COMPACT AND WEAKLY COMPACT OPERATORS ON $C(S)_\beta$

BY

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In 1958, R. C. Buck [1] introduced the β or strict topology on the linear space $C(S)$ of bounded continuous functions on a locally compact Hausdorff space S. This topology is defined by the seminorms

$$P_\phi(f) = \sup\{|f(x)\phi(x)| : x \in S\} = \|\phi f\|$$

where $\phi \in C_0(S)$, the subspace of functions in $C(S)$ which vanish at infinity. Since this time several authors have studied and made use of the strict topology in various settings. One may consult [2] for more specific references. In this paper a study will be made of the compact and weakly compact linear operators on this space.

The strict topology is a complete locally convex topology which is neither barrelled, bornological nor metrizable. In fact, any of these is equivalent to the compactness of S. On the other hand the strong dual of $C(S)_\beta$ is the space $M(S)$ of bounded regular Borel measures on S as was shown in [1], and furthermore, the β and supremum norm bounded sets in $C(S)$ coincide. These two facts along with the integral representation of the continuous operators on $C(S)_\beta$ into a space $C(T)_\beta$ obtained in [8] allows us to obtain the following principal result.

Let us call an operator A on $C(S)$ into a topological vector space X compact (weakly compact) if A maps β-bounded subsets of $C(S)$ into relatively compact (weakly relatively compact) subsets of X, and call A β-compact (β-weakly compact) if A maps a β neighborhood of 0 into a relatively compact (weakly relatively compact) subset of X. It will be shown that when $X = C(T)_\beta$, then A is β-compact (β-weakly compact) if and only if A is continuous with the norm topology on $C(T)$ and compact (weakly compact). As a consequence it will be shown that these two properties coincide when X is a Banach space.

In closing this introduction the author wishes to acknowledge the aid of the referee in improving the paper, particularly with regard to the considerably shortened proofs of Corollaries 2 and 4.

Our notation will be taken from [8] and [9] and we rely on [8] for the following result.

If A is a continuous linear operator from $C(S)_\beta$ into $C(T)_\beta$ then there is a unique mapping $\lambda : T \to M(S)$, henceforth called the kernel of A, such that

$$[Af](x) = \int_S f(y)\lambda(x)(dy) = \int_S f(y)\lambda(x, dy)$$

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for all $f \in C(S)$ and $x \in T$. This last integral will be denoted by $\lambda(f)(x)$. Furthermore, $(A^*\mu)(E) = \int_T \lambda(x, E)\mu(dx)$ when T is locally compact and Hausdorff and $\mu \in M(T)$, while for a bounded Borel measurable function $f \in M(S)^*$, $[A^{**}f](x) = \lambda(f)(x)$ is a bounded Borel function on T. When A is a weakly compact operator, the function $A^{**}f \in C(T)$ and consequently the kernel λ is a weakly continuous kernel as defined in [9]. This, along with the work in [9] on such kernels, motivates the work herein. Finally, we remark that the topology on $C(S)$ denoted by β' in [8] coincides with β as was shown by Dorroh [4].

In the sequel, T is a locally compact Hausdorff space as is S. We begin with a classification of the operators on a space $C(S)_\beta$ into $C(T)_\beta$ essentially established in [8] and [9]. Briefly, the type of operator with kernel λ is determined by the peoperties of the sets $\lambda(K) = \{\lambda(x) : x \in K\}$ where K is a compact subset of T.

THEOREM 1. *Let A be a linear mapping of $C(S)_\beta$ into $C(T)_\beta$ given by a kernel λ. Then*

(1) *A is continuous on $C(S)_\beta$ into $C(T)_\beta$ if and only if each set $\lambda(K)$, for K compact in T, is β-equicontinuous. That is, gien $\varepsilon > 0$ and K compact in T there is a compact set $Q \subset S$ such that $|\lambda|(x, S\backslash Q) < \varepsilon$ for all $x \in K$.*

(2) *A is weakly compact if and only if each set $\lambda(K)$ is weakly compact in $M(S)$ for K a compact subset of T. That is, given $\varepsilon > 0$, K compact in T and U any open subset of S, there is a compact set $Q \subset U$ such that $|\lambda|(x, U\backslash Q) < \varepsilon$ for all $x \in K$.*

(3) *A is compact if and only if each set $\lambda(K)$, for K compact in T, is compact in $M(S)$.*

Proof. (1) follows immediately from [8, Theorem 5]. That the set of measures $\lambda(K)$ satisfies the measure theoretic properties stated in (1) follows from [3, Theorem 2].

