumption $G = O^2(G)$, u must be conjugate to an element of $J(D)$ and so, we get $u \sim z$ or z_1 in G. Thus G has two classes of involutions.

If $u \sim z$ in G, we have $z \sim b_1 b_2$ or $c_1 c_2$ in G and $b_1 b_2 \not\sim c_1 c_2$ by (3.8). Because of the symmetric role of B and C, we may assume $z \sim b_1 b_2$ in G and so $z_1 \sim c_1 c_2$ in G. Then Case I occurs. If $u \sim z_1$ in G, we have $z_1 \sim b_1 b_2 \sim c_1 c_2$ by (3.9). Then Case II occurs. The third statement follows from (4.3) and (4.5).

(4.7) LEMMA. *$N_H(B)/C_H(B)$ (resp. $N_H(C)/C_H(C)$) operates faithfully on $\mathrm{ccl}_D(b_1)$ (resp. $\mathrm{ccl}_D(c_1)$). Furthermore $N_H(F_1)/C_H(F_1)$ is an elementary subgroup of order* 4 *and $N_G(F_1)/C_G(F_1)$ has normal 2-complement.*

Proof. Assume there exists an element x of $N_H(B)$ such that $b_1^x = z_1$ or $b_1 b_2$. If $b_1^x = z_1$, we have $(b_1 z)^x = z_1 z = z_2$. This is impossible because of (4.6). If $b_1^x = b_1 b_2$, we have $(b_1 z)^x = b_1 b_2 z$, again impossible. Then (4.6) implies the first statement of our lemma.

Similarly we can prove that $N_H(F_1)/C_H(F_1)$ operates faithfully on the set $\{b_1, b_1 z_1, c_2, c_2 z_2\}$. Then $N_H(F_1)/C_H(F_1)$ is isomorphic to the four group or the alternating group of degree four, since an S_2-subgroup of $N_H(F_1)/C_H(F_1)$ is $\langle \bar{c}_1, \bar{b}_1 \rangle$ because of (3.4). On the other hand, c_1 and $b_2 c_1$, regarded as linear transformations on F_1, are not conjugate. This implies that $N_H(F_1)/C_H(F_1)$ must be isomorphic to the four group and $N_G(F_1)/C_G(F_1)$ has normal 2-complement.

(4.8) LEMMA. *If $N_H(B)/C_H(B)$ is isomorphic to a dihedral group of order* 8, *we have Case* I *for the fusion of involutions of G and $N_H(C)/C_H(C) \cong S_4$. If $N_H(B)/C_H(B)$ isomorphic to S_4, we have Case* II *and $N_H(C)/C_H(C) \cong S_4$.*
(Remark that $N_H(B)/C_H(B)$ (or $N_H(C)/C_H(C)$) is isomorphic to a dihedral group of order 8 or S_4 by (4.7).)

Proof. Suppose that $[N_H(B) : C_H(B)] = 8$. If we have Case II, we get $[N_G(B) : C_G(B)] = 8 \cdot 5$ from (4.6) and (3.5). This is impossible since $N_G(B)/C_G(B)$ is a subgroup of $A_8 \cong GL(4, 2)$ with a dihedral group of order 8 as an S_2-subgroup. Thus we have Case I. If $[N_H(C) : C_H(C)] = 8$, we get $[N_G(C) : C_G(C)] = 8 \cdot 5$ from (4.6) and (3.5), again impossible. Similarly the second statement can be proved.

(4.9) LEMMA. *$b_1 c_2$ is conjugate to z in G in both Cases* I *and* II. *Furthermore, we have $c_1^\beta = c_1$ and $b_1^\gamma = b_1$, where β and γ are elements defined in* (4.3) *and* (4.5).

Proof. Firstly we note that $\bar{F}_2 = N_G(F_2)/C_G(F_2)$ is of order $4 \cdot 3$ or $4 \cdot 3^2$, since $\bar{F}_2$ has normal 2-complement by (4.7) and is isomorphic to a subgroup of A_8 with the four group as an S_2-subgroup, and $\beta \in N_G(F_2) - C_G(F_2)$ by (4.3). Assume by way of contradiction that $b_2 c_1 \not\sim z$ in G and so $b_2 c_1 \sim z_1$ in G. Then we have

$$| \bar{F}_2 | = [N_G(F_2) : N_H(F_2)] \cdot [N_H(F_2) : C_H(F_2)] = 3 \cdot 4$$

