SPECTRA OF ALGEBRAS OF HOLOMORPHIC GERMS

BY

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1. Introduction

If X is an analytic complex manifold, the space $\mathcal{O}(X)$ of all analytic functions defined on X can be given the locally convex topology defined by uniform convergence on compact sets (a thorough study of this topology can be found in [4]). The ordinary product of functions makes $\mathcal{O}(X)$ into a topological algebra and this structure has been exploited by Rossi (cf. [6]) in order to produce the envelope of holomorphy of X, when X is a Riemann Domain. If $K \subset X$ is a subset of X, the space $H(K)$ of germs on K of holomorphic functions defined on neighborhoods of K can be identified with the inductive limit $H(K) = \text{ind lim } \mathcal{O}(U)$ where $K \subset U \subset X$, U is open and if $U \subset V$, the mapping $\mathcal{O}(V) \to \mathcal{O}(U)$ is given by restriction to U of functions in $\mathcal{O}(V)$. Consequently, $H(K)$ can also be given a locally convex topology, namely the inductive limit of the topologies in $\mathcal{O}(U)$ (cf. [4]). Thus, properties that are preserved under inductive limits are inherited by $H(K)$ from the $\mathcal{O}(U)$. If K is compact, denote by $C(K)$ the Banach algebra of all continuous functions on K under the norm $\| f \|_K = \text{Sup } \{|f(x)|; x \in K\}$ and by $A(K)$ the closure in $C(K)$ of the set of germs in $H(K)$. The canonical map $H(K) \to A(K)$ is obviously continuous and $A(K)$ is also a Banach algebra. Therefore the spectrum $\mathfrak{M}(A(K))$ of $A(K)$ is contained in the spectrum $\mathfrak{M}(H(K))$ of $H(K)$. On the other hand, the evaluations at points of K are characters of $A(K)$ and then we have $K \subset \mathfrak{M}(A(K)) \subset \mathfrak{M}(H(K))$. Rossi (cf. [5]) has shown that $K = \mathfrak{M}(H(K))$ if K is meromorphically convex in X. Our aim is to furnish a description of $\mathfrak{M}(H(K))$ in the particular case where X is a Riemann Domain (Prop. 3.5) and show that under suitable hypothesis, $\mathfrak{M}(H(K))$ can be identified with K, whence it will follow that $\mathfrak{M}(A(K)) = K$ also. Therefore Cor. 3.7 below generalizes Th. 2.12 in [5]. Moreover it is shown that if K is the closure of a domain D, then the "Nebenhülle" $\mathfrak{N}(D)$ of D (cf. [1]) can be injected in $\mathfrak{M}(H(K))$.

2. Notations

If X is a topological space, $Y \subset\subset X$ means that $Y \subset X$ and that the closure $\bar{Y}$ of Y is compact. If A is any (complex) topological algebra with identity, the *spectrum* $\mathfrak{M}(A)$ of A is the set of all continuous homomorphisms of algebras (called *characters*) $\chi : A \to \mathbf{C}$ ($\mathbf{C}$ is the complex field) that preserve the identities. $\mathfrak{M}(A)$ will be topologized by simple convergence on the elements of A. By a *Riemann domain* (see [3]) we shall understand a pair (X, φ) where

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X is an analytic complex manifold of dimension n and $\varphi : X \to \mathbf{C}^n$ is a holomorphic mapping such that (1) $\mathcal{O}(X)$ separates points in X; (2) $\varphi : X \to \mathbf{C}^n$ is locally biholomorphic. If (X, φ) is a Riemann domain, $(E(X), \hat{\varphi})$ will denote its envelope of holomorphy ([3]). It is known (cf. [3]) that there is an isomorphism $f \to \hat{f}$ between $\mathcal{O}(X)$ and $\mathcal{O}(E(X))$ whose inverse is the mapping defined by restriction $g \to g|_X$, $g \in \mathcal{O}(E(X))$. Also ([3], [6]), if (X, φ) is a Riemann domain, $E(X)$ can be identified with the spectrum $\mathfrak{M}(\mathcal{O}(X))$ of $\mathcal{O}(X)$ as follows: $z \in E(X)$ corresponds to the character $f \to \hat{f}(z)$.