(2) follows immediately from [9, Theorem 2] and the fact that the norm and β-bounded sets coincide. The stated measure theoretic property is a consequence of [9, Theorem 2, part 3].

(3) follows from [9, Theorem 3].

As a consequence of the above, if $\mu \in M(S)$ and k is a real or complex function on $T \times S$ such that $k(x, \cdot) \in L^1(\mu)$ for all $x \in T$ and k is uniformly bounded on $K \times S$ for each compact set $K \subset T$ and

$$[Af](x) = \int_S f(y)k(x, y)\mu(dy)$$

is continuous on T, then A is a continuous weakly compact operator on $C(S)_\beta$ into $C(T)_\beta$. For its kernel, $\lambda(x, E) = \int_E k(x, y)\mu(dy)$, satisfies the measure theoretic properties in (2) since given K compact in T, $\varepsilon > 0$, and U open in S there is a compact set $Q \subset U$ such that $|\mu|(U\backslash Q) < \varepsilon/\gamma$, where

$$\gamma = \sup\{|k(x, y)| : (x, y) \in K \times S\}$$

so that

$$|\lambda|(x, U\backslash Q) \leq \int_{U\setminus Q} |k(x, y)||\mu|(dy) < \varepsilon \quad \text{for all} \quad x \in K.$$

THEOREM 2. *Let A be a linear mapping of $C(S)$ into $C(T)$. Then A is β-weakly compact if and only if A is weakly compact and continuous from $C(S)_\beta$ to $(C(T), \|\ \ \|)$.*

Proof. Suppose A is β-weakly compact. Then there is a β-neighborhood V of 0 such that $A(V)$ is weakly relatively compact in $C(T)$ and hence weakly bounded and therefore norm bounded. Consequently, A is continuous from $C(S)_\beta$ to $(C(T), \|\ \ \|)$. Furthermore, since β-bounded sets are absorbed by V, the image under A of a β-bounded set is absorbed by the weakly relatively compact set $A(V)$ making A weakly compact.

Conversely, suppose A is weakly compact and continuous into $C(T)$ with the supremum norm topology. Then A^* maps equicontinuous sets of $(C(T), \|\ \ \|)^*$ into equicontinuous sets of $C(S)_\beta^*$. Let λ be the kernel of A as described above. With $\mathring{x}$ denoting the unit point measure concentrated at $x \in T$ the set $\{\mathring{x} : x \in T\}$ is equicontinuous in $(C(T), \|\ \ \|)^*$ and consequently

$$\{\lambda(x) : x \in T\} = A^*\{\mathring{x} : x \in T\}$$

is β-equicontinuous in $M(S)$.

It follows from [3, Theorem 1] that there is a non-negative function $\phi \in C_0(S)$ such that each measure $\lambda(x)$ vanishes off the non-zeroes of ϕ and

$$\|(1/\phi)\cdot\lambda(x)\| \leq 1$$

where the symbol on the left is the total variation on S of the measure $(1/\phi)\cdot\lambda(x)$ defined by

$$\left[\frac{1}{\phi}\cdot\lambda(x)\right](E) = \int_E \frac{1}{\phi(y)}\,\lambda(x, dy)$$

for Borel sets E.

