CLOSED PARTITIONS OF MAXIMAL IDEAL SPACES

BY

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1. Statement of results

Let A be a uniform algebra on the compact Hausdorff space X, that is, A is a closed subalgebra of $C(X)$ which separates the points of X and which contains the constants. The maximal ideal space of A will be denoted by M_A. The Shilov boundary of A will be denoted by bA.

The A-convex hull $\hat{E}$ of a closed subset E of X is the set of all $\phi \in M_A$ which extend continuously to the uniform closure $(A|_E)^-$ of the restriction algebra $A|_E$ in $C(E)$. Equivalently, $\phi \in \hat{E}$ if and only if $|\phi(f)| \leq \|f\|_E$ for all $f \in A$. The maximal ideal space of $(A|_E)^-$ can be identified with $\hat{E}$.

The first two results extend theorems in [9] and [14].

THEOREM 1. *If $\{E_j\}_{j=1}^{\infty}$ is a closed cover of X such that $A|_{E_j}$ is closed in $C(E_j)$, $1 \leq j < \infty$, then $M_A = \bigcup_{j=1}^{\infty} \hat{E}_j$.*

THEOREM 2. *If $\{E_j\}_{j=1}^{\infty}$ is a closed cover of X such that $A|_{E_j} = C(E_j)$, $1 \leq j < \infty$, then $A = C(X)$.*

The next theorem extends a result of Stolzenberg [11]. If $f \in C(M_A)$, let $[A, f]$ denote the uniform algebra generated by A and f on M_A.

THEOREM 3. *Let $\{E_j\}_{j=1}^{\infty}$ be a closed cover of M_A. If $f \in C(M_A)$ satisfies $f|_{E_j} \in A|_{E_j}$, $1 \leq j < \infty$, then $b[A, f] = bA$ and $M_{[A,f]} = M_A$.*

The remaining results deal with the situation in which, instead of assuming that $f|_{E_j} \in A|_{E_j}$, we assume that $f|_{E_j} \in (A|_{E_j})^-$. That is, instead of assuming that f belongs to A on E_j, we wish to assume that f is uniformly approximable on E_j by functions in A.

THEOREM 4. *Let $\{E_1, E_2\}$ be a closed cover of M_A, and let $g \in A$. If $f \in C(M_A)$ satisfies $f|_{E_j} \in (A|_{E_j})^-$, $j = 1, 2$, and $f|_X = g|_X$, then $f \equiv g$.*

THEOREM 5. *Let $\{E_1, E_2\}$ be a closed cover of M_A. If $f \in C(M_A)$ satisfies $f|_{E_j} \in (A|_{E_j})^-$, $j = 1, 2$, then $M_{[A,f]} = M_A$.*

In Theorem 5, it is not necessarily true that $b[A, f] = bA$, even if A is the algebra $P(\Delta)$ of functions continuous on the disc $\Delta = \{|z| \leq 1\}$ and analytic on the interior of Δ. Also, Theorems 4 and 5 cannot be extended to closed covers of M_A by three sets, even if $A = P(\Delta)$.

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2. Proofs of the theorems

It will be convenient to introduce first some of the ingredients of the proofs.

A positive measure μ on X is a *Jensen measure* for $\varphi \in M_A$ if Jensen's inequality is valid:

$$\log |\varphi(f)| \leq \int \log |f| \, d\mu, \qquad f \in A.$$

Each $\varphi \in M_A$ has a Jensen measure on X, and every Jensen measure for φ is a representing measure for φ, that is, satisfies $\varphi(f) = \int f \, d\mu, f \in A$ (cf. [3]).