In general, we refer the reader to [3] for definitions and results in the theory of several complex variables.

We shall also use the notions of direct (= inductive) and inverse (= projective) limits in the categories of topological spaces and locally convex spaces (to be denoted by $\operatorname{ind\,lim}_{\alpha\in\Lambda} X_\alpha$ and $\operatorname{proj\,lim}_{\alpha\in\Lambda} X_\alpha$, where $\{X_\alpha ; \alpha \in \Lambda\}$ is a direct or inverse system, respectively). The reader is referred to *Les Éléments* of N. Bourbaki. Also, if f is a function defined near a point x of a topological space X, the *germ* of f at x will be denoted $\gamma_x(f)$. Finally, if F is a complex locally convex space, F' will denote its dual space, i.e., $F' = \{\Psi : F \to \mathbf{C}; \Psi$ is linear, continuous$\}$

3. Spectra

All throughout this section we shall assume that (X, φ) is a Riemann domain of dimension n. Let $U \subset V$ be two open submanifolds of X, $j : U \to V$ the natural injection and $p : U \to \mathbf{C}^n$ and $q : V \to \mathbf{C}^n$ the restrictions of φ to U and V. Denote by $J : \mathcal{O}(V) \to \mathcal{O}(U)$ the map $J(f) = f \circ j$. Clearly J is a continuous homorphism of algebras. Its transpose ${}^tJ : \mathcal{O}(U)' \to \mathcal{O}(V)'$ preserves the multiplicative functionals and therefore induces a map $\hat{j} : \mathfrak{M}(\mathcal{O}(U)) \to \mathfrak{M}(\mathcal{O}(V))$. Since $\mathfrak{M}(\mathcal{O}(U)) = E(U)$ and $\mathfrak{M}(\mathcal{O}(V)) = E(V)$, we have (cf. [2]):

3.1. LEMMA. *If $(E(U), \hat{p})$ and $(E(V), \hat{q})$ are the envelopes of holomorphy of U and V, then the following diagram is commutative*

$$\begin{array}{ccc} E(U) & \xrightarrow{\ \hat{j}\ } & E(V) \\ & \searrow^{\hat{p}} \quad {}^{\hat{q}}\swarrow & \\ & \mathbf{C}^n & \end{array}$$

and therefore $\hat{j}$ is holomorphic (and in fact, locally biholomorphic). Moreover $\hat{j} = j$ on $U \subset E(U)$.

Remark. $\hat{j}$ is not necessarily an injection; example: $V = X = \mathbf{C}^n$, U a domain with non-schlicht envelope of homolorphy.

If we assume furthermore that $U \subset\subset V$, then $\bar{U}$ can be considered as a compact subset of $E(V)$. Denote by $\Delta(U, V)$ the holomorphic hull of $\bar{U}$ in

$E(V)$, i.e.:

$$\Delta(U, V) = \{z \in E(V); |f(z)| \leqq \|f\|_{\bar{U}}, \quad \text{for all} \quad f \in \mathcal{O}(E(V))\}.$$

3.2. LEMMA. *Assume that $U \subset\subset V$ and let $A(U, V)$ be the closure in $C(\bar{U})$ of the algebra of restrictions to $\bar{U}$ of the functions in $\mathcal{O}(V)$. Then $A(U, V)$ is a Banach algebra whose spectrum $\mathfrak{M}(A(U, V))$ can be topologically identified with the holomorphic hull $\Delta(U, V)$ of $\bar{U}$ in $E(V)$.*

Proof. The algebra of restrictions to $\bar{U}$ of functions in $\mathcal{O}(V)$ coincides with the algebra of restrictions of functions in $\mathcal{O}(E(V))$, because $f|_{\bar{U}} = \hat{f}|_{\bar{U}}$ for every $f \in \mathcal{O}(V)$. Therefore, since $E(V)$ is a Stein manifold, it follows from Theorem 2.3 in [5] that $\mathfrak{M}(A(U, V)) = \Delta(U, V)$, Q.E.D.

Consider now three open submanifolds U, V, W of X such that

$$U \subset\subset V \subset\subset W \subset\subset X.$$

Let $k : V \to W$ be the natural injection.