We set $\sigma(s) = \phi(s)^{1/3}$ for all $s \in S$ and define $\mu : T \to M(S)$ by

$$\mu(x, E) = \int_E \frac{1}{\sigma(y)}\,\lambda(x, dy) = \left[\frac{1}{\sigma}\cdot\lambda(x)\right](E).$$

We first show that $\sup\{\|\mu(x)\| : x \in T\} < \infty$. To see this let

$$W = \{s : \phi(s) \geq 1\}.$$

Then

$$\begin{aligned}\|\mu(x)\| &= |\mu|(x, S)\\ &= \int_W \frac{1}{\sigma(y)}|\lambda|(x, dy) + \int_{S\setminus W} \frac{1}{\sigma(y)}|\lambda|(x, dy)\\ &\leq |\lambda|(x, W) + \int_{S\setminus W} \frac{1}{\phi(y)}|\lambda|(x, dy)\end{aligned}$$

$$\leq |\lambda|(x, W) + \left\| \frac{1}{\sigma} \cdot \lambda(x) \right\|$$

$$\leq \sup \{\| \lambda(x) \| : x \in T\} + 1 < \infty$$

this last inequality following from the uniform boundedness principle applied to the measures $\lambda(x)$ as functionals on $C_0(S)$.

We will now show that the function $\mu(\cdot, E)$ is continuous on T for each Borel set $E \subset S$.

Let $N(\phi) = \{s \in S : \phi(s) > 0\}$ and

$$W_n = \{s \in S : 1/(n+1)^3 \leq \phi(s) < 1/n^3\} \quad \text{for} \quad n = 1, 2, \cdots$$

and let W be defined as above. Then,

$$S = W \cup \bigcup_{k=1}^{\infty} W_k \cup S \backslash N(\phi), \qquad |\lambda|(x, S\backslash N(\phi)) = 0$$

as noted above, and if $s \in W_n$, then $1/(n+1) \leq \sigma(s) < 1/n$.

If $V_n = \{s \in S : n^3 < 1/\phi(s)\}$ then $W_n \subset V_n$ and $|\lambda|(x, V_n) \leq 1/n^3$ as a consequence of the inequality $\|(1/\phi) \cdot \lambda(x)\| \leq 1$. Consequently,

$$|\lambda|(x, W_n) \leq 1/n^3.$$

For any Borel set E,

$$\begin{aligned}
\mu(x, E) &= \int_E \frac{1}{\sigma(y)} \lambda(x, dy) \\
&= \int_{E \cap W} \frac{1}{\sigma(y)} \lambda(x, dy) + \sum_{k=1}^{\infty} \int_{E \cap W_n} \frac{1}{\sigma(y)} \lambda(x, dy) \\
&= \int_S \left(\frac{\chi_{E \cap W}}{\sigma} \right)(y) \lambda(x, dy) + \sum_{k=1}^{\infty} \int_S \left(\frac{\chi_{E \cap W_k}}{\sigma} \right)(y) \lambda(x, dy)
\end{aligned}$$

where $\chi_F(s) = 1$ if $s \in F$, 0 if $s \notin F$.

The function $g_n(s) = (\chi_{E \cap W_n}/\sigma)(s)$ for $\sigma(s) \neq 0$ and $g_n(s) = 0$ if $\sigma(s) = 0$ is a bounded Borel measurable function on S. Similarly for

$$g(s) = (\chi_{E \cap W}/\sigma)(s)$$

if $\sigma(s) \neq 0$, 0 if $\sigma(s) = 0$. Finally note that

$$\mu(x, E) = \lambda(g)(x) + \textstyle\sum_{k=1}^{\infty} \lambda(g_k)(x).$$

We will show that the functions $\lambda(g)$ and $\lambda(g_n)$, $n = 1, 2, \cdots$ are continuous on T and that the convergence is uniform.

First, A being weakly compact with range $C(T)_\beta$ implies that the kernel λ of A satisfies condition (5) of Theorem 2 in [9] and hence also condition (7) which says that $\lambda(f)$ is continuous on T for any bounded Borel function f on S. Hence $\lambda(g)$ and $\lambda(g_n)$ are continuous.