by (4.3) and so $N_G(F_2) = \langle c_2, b_1, \beta, C_G(F_2) \rangle$ and $z_1 \in Z(N_G(F_2))$. Denote by T_1 an S_2-subgroup of $C_G(b_2 c_1)$ with $F_2 \subset T_1$. Then we have $|T_1| = 64$ and $T_1 \subset N_G(F_2)$. Hence we get $Z(T_1) = \langle b_2 c_1, z_1 \rangle$. Since $b_2 c_1 \sim z_1$ by the assumption, we must have $b_2 c_1 z \sim z$. This is impossible since $b_2 c_1 \sim b_2 c_1 z_1$ in D.

We have proved that $b_2 c_1 \sim z$ in G. Since, by (4.3), β is of odd order and normalizes F_2, $\langle b_2, z_2, z_1 \rangle$ and $z_1^\beta = z_1$, β must centralize an element of $F_2 - \langle b_2, z_2, z_1 \rangle$. If β centralizes one of

$$\{b_2 c_1, b_2 c_1 z_1, b_2 c_1 z_2, b_2 c_1 z, c_1 z, c_1 z_2\},$$

we have $(b_2 c_1)^\beta = b_2 c_1$ or $(b_2 c_1 z_2)^\beta = b_2 c_1 z_2$ or $(c_1 z_2)^\beta = c_1 z_2$. If $(b_2 c_1)^\beta = b_2 c_1$, we get $c_1^\beta = c_1 z_2$ by (4.3). This is impossible because of (4.6). If $(b_2 c_1 z_2)^\beta = b_2 c_1 z_2$, we get $c_1^\beta = b_2 c_1$ which is impossible because of (4.6) and the fact that $b_2 c_1 \sim z$ in G. If $(c_1 z_2)^\beta = c_1 z_2$, we get $c_1^\beta = b_2 c_1 z_2$, which is impossible. Hence β must centralize c_1. Analogously, by using (4.5) and (4.6), we get $b_1^\gamma = b_1$.

(4.10) *Proof of* (iv) *of Theorem* A. Put $Q = C_G(z_1)$. In order to apply Grün's first theorem, we shall compute

$$W = \langle J \cap N_Q(J)', J \cap J'^x \mid x \in Q \rangle,$$

where $J = J(D)$. Since the automorphism group of J is 2-group, we have $N_Q(J) = JC_Q(J)$ and so $J \cap N_Q(J)' = J' = \langle z_1, z_2 \rangle$. J has fifteen conjugate classes of involutions with the representatives $b_1, c_1, z_1, b_2, c_2, z_2, b_1 z_2, c_1 z_2, b_1 b_2, c_1 c_2, b_1 c_2, b_2 c_1, b_2 z_1, c_2 z_1$ and z. Then it is easy to see that, if $z^x \in J$ for some $x \in Q$, we must have

$$z^x \in \{b_2 z_1, b_2 z, c_2 z_1, c_2 z, z\}$$

by using (4.6) and (4.9). From this, it follows that $W \subset \langle z_1, z_2, b_2, c_2 \rangle$. On the other hand, by (4.3) and (4.5) we have $\beta, \gamma \in Q$ and $z_2^\beta = b_2$ and $z_2^\gamma = c_2$. This yields that $W \supset \langle z_1, z_2, b_2, c_2 \rangle$ and so, $W = \langle z_1, z_2, b_2, c_2 \rangle$. Hence by Grün's first theorem [3] there exists a normal subgroup M of Q with W as an S_2-subgroup. Since $W = \langle z_1 \rangle \times \langle b_2, c_2 \rangle$, a theorem of Gaschütz [2] yields that there exists a normal subgroup K_1 of M such that $M = \langle z_1 \rangle \times K_1$. Since β and γ are of odd order, we have $\beta, \gamma \in K_1$. Moreover, the fact that $z_2^\beta = b_2$ and $z_2^\gamma = c_2$ yields that $K_1 \supset \langle b_2, c_2 \rangle$ and K_1 has one class of involutions, since $[Q : K_1] = 8$ and so $\langle b_2, c_2 \rangle$ is an S_2-subgroup of K_1. This completes the proof.