3.3. LEMMA. *If $\hat{k} : E(V) \to E(W)$ is the mapping induced by $k : V \to W$ (as in Lemma 3.1), then $\hat{k}(\Delta(U, V)) \subset \Delta(V, W)$.*

Proof. If $h \in \Delta(U, V)$ is considered as a homomorphism $h : \mathcal{O}(V) \to \mathbf{C}$ and $f \in \mathcal{O}(E(W))$, we have

$$|[\hat{k}(h)](f)| = |h(f \circ k)| \leqq \|f \circ k\|_{\bar{U}} = \|f\|_{k(\bar{U})} \leqq \|f\|_{\bar{V}}$$

and therefore $\hat{k}(h) \in \Delta(V, W)$, Q.E.D.

It is clear that the transpose ${}^t\gamma$ of $\gamma : A(V, W) \to A(U, V)$ defined as $\gamma(f) = f|_{\bar{U}}$ coincides with $\hat{k}$ on $\mathfrak{M}(A(U, V)) = \Delta(U, V)$.

Suppose now that $K \subset X$ is a compact subset of X and choose a fundamental system of open neighborhoods $X \supset\supset U_1 \supset\supset U_2 \supset\supset \cdots$ of K.

3.4. LEMMA. *There are topological algebraic isomorphisms (for each n):*

$$\text{ind lim}\, A(U_{n+1}, U_n) = \text{ind lim}\, \mathcal{O}(U_n) = H(K).$$

Proof. It is easy to see that $\text{ind lim}\, \mathcal{O}(U_n) = H(K)$ [4, Chap. I, §1]. Moreover the restriction mappings

$$A(U_{n+1}, U_n) \to \mathcal{O}(U_{n+1}), \qquad \mathcal{O}(U_n) \to A(U_{n+1}, U_n)$$

are certainly continuous. The lemma follows.

Let us observe that if $\{A_n, \psi_{n,m}\}$ is a direct system of topological algebras with homomorphisms $\psi_{n,m} : A_n \to A_m$ for $n \leqq m$, and $\mathfrak{M}_n = \mathfrak{M}(A_n)$. Then the spectrum $\mathfrak{M} = \mathfrak{M}(A)$ of the inductive limit $A = \text{ind lim}\, A_n$ can be identified to the projective limit $\mathfrak{M} = \text{proj lim}\, \mathfrak{M}_n$ of the spaces $\mathfrak{M}_n$ (with mappings $\eta_{n,m} : \mathfrak{M}_m \to \mathfrak{M}_n$ defined as the transposes of the $\psi_{n,m}$). The proof of this fact readily follows from the definitions. From this remark and Lemmas 3.2, 3.3

and 3.4 we conclude:

3.5. PROPOSITION. *Let $K \subset X$ be a compact subset of a Riemann domain X and $X \supset\supset U_1 \supset\supset U_2 \supset\supset \cdots$ a fundamental system of neighborhoods of K. Denote by K_n the holomorphic hull of $\bar{U}_{n+1}$ considered as a compact subset of the envelope of holomorphy $E(U_n)$ of U_n and let $\hat{j}_{n,m} : E(U_m) \to E(U_n)$ (for $n \leqq m$) be the holomorphic mapping induced by the injection $j_{n,m} : U_n \to U_m$. Then $\hat{j}_{n,m}(K_m) \subset K_n$, and $\mathfrak{M}(H(K))$ can be naturally identified as a topological space with the projective limit* proj lim K_n *of the compact spaces K_n (with the mappings $\hat{j}_{n,m}|_{K_m}$).*

3.6. PROPOSITION. $\mathfrak{M}(H(K)) = \mathfrak{M}(A(K))$.

Proof. Clearly (see §1) $\mathfrak{M}(A(K)) \subset \mathfrak{M}(H(K))$. So let $\varphi \in M(H(K))$ and $f \in H(K)$. We need to show that $|\varphi(f)| \leqq \|f\|_K$. But if this inequality does not hold then $g = f - \varphi(f)$ has these contradictory properties:

(i) g is invertible in $H(K)$
(ii) $\varphi(g) = 0$.

Proposition 3.6 was suggested to us by the referee.