Finally,

$$\begin{aligned}
\lambda(g_n)(x) = \int_{E \cap W_n} \frac{1}{\sigma(y)} \lambda(x, dy) &\leq \int_{E \cap W_n} (n+1) |\lambda| (x, dy) \\
&\leq (n+1) |\lambda| (x, W_n) \leq 1/n^2 + 1/n^3
\end{aligned}$$

and the convergence of $\sum_{k=1}^{\infty} \lambda(g_k)$ is uniform making $\mu(\cdot, E)$ continuous on T.

From this it easily follows that $\mu(f)(x) = \int_S f(y)\mu(x, dy)$ is continuous on T for each bounded Borel function f on S. Hence μ is a kernel on T into $M(S)$ which satisfies condition (7) of Theorem 2 in [9] and so also condition (5). That is,

$$C = \{\mu(f) : f \in C(S), \|f\| \leq 1\}$$

is a weakly relatively compact set in $C(T)_\beta$.

Let $V = \{f \in C(S) : \|f\sigma\| \leq 1\}$. Then V is a β-neighborhood of 0 in $C(S)$ and if $f \in V$ then

$$Af = \lambda(f) = \lambda(f\sigma/\sigma) = \mu(f\sigma) \in C.$$

Consequently, the set $A(V) \subset C$ is weakly relatively compact in $C(T)_\beta$ completing the proof.

Remark 1. It is easy to see that the condition that A be continuous into $(C(T), \|\ \ \|)$ and weakly compact could be replaced by the condition that A have a kernel λ which satisfies any one of the conditions of Theorem 2 in [9] and such that $\{\lambda(x) : x \in T\}$ be β-equicontinuous.

When the underlying space S is compact, the β and norm topologies coincide and consequently so do the β-weakly compact and weakly compact operators. Surprisingly, it is easy to see that the same condition holds when T, rather than S, is compact.

COROLLARY 1. *Let T be compact and let A be a continuous linear operator from $C(S)_\beta$ into $C(T)$. Then A is β-weakly compact if and only if A is weaklu compact.*

We now replace the space $C(T)$ by an arbitrary Banach space X to achieve the same result.

COROLLARY 2. *Let A be a continuous linear operator from $C(S)_\beta$ into a Banach space X. Then A is β-weakly compact if and only if A is weaklu compact.*

Proof. Let T denote the unit ball in X^* with the weak* topology so that T is compact. For $x \in X$ let $\gamma(x)$ be the restriction of x, as a function on X^*, to the space T. Let $Bf = \gamma(Af)$ for $f \in C(S)$. Then B is a continuous weakly compact operator on $C(S)_\beta$ into $C(T)$. By Corollary 1, B is β-weakly compact. There then is a β-neighborhood V of 0 in $C(S)$ such that $B(V)$ is weakly relatively compact in $C(T)$. Since γ is an isometry and consequently $\gamma(X)$ is weakly closed in $C(T)$, this means $A(V) = \gamma^{-1}(B(V))$ is weakly relatively compact in X. Since the converse is clear, this completes the proof.

It is easy to see that the hypothesis of continuity of A on $C(S)_\beta$ cannot be dropped. All one need have is a bounded linear functional on $(C(S), \|\ \ \|)$ which is not continuous on $C(S)_\beta$. Such a functional is even compact but not β-weakly compact and is easily found.

Certainly the above proof will not hold unless X is a complete topological vector space. The proof also strongly uses the hypothesis that X be a normed space. The following corollary shows that this hypothesis is not necessary, provided that S is paracompact.

COROLLARY 3. *Suppose S is a paracompact space and X is a Banach space. Let Y denote the space X with the weak topology. Then any weakly continuous weakly compact linear mapping A of $C(S)_\beta$ into Y is β-weakly compact.*

Proof. For when S is paracompact the β topology on $C(S)$ is the Mackey topology on $C(S)$ as was shown by Conway [3] and so by [7, p. 62] A is continuous on $C(S)_\beta$ into the Banach space X. Since A is weakly compact an appeal to Corollary 2 completes the proof.