Glicksberg's Lemma [7]. *Let U be a relatively open subset of bA. If $f \in A$ vanishes on U, then f also vanishes on the set $V = M_A \setminus (bA \setminus U)^\wedge$. Moreover, V is an open subset of M_A, and $V \cap U$ is dense in U.*

The lemma is proved by noting that if μ is a Jensen measure on bA for $\varphi \in M_A$, and if $\varphi \notin (bA \setminus U)^\wedge$, then μ cannot be supported on $bA \setminus U$. Consequently $\mu(U) > 0$, and

$$|\varphi(f)| \leq \exp \int \log |f| \, d\mu = 0$$

whenever $f \in A$ vanishes on U. The last statement of the lemma follows from the fact that bA is a minimal closed boundary for A.

Another prime ingredient is Rossi's local maximum modulus principle [8], which states that for any closed subset E of M_A we have

$$\text{(LMM)} \qquad b[(A|_E)^-] \subseteq (E \cap bA) \cup bE.$$

Proof of Theorem 1. Let $\varphi \in M_A$, and let μ be a Jensen measure on X for φ. Since $X = \bigcup_{j=1}^\infty E_j$, there is an index m such that $\mu(E_m) > 0$. With A_φ denoting the kernel of φ, either $A_\varphi|_{E_m} = A|_{E_m}$ or $A_\varphi|_{E_m}$ is a maximal ideal in $A|_{E_m}$.

If $A_\varphi|_{E_m} = A|_{E_m}$, there is a function $g \in A_\varphi$ such that $g|_{E_m} = 1$. Then $\log|\varphi(1-g)| = 0$, while $\int \log|1-g| \, d\mu = -\infty$. This contradicts Jensen's inequality.

It follows that $A_\varphi|_{E_m}$ is a maximal ideal in $A|_{E_m}$, and so $\varphi \in \hat{E}_m$.

Proof of Theorem 2. By the theory of antisymmetric sets [6], we can assume, restricting A to a maximal set of antisymmetry, that the only real-valued functions in A are the constants.

From $A|_{E_j} = C(E_j)$, it follows that $\hat{E}_j = E_j$. By Theorem 1,

$$M_A = \bigcup_{j=1}^\infty E_j = X.$$

By the Baire category theorem, there is an index m such that int (E_m) is not empty. Since $A|_{E_m} = C(E_m)$, every compact G_δ-set $F \subseteq \text{int}(E_m)$ is a local peak set for A. By Rossi's local peak set theorem [8], F is a peak set of A. By [6], ν_F belongs to $A^\perp$ whenever $\nu \in A^\perp$, this for all compact G_δ-sets $F \subseteq \text{int}(E_m)$. It follows that every $\nu \in A^\perp$ satisfies $|\nu|(\text{int}(E_m)) = 0$.

Let f be a real-valued function in $C(X)$ such that $f = 0$ off int (E_m), while f does not vanish identically. Then $\int f\,d\nu = 0$ for all $\nu \in A^{\perp}$, so $f \in A$. By antisymmetry, f is constant, and this constant is not zero. Consequently $X \backslash \text{int}\ (E_m)$ is empty, and $X = \text{int}\ (E_m)$ must consist of just one point. Evidently $A = C(X)$.

Proof of Theorem 3. Suppose that $b[A, f] \neq bA$. By the Baire category theorem, there is an index m and a non-empty relatively open subset U of $b[A, f] \backslash bA$ such that $U \subseteq E_m$. Choose $f_m \in A$ such that $f = f_m$ on E_m. By the Glicksberg lemma, $f - f_m$ vanishes on an open subset V of M_A such that $V \cap U$ is dense in U. By LMM, $b[A, f]$ cannot meet $V \backslash bA$. This contradicts the fact that $V \backslash bA$ contains the non-empty subset $U \cap V$ of $b[A, f]$. Hence $b[A, f] = bA$.