(4.11)[2] *Proof of* (i) *of Theorem* A. We may assume that G has no normal subgroup of odd order. Let G_1 be a minimal normal subgroup of G. Then G_1 is of even order and z is contained in G_1. Since $z \sim b_1 z$ in G, we get $b_1 z \in G_1$ and so $b_1 \in G_1$. Since $b_1 \sim z_1$ in G, we get $z_1 \in G_1$. Hence all involutions of G are contained in G_1. This implies that $[G : G_1]$ is odd, since D is generated

[2] For the purpose of the proof of Theorem B, Lemmas (4.11)–(4.14) are not necessary. So the reader who is interested only in the characterization of A_{11} may omit the subsequent lemmas in §4.

by involutions. The minimality of G_1 yields that G_1 has no normal subgroup of odd index.

Moreover, G has no normal subgroup of index 2, because otherwise G would have a normal subgroup of index 2. The same argument applied to G_1 yields that G_1 is a simple group.

(4.12) *Proof of* (v) *of Theorem* A For an element x of $N_G(B)$ (resp. $N_G(C)$), we denote by $\bar{x}$ the image by the canonical homomorphism of $N_G(B)$ (resp. $N_G(C)$) onto $N_G(B)/C_G(B)$ (resp. $N_G(C)/C_G(C)$). From (4.8), it follows that there exists an element γ' in $N_H(C)$ such that

$$c_1^{\gamma'} = c_1 z_1, \quad (c_1 z_1)^{\gamma'} = c_2, \quad \overline{b_1 b_2}^{\bar{\gamma}'} = \bar{u}, \quad \bar{u} = \overline{ub_1 b_2}^{\bar{\gamma}'} \quad \text{and} \quad \bar{\gamma}'^{\bar{b}_1} = \bar{\gamma}'^{-1}.$$

Let γ be an element defined in (4.5). From the actions of $x_1 = \bar{b}_1$, $x_2 = \overline{b_1 \gamma'}$, $x_3 = \overline{b_2}$ and $x_4 = \overline{\gamma b_2}$ on C, it follows that x_1, x_2, x_3 and x_4 satisfy the relations

$$x_1^2 = x_2^2 = x_3^2 = x_4^2 = 1, \qquad (x_i x_{i+1})^3 = 1 \qquad (i = 1, 2, 3),$$

and

$$(x_i x_j)^2 = 1 \qquad (\,|\,i - j\,| > 1).$$

This implies that $N_G(C)/C_G(C)$ is isomorphic to S_5, since

$$[N_G(C) : C_G(C)] = 120$$

by (4.6), (3.5) and (4.8).

If we have Case II, from (4.8) it follows that there exists an element β' in $N_H(B)$ such that

$$b_1^{\beta'} = b_1 z_1, \quad (b_1 z_1)^{\beta'} = b_2, \quad \overline{c_1 c_2}^{\bar{\beta}'} = \bar{u}, \bar{u}^{\bar{\beta}'} = \overline{uc_1 c_2} \text{ and } \bar{\beta}'^{\bar{c}1} = \bar{\beta}'^{-1}.$$

By using an element β defined in (4.3), it follows that $N_G(B)/C_G(B)$ in Case II is isomorphic to S_5. If we have Case I, from (4.6), (3.5) and (4.8) it follows that $[N_G(B) : C_G(B)] = 8 \cdot 9$. An element β defined in (4.3) satisfies the relations $b_1^\beta = b_1$, $z_1^\beta = z_1$, $z_2^\beta = b_2$ and $b_2^\beta = b_2 z_2$. In the same way as the construction of β, we get that there exists an element β' in $N_G(B)$ such that $\beta'^3 \in C_G(B)$, $b_2^{\beta'} = b_2$, $b_2^{\beta'} = z_2$, $z_2^{\beta'} = b_1$ and $b_2^{\beta'} = b_1 z_1$. (Remark that we must use $z \sim b_1 b_2$ in G.). Then from the action of $\bar{u}$, $\overline{c_1 c_2}$, $\bar{c}_1$, $\bar{\beta}$ and $\bar{\beta}'$ on B, it follows that $[\bar{\beta}, \bar{\beta}'] = 1$ and $\langle \bar{u}, \overline{c_1 c_2}, \bar{c}_1 \rangle$ normalizes $\langle \bar{\beta}, \bar{\beta}' \rangle$. This implies that $N_G(B)/C_G(B)$ in Case I is isomorphic to a 3-Sylow normalizer of A_8.