4. The "Nebenhülle"

The system $\{E(U_m), \hat{j}_{n,m}\}$ in Prop. 3.5 is a particular case of the following situation. Let $\{X_\alpha, \varphi_{\alpha,\beta}\}$, $\alpha, \beta \in \Lambda$ be an inverse system of topological spaces and assume that

(a) *each X_α is a Riemann domain of dimension n (with projection denoted π for all α),*
(b) *for all $\alpha \leqq \beta$, $\varphi_{\alpha,\beta} : X_\beta \to X_\alpha$ commutes with the projections (and is therefore a locally biholomorphic mapping)*

One such system might be called an *inverse system of Riemann Domains.*

The universal property of inverse limits implies the existence of a projection $\pi :$ proj lim $X_\alpha \to \mathbf{C}^n$.

To every open set $U \subset \mathbf{C}^n$ we can associate now the set $F(U)$ of all families $\{h_\alpha\}_{\alpha \in \Lambda}$ of holomorphic functions $h_\alpha : U \to X_\alpha$ that satisfy: if $\alpha \leqq \beta$, then $\varphi_{\alpha,\beta} h_\beta = h_\alpha$. It is well known (see [2]) that F is a presheaf that generates an analytic sheaf, denoted $\mathfrak{R}$ in the following. Let $w = \{z_\alpha\}_{\alpha \in \Lambda}$ be an element of proj lim X_α and $z_0 \in \mathbf{C}^n$ the point $z_0 = \pi(w)(= \pi(z_\alpha)$, all $\alpha)$. If $U \subset \mathbf{C}^n$ is open and $z_0 \in U$, define $h_\alpha : U \to X_\alpha$ by $h_\alpha(z) = z_\alpha$. Clearly $\{h_\alpha\} \in F(U)$ and therefore w determines an element $j(w) \in \mathfrak{R}$ such that $\pi(j(w)) = z_0$, (namely the family of germs of the h_α at z_0). The mapping $j :$ proj lim $X_\alpha \to \mathfrak{R}$ is continuous and commutes with projections. On the other hand, if $\gamma \in \mathfrak{R}$, then there exist functions h_α defined on some neighborhood of $\pi(\gamma)$ with values in X_α such that γ is the family of germs of the h_α at $\pi(\gamma)$. The family $j'(\gamma)$ of all values $h_\alpha(\pi(\gamma))$, $\alpha \in \Lambda$, is of course an element of proj lim X_α. It is clear that $j' : \mathfrak{N} \rightarrow \text{proj lim } X_\alpha$ is continuous, and $j'j$ is the identity on proj lim X_α. Moreover if $j'_\alpha : \mathfrak{N} \rightarrow X_\alpha$ is the composition of j' with the canonical mapping proj lim $X_\alpha \rightarrow X_\alpha$, then j'_α is holomorphic and commutes with projections.

If $C \subset \text{proj lim } X_\alpha$ is any non-empty set, the *intersection relative to* C *of the Riemann domains* X_α is defined to be the union of the connected components of $\mathfrak{N}$ that intersect $j(C)$ (cf. [2]). It is not hard to see that it is the inverse limit of the X_α in a suitable category.

In particular, if $D \subset \mathbf{C}^n$ is a bounded open set and U_α is the family of all open sets in $\mathbf{C}^n$ that contain the closure $\bar{D}$ of D, then the envelopes of homomorphy X_α of the U_α form an inverse system of Riemann domains (with the obvious mappings, see Lemma 3.1). The intersection $\mathfrak{N}(D)$ of the X_α relative to D is called (in [1]) the "Nebenhülle" of D. It can be proven [1] that it is enough to consider a fundamental system $U_1, U_2, \cdots$ of neighborhoods of $\bar{D}$, which can of course be chosen to satisfy also $U_1 \supset\supset U_2 \supset\supset U_3 \supset\supset \cdots$ Therefore the situation is exactly the same as in §3 and it is very natural to compare $\mathfrak{M}(H(\bar{D}))$ with $\mathfrak{N}(D)$.