To show that $M_{[A,f]} = M_A$, we employ an elegant argument due to Quigley (unpublished). Let $\pi : M_{[A,f]} \rightarrow M_A$ be the natural projection, which restricts the homomorphisms in $[A, f]$ to A. Let $\hat{g}$ be the Gelfand transform of the function $g \in [A, f]$. If $g \in A$, then $\hat{g} = g \circ \pi$. Now $\hat{f} \circ \pi$ belongs to $[A, f]$ on each set of the closed cover $\{\pi^{-1}(E_j)\}_{j=1}^{\infty}$ of $M_{[A,f]}$. Applying what we have already proved to the algebras $[A, \hat{f}]$, and $[A, \hat{f}, f \circ \pi]$, considered as uniform algebras on $M_{[A,f]}$, we find that $b[A, \hat{f}, f \circ \pi] = b[A, \hat{f}] = bA$. Since $\hat{f} - f \circ \pi$ vanishes on $b[A, \hat{f}, f \circ \pi]$, $\hat{f} - f \circ \pi$ must vanish on $M_{[A,\hat{f},f\circ\pi]} \supseteq M_{[A,f]}$. Consequently $\hat{f} = f \circ \pi$, and $\hat{f}$ is constant on each fiber $\pi^{-1}(x)$. It follows that π is one-to-one, and $M_{[A,f]} = M_A$.

Proof of Theorem 4. We can assume that $X = bA$. Replacing f by $f - g$, we can assume that f vanishes on bA. We must show that $f \equiv 0$.

Let $E = M_A \backslash \text{int} f^{-1}(0)$, and let $B = (A\,|_{\hat{E}})^-$. By LMM, $bB \subseteq bA \cup b\hat{E}$. So f vanishes on bB. Replacing A by B and E_j by $E_j \cap \hat{E}$, we can assume that M_A is the A-convex hull of the set $\{f \neq 0\}$.

By LMM, $b(A\,|_{E_2})^- \subseteq (E_2 \cap bA) \cup bE_2$. Consequently $bA \cap \text{int}\, E_2$ is a relatively open subset of $b(A\,|_{E_2})^-$ on which f vanishes. Since $f \in (A\,|_{E_2})^-$, we can apply the Glicksberg lemma and deduce that f vanishes on the subset of E_2 which does not meet the $(A\,|_{E_2})^-$-convex hull of bE_2. Consequently f vanishes on $E_2 \backslash (bE_2)^{\wedge} \subseteq E_2 \backslash \hat{E}_1$. So f vanishes off $\hat{E}_1$. Since the A-convex hull of $\{f \neq 0\}$ is M_A, we must have $M_A = \hat{E}_1$. Consequently $bA \subseteq E_1$.

Now take $f_n \in A$ such that f_n converges uniformly to f on E_1. Since $f_n \rightarrow 0$ uniformly in bA, we obtain $f_n \rightarrow 0$ uniformly on M_A. Hence $f \equiv 0$.

Proof of Theorem 5. Theorem 5 follows from Theorem 4 and Quigley's argument, as in the proof of Theorem 3. In fact, let π be the natural projection of $M_{[A,f]}$ onto M_A. Then $f \circ \pi$ is uniformly approximable by $[A, f]$ on the closed sets $\pi^{-1}(E_1)$ and $\pi^{-1}(E_2)$, and $f \circ \pi$ agrees with the function $\hat{f} \in [A, f]$ on $M_A \supseteq b[A, f]$. Consequently $f \circ \pi = \hat{f}$, and $\hat{f}$ is constant on each fiber $\pi^{-1}(x)$, $x \in M_A$. So $M_A = M_{[A,f]}$.

3. Examples

Let K be a compact subset of the complex plane. By $P(K)$ we denote the subalgebra of $C(K)$ of functions uniformly approximable on K by polynomials in z. Mergelyan's theorem states that $P(K)$ consists of precisely the functions in $C(K)$ which are analytic on int (K), providing the complement of K is connected.

A *dirichlet algebra* on X is a uniform algebra A on X such that Re (A) is uniformly dense in $C_R(X)$. By the Walsh-Lebesgue theorem, $P(K)$ is a dirichlet algebra on bK whenever the complement of K is connected.

We will base the examples on the following theorem.