(4.13) LEMMA. *There exists an element γ' of $N_G(C)$ such that $\gamma'^3 \in C_G(C)$, $z\gamma' = z$, $c_1^{\gamma'} = c_1 z_1$, $(c_1 z_1)^{\gamma'} = c_2$, $b_1 b_2 \gamma' = u$ and $u^{\gamma'} = ub_1 b_2$.*

Proof. We know that D splits over C. It is easy to see that the complement of D over C is conjugate to $\langle u, b_1 b_2, b_1 \rangle$ or $\langle u, b_1 b_2, b_1 z \rangle$ in D. A theorem of Gaschütz [2] yields that the extension of $N_G(C)$ over C splits. Let N be a complement of $N_G(C)$ over C. We may assume that

$$N \supset \langle u, b_1 b_2, b_1 \rangle \text{ or } \langle u, b_1 b_2, b_1 z \rangle.$$

In particular, we have $N \supset \langle u, b_1 b_2\rangle$ in any case. On the other hand, we know that $C_G(C) = C \times U$ and $|U| =$ odd. Then we have $N \triangleright U$. From (4.7), it follows that there exists an element γ'' of N such that $\gamma''^3 \in C_G(C)$, $z^{\gamma''} = z$, $c_1^{\gamma''} = c_1 z_1 (c_1 z_1)^{\gamma''} = c_2$, $(b_1 b_2)^{\gamma''} \equiv u \bmod U$ and $u^{\gamma''} \equiv u b_1 b_2 \bmod U$. Frattini argument yields

$$\langle \gamma'', u, b, b_2\rangle U = N_N(\langle u, b_1 b_2\rangle)\cdot U.$$

Then an element γ' of $\gamma'' U \cap N_N(\langle u, b_1 b_2\rangle)$ satisfies the required properties.

Remark. The complement N of $N_G(C)$ over C must have a subgroup conjugate in G to $\langle u, b_1 b_2, b_1\rangle$ as an S_2-subgroup. In fact, we may assume that

$$N \supset \langle u, b_1 b_2, b_1\rangle \text{ or } \langle u, b_1 b_2, b_1 z\rangle.$$

If $N \ni b_1 z$, the action of $(b_1 z)^{-1}(b_1 z)^{\gamma}$ on C is trivial and so $(b_1 z)^{-1}(b_1 z)^{\gamma} \in U$, where γ is an element defined in (4.5) (Remark that γ can be taken in N.) and U is the complement in $C_G(C)$ of C. However, we have $(b_1 z)^{-1}(b_1 z)^{\gamma} = c_2 z \in U$, which is impossible because of $|U| =$ odd. Similarly, the complement of $N_G(B)$ over B has a subgroup conjugate in G to $\langle u, c_1 c_2, c_1\rangle$ as an S_2-subgroup.

(4.14) *Proof of* (iii) *of Theorem* A. Put

$$W = \langle D \cap N_H(D)', D \cap D'^x \mid x \in H\rangle,$$

where $H = C_G(z)$. Then it is easy to see from (4.6) that W is contained in

$$U = \langle z_1, z_2, b_1 b_2, c_1 c_2, u, b_1 c_2\rangle.$$

Grün's first theorem yields that H has a normal subgroup M of index 2 with U as an S_2-subgroup. Put $\bar{M} = M/\langle z\rangle$. Then an S_2-subgroup of $\bar{M}$ is isomorphic to a 2-group of (1.3) by a mapping defined by

$$x_1 \leftrightarrow \overline{b_1 b_2}, \quad x_2 \leftrightarrow \bar{u}, \quad y_1 \leftrightarrow \overline{b_1 b_2 z_1}, \quad y_2 \leftrightarrow \overline{u a_1 a_2} \quad \text{and} \quad w \leftrightarrow \overline{b_1 c_2},$$

where x_i, y_i and w are as in (1.3). Hence $\bar{M}$ has a normal subgroup of index 2 with

$$\langle \bar{z}_1, \overline{b_1 b_2}, \overline{c_1 c_2}, \bar{u}\rangle$$

as an S_2-subgroup and so M has a normal subgroup K of index 2 with

$$\langle z_1, z_2, b_1 b_2, c_1 c_2, u\rangle$$

as an S_2-subgroup. The existence of an element γ' in (4.13) yields that K has no normal subgroup of index 2.