All throughout this section $D \subset \mathbf{C}^n$ will denote a fixed open bounded set, $\mathbf{C}^n \supset\supset U_1 \supset\supset U_2 \supset\supset \cdots$ a fundamental system of neighborhoods of the compact set $\bar{D}$, $\{E_n, \hat{\jmath}_{n,m}\}$ the inverse system of the envelopes of holomorphy $E_n = E(U_n)$ and $\mathfrak{N}$ the sheaf constructed as above with the E_n as data. It follows from the definition that $\mathfrak{N}(D)$ is an open set of $\mathfrak{N}$. The mappings $j : \text{proj lim } E_n \rightarrow \mathfrak{N}$, $j' : \mathfrak{N} \rightarrow \text{proj lim } E_n$ and $j'_n : \mathfrak{N} \rightarrow E_n$ will have the same meaning as above. We want to prove that j' is one-to-one on $N(D)$. For that, consider the set $V \subset \mathfrak{N}$ of all elements in $\mathfrak{N}$ that are sequences $\{\gamma(s_n)\}$ of germs of *local sections* s_n of the Riemann domains E_n.

4.1. Lemma. *V is open and closed in $\mathfrak{N}$.*

Proof. It is clear that V is open in $\mathfrak{N}$. Let us show that it is also closed. Suppose that $\gamma \in \mathfrak{N}$ and let $\{h_n\}, n = 1, 2, \cdots$ be a family of holomorphic functions $h_n : U \rightarrow E_n$ such that $\gamma = \{\gamma_z(h_n)\}$, where $z = \pi(\gamma)$. We can assume that U is an open policylinder. Now the set W of all sequences $\{\gamma_t(h_n)\}$ with $t \in U$ is an open neighborhood of γ. Therefore, if γ belongs to the closure of V, there exist a point $w \in U$, an open set $U' \subset \mathbf{C}^n$ and sections $s_n : U' \rightarrow E_n$ such that $\gamma_w(s_n) = \gamma_w(h_n)$. But this implies $s_n = h_n$ on some neighborhood of $w \in U$ and U being a policylinder it follows that h_n is a section of E_n on U. Therefore $\gamma \in V$ and V is closed.

4.2. Corollary. $\mathfrak{N}(D) \subset V$.

Proof. Since $j(D) \subset V$ and V, being open and closed, is a union of components of $\mathfrak{N}$, the inclusion $\mathfrak{N}(D) \subset V$ follows.

Remark. $\mathfrak{N}(D)$ is in fact a union of components of V (or $\mathfrak{N}$).

According to Lemma 3.4, there is a topological identification proj lim $E_n = \mathfrak{M}(H(\bar{D}))$ and therefore j' induces a map $j'' : \mathfrak{N} \rightarrow \mathfrak{M}(H(\bar{D}))$.

4.3. PROPOSITION. *j'' is one-to-one on V.*

Proof. Assume $j''(v) = j''(v')$ where $v, v' \in V$. It follows that $\pi(v) = \pi j''(v) = \pi j''(v') = \pi(v')$. Denote $\pi(v) = \pi(v')$ by z. Now if s_n, s_n' are local sections of E_n such that $\{\gamma_z(s_n)\} = v$, $\{\gamma_z(s_n')\} = v'$ it follows that $\{s_n(z)\} = j''(v) = j''(v') = \{s_n'(z)\}$ or $s_n(z) = s_n'(z)$ for all $n = 1, 2, \cdots$. But since s_n and s_n' are sections we can conclude that $\gamma_z(s_n) = \gamma_z(s_n')$ and therefore $v = v'$ as desired.

4.4. THEOREM. *If $D \subset \mathbf{C}^n$ is open and bounded, there is a natural one-to-one mapping $\mathfrak{N}(D) \to \mathfrak{M}(H(\bar{D}))$.*

4.5. COROLLARY (cf. Rossi, [5, Th. 2.14]). *If $D \subset \mathbf{C}^n$ is bounded and closed a necessary condition for $\bar{D}$ to be the spectrum of $H(\bar{D})$ is that $D = \mathfrak{N}(D)$.*

Remark. The condition $D = \mathfrak{N}(D)$ implies that D is a domain of holomorphy, but is in general stronger.

Under suitable conditions one can prove that $\mathfrak{N}(D) = V$ or that $j''(V)$ is dense in $\mathfrak{M}(H(\bar{D}))$. Details will be given elsewhere.

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