THEOREM (Browder-Wermer [5]). *Suppose A and B are dirichlet algebras on X, such that every measure in $A^\perp$ is mutually singular with every measure in $B^\perp$. Then $A \cap B$ is a dirichlet algebra on X, and $(A \cap B)^\perp$ is the vector space direct sum of $A^\perp$ and $B^\perp$.*

Let Δ be the closed unit disc $\{|z| \leq 1\}$, and let $b\Delta$ be its boundary $\{|z| = 1\}$. Then $P(b\Delta)$ has maximal ideal space Δ, and we can identify $P(b\Delta)$ and $P(\Delta)$.

LEMMA 1. *There is an arc Γ joining -1 to $+1$ through* int (Δ), *dividing* int $(\Delta)\backslash\Gamma$ *into two components D_+ and D_-, such that the harmonic measures on bD_+ for points in D_+ is mutually singular with the harmonic measures on bD_- for points in D_-.*

This lemma will not be proved, as the same type of construction was made for essentially the same purpose by Browder and Wermer [4]. It is a consequence of the existence of a quasiconformal homeomorphism of the upper half-plane whose derivative vanishes almost everywhere along the real axis [1], together with the fact that one can weld conformal structures together if the boundary identification arises from a quasiconformal homeomorphism (cf. [10] for the welding problem, especially Lemma 3).

LEMMA 2. *Let Γ be the arc of Lemma* 1, *and let $X = \Gamma \cup b\Delta$. Let B be the algebra of functions in $C(X)$ which extend continuously to Δ to be analytic on $\Delta\backslash X$. Then B is a dirichlet algebra on X.*

Proof. Let B_+ be the algebra of functions in $C(X)$ which extend analytically to D_+. Then B_+ consists of the functions in $P(\bar{D}_+)$, extended in all possible continuous ways to $X\backslash bD_+$. Consequently B_+ is a dirichlet algebra on X. Every measure on X orthogonal to B_+ is absolutely continuous with respect to harmonic measure for D_+ (cf. [2]).

The algebra B_- is defined analogously. Then $B = B_+ \cap B_-$. In view of Lemma 1, every measure in $B_+^\perp$ is singular to every measure in $B_-^\perp$. By the Browder-Wermer theorem, B is a dirichlet algebra on X.

Example 1. Let Γ be the arc of Lemma 1. In view of Lemma 2, there is $f \in C(\Delta)$ such that f is analytic on $D_+ \cup D_-$, while f attains its maximum modulus on a subset of $X \backslash b\Delta$. Then the Shilov boundary of $[P(\Delta), f]$ must be strictly larger than $b\Delta = bP(\Delta)$. By Mergelyan's theorem, f can be approximated uniformly by functions in $P(\Delta)$ on each of the sets $\bar{D}_+$, $\bar{D}_-$. This shows that no statement about the Shilov boundaries can be made in Theorem 5.

For the second example, we start with three half-open arcs Γ_1, Γ_2, Γ_3 in $\operatorname{int} \Delta$, such that each Γ_j begins at $z = 0$ and continues through the sector $(j - 1)2\pi/3 < \arg z < j2\pi/3$ so that it clusters on the arc

$$\{e^{i\theta} : (j - 1)2\pi/3 \leq \theta \leq j2\pi/3\}$$

on the circle. Then $\operatorname{int} \Delta \backslash (\Gamma_1 \cup \Gamma_2 \cup \Gamma_3)$ consists of three components U_1, U_2, U_3.

The sets $\bar{U}_1$, $\bar{U}_2$, $\bar{U}_3$ form a closed cover of Δ. The complement of each $\bar{U}_j$ is connected. By Mergelyan's theorem, every function in $P(\bar{U}_j)$ can be approximated uniformly on $\bar{U}_j$ by functions in $P(\Delta)$.

The harmonic measure on bU_j for points in U_j is supported on the set of points in bU_j which are accessible from U_j (cf. [2]). By construction, only three points on $b\Delta$ are accessible from the sets U_j. Consequently the harmonic measures for the U_j are supported on the arcs Γ_j.