5. A characterization of A_{11}

(5.0) In §5, we assume that G satisfies the following conditions:

(i) G has no normal subgroup of index 2, and

(ii) G contains an involution z_0 such that $C_G(z_0)$ is isomorphic to the centralizer of an involution in the center of an S_2-subgroup of A_{11}.

Then G has an S_2-subgroup isomorphic to D. We may assume that $G \supset D$ and identify z_0 with z. Then $H = C_G(z)$ is generated by D, γ' and v subject to the following relations:

$$v^3 = \gamma'^3 = 1, \qquad [v, \langle B, c_1 c_2, u, \gamma' \rangle] = 1, \qquad v^{c_1} = v^{-1}$$

$$b_1^{\gamma'} = u, \qquad u^{\gamma'} = b_1 b_2, \qquad \gamma'^{b_1} = \gamma'^{-1}, \qquad c_1^{\gamma'} = c_1 z_1, \qquad (c_1 z_1)^{\gamma'} = c_2.$$

Then we have $N_H(B) = D \cdot C_H(B)$ and $C_H(B) = B \times \langle v \rangle$. Hence we have Case I for the fusion of involutions of G by (4.8).

(5.1) LEMMA.

$$C_G(z_1) = (\langle b_1, z_1 \rangle \times K_1) \langle c_1 \rangle \qquad K_1 = \langle b_2, c_2, \beta, \gamma, v \rangle \cong A_7$$

and $K_1\langle c_1 \rangle = S_7$, *where* K_1 *is a subgroup of* $C_G(z_1)$ *defined in* (4.10), *and* β *and* γ *are elements defined in* (4.3) *and* (4.5).

Proof. Put $Q = C_G(z_1)$ and $W = \langle b_2, c_2, \beta, \gamma, v \rangle$. From the structure of H, it follows that $C_Q(z_2) = C_H(z_2) = \langle J(D), v \rangle$. This yields $C_{K_1}(z_2) = C_W(z_2) = \langle b_2, c_2, v \rangle$ which is isomorphic to $C_{A_7}((1\ 2)(3\ 4))$. K_1 and W have no normal subgroup of index 2 because of $z_2^\beta = b_2$ and $z_2^\gamma = c_2$. Then a theorem of Suzuki [6] yields that K_1 and W are isomorphic to A_7 respectively. Hence we get $K_1 = \langle b_2, c_2, \beta, \gamma, v \rangle$. Then it is clear that $\langle b_1, z_1 \rangle$ centralizes K_1 because of (4.6) and (4.9). We shall show that $K_1\langle c_1 \rangle \cong S_7$. Suppose false. Then c_1 induces an inner automorphism of $K_1 \cong A_7$ and so, there exists an element x of K_1 of order 2 such that $[c_1 x, K_1] = 1$. Since c_1 centralize $\langle z_2, c_2 \rangle$, so does x. Hence we get $x \in \langle z\ \ c_2 \rangle$ and so $[x\ \ v] = 1$. On the other hand, we have $v^{c_1} = v^{-1}$ and so $v^x = v^{-1}$. This is a contradiction. Thus we have proved that $K_1\langle c_1 \rangle \cong S_7$.

(5.2) Now the proof of Theorem B can be accomplished by using a theorem in [5], which is a generalization of W. J. Wong's theorem [8]. Let ρ be a mapping from

$C_G(z) \cup C_G(z_1)$ onto

$$C_{A_{11}}((1\ 2)(3\ 4)(5\ 6)(7\ 8)) \cup C_{A_{11}}((1\ 2)(3\ 4))$$

defined as follows:

$$\begin{gathered} b_1 \leftrightarrow (1\ 3)(2\ 4), \qquad c_1 \leftrightarrow (1\ 2)(9\ 10), \\ b_2 \leftrightarrow (5\ 7)(6\ 8), \qquad c_2 \leftrightarrow (5\ 6)(9\ 10), \\ u \leftrightarrow (1\ 5)(2\ 6)(3\ 7)(4\ 8), \qquad \beta \leftrightarrow (5\ 6\ 7), \\ \gamma \leftrightarrow (5\ 9\ 7)(6\ 10\ 8), \qquad v \leftrightarrow (9\ 10\ 11), \\ \gamma' \leftrightarrow (1\ 3\ 5)(2\ 4\ 6). \end{gathered}$$

Then from Lemma (5.1) it follows that ρ satisfies the condition of a theorem in [5]. Hence G is isomorphic to A_{11}.

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University of Tokyo
Tokyo, Japan