LEMMA 3. *The arcs Γ_1, Γ_2, Γ_3 can be chosen so that, in addition to the topological properties described above, the harmonic measure on bU_j for points in U_j is mutually singular with harmonic measure on bU_k for points on U_k, $j \neq k$.*

Proof. This can be accomplished by first drawing smooth guide arcs with the desired topological properties, covering the guide arcs with discs which overlap only at their boundary points, and then replacing the segment inside each disc by the arc Γ of Lemma 1. We have already shown that $b\Delta$ cannot carry any harmonic mass. Neither can the countable set at which the discs used in the construction touch. Since the harmonic measures are locally mutually singular on the remainder, they are mutually singular.

LEMMA 4. *Let the arcs Γ_1, Γ_2, Γ_3 be as in Lemma* 3. *Let*

$$X = b\Delta \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3,$$

and let A be the subalgebra of $C(X)$ of functions which extend continuously to Δ to be analytic on $\Delta \backslash X$. Then A is a dirichlet algebra on X. Moreover, $b\Delta$ is a peak set for A. (In fact, $b\Delta$ is a peak interpolation set for A.]

Proof. Let A_j be the subalgebra of $C(X)$ of functions which extend analytically to U_j, so that $A = A_1 \cap A_2 \cap A_3$. Each A_j is a dirichlet algebra on X. By [2], every measure in $A_j^\perp$ is absolutely continuous with respect to harmonic measure on bU_j for points of U_j. By the Browder-Wermer theorem, $A^\perp = A_1^\perp + A_2^\perp + A_3^\perp$. Every $\nu \in A^\perp$ satisfies $|\nu|(b\Delta) = 0$, because of the corresponding fact for the harmonic measures. By [6], $b\Delta$ is a peak set for A.

Example 2. Take $g \in A$ to peak on $b\Delta$, and set $f = 1 - g$. Then f vanishes on $b\Delta$. Mergelyan's theorem shows that f is approximable on each of the sets $\bar{U}_1$, $\bar{U}_2$ and $\bar{U}_3$ by functions in $P(\Delta)$. However, $f^{-1}(0) = b\Delta$ is not $P(\Delta)$-convex. By the following lemma, $M_{[P(\Delta),f]}$ is strictly larger than Δ. And so Theorems 4 and 5 do not extend to closed covers by three sets.

LEMMA [13]. *Let A be a uniform algebra, and let $f \in C(M_A)$. If $M_{[A,f]} = M_A$, then each of the level sets of f is A-convex.*

Proof. Suppose the level set $f^{-1}(\alpha)$ is not A-convex. If

$$\varphi \in (f^{-1}(\alpha))^{\wedge} \backslash f^{-1}(\alpha),$$

then the functional $\Psi(\sum_{k=0}^{n} g_k f^k) = \sum_{k=0}^{n} \varphi(g_k)\alpha^k$, $g_k \in A$, determines an extension $\Psi \in M_{[A,f]}$ of φ which does not coincide with evaluation at φ.

It might be conjectured that if each level set of f were A-convex, then $M_{[A,f]}$ would coincide with M_A. That this is not the case is demonstrated by the function $f(z) = z^2\bar{z}$, $z \in \Delta$, which is a homeomorphism of the unit disc Δ, but which satisfies $M_{[P(\Delta),f]} \neq \Delta$.

We remark that S. Scheinberg has modified an unpublished example of J. Wermer, connected with the main theorem of [12], to produce a polynomial f on degree five in z and $\bar{z}$ such that f is one-to-one on Δ, $\partial f/\partial \bar{z}$ does not vanish, and $M_{[P(\Delta),f]}$ is strictly larger than Δ. The polynomial is

$$f(z) = z + \bar{z}(z\bar{z} - 1)(z\bar{z} + i)/1000.$$

Note added in proof. R. Mullins had independently obtained Theorem 2 (unpublished).

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