Before considering the case of compact operators on $C(S)$ we state two results on weakly compact operators on $C_0(S)$ and their extension to $C(S)$.

THEOREM 3. *Let A be a weakly compact operator on $C_0(S)$ into $C(T)_\beta$. Then A has a unique extension to a continuous operator on $C(S)_\beta$. Furthermore, this extension is weakly compact on $C(S)_\beta$.*

Proof. By [8, Theorem 3] the operator A can be represented by a kernel $\lambda : T \to M(S)$ such that $Af = \lambda(f)$ for all $f \in C_0(S)$. Since A is weakly compact, the kernel λ satisfies condition (6) in [9, Theorem 2] and consequently condition (3). But this means λ satisfies E (See [8, Remark 5]) and consequently the map $f \to \lambda(f)$ for $f \in C(S)$ defines a continuous operator B on $C(S)_\beta$ into $C(T)_\beta$ which extends A uniquely. Since λ satisfies condition (5) in [9, Theorem 2] and the β and norm bounded sets in $C(S)_\beta$ coincide, the operator B is weakly compact.

COROLLARY 4. *Let A be a continuous weakly compact mapping of $C_0(S)$ into a Banach space X. Then A has a unique extension to a β-weakly compact operator on $C(S)_\beta$ into X.*

Proof. Define the space T and the operators γ and B as in the proof of Corollary 2. By Theorem 3 and Corollary 1, B has a unique extension to a β-weakly compact operator B' on $C(S)_\beta$ into $C(T)$. Furthermore, since $\gamma(X)$ is closed in $C(T)$ and $C_0(S)$ is β-dense in $C(S)$ one has $B'(C(S)) \subset \gamma(X)$ and hence that $A'f = \gamma^{-1}B'f$ is a unique β-weakly compact extension of A to $C(S)_\beta$ into X.

Finally, certain known results for weakly compact operators on $C_0(S)$ and on the space of continuous functions on S with the compact open topology have analogues for $C(S)_\beta$. One of these is that if T is σ-compact and A has kernel λ and is a weakly compact operator, then $\lambda(x, E) = \int_E k(x, y)\mu(dy)$ for some nonnegative bounded Borel measure μ on S and function k on $T \times S$ such that $k(x, \cdot) \in L^1(\mu)$ for all $x \in T$; consequently

$$[Af](x) = \int_S f(y)k(x, y)\mu(dy).$$

This result is the analogue of the result in [6, p. 665] and follows from [9, Theorem 2, part 2] and [6, p. 287].

If A maps real functions into real functions then $\lambda(x)$ is a real signed measure on S for each $x \in T$. Hence $\lambda(x) = \lambda(x)^+ - \lambda(x)^-$ where $\lambda(x)^+$, $\lambda(x)^-$ are non-negative measures such that $|\lambda|(x, S) = \lambda^+(x, S) + \lambda^-(x, S)$. Setting $A^+f = \lambda^+(f)$, $A^-f = \lambda^-(f)$ defines positive operators on $C(S)$ such that $A = A^+ - A^-$ and such that $|\lambda|(x, S) = (A^+1)(x) + (A^-1(x)$. If this last function is continuous on T then $|\lambda|(\cdot, E)$ is continuous on T for all Borel sets E since $|\lambda|(\cdot, E)$ is lower semicontinuous because

$$|\lambda|(\cdot, E) = \sup \{\textstyle\sum_{i=1}^n |\lambda(\cdot, E_i)| : \{E_i\}_{i=1}^n \text{ is a partition of } E \text{ by Borel sets}\}$$

and also upper semicontinuous because

$$|\lambda|(\cdot, E) = |\lambda|(\cdot, S) - |\lambda|(\cdot, S\backslash E).$$

Hence by [9, Theorems 1 and 2],

$$\lambda^+(x) = (\lambda(x) + |\lambda|(x))/2 \quad \text{and} \quad \lambda^-(x) = (|\lambda|(x) - \lambda(x))/2$$

are kernels defining weakly compact operators on $C(S)_\beta$ into $C(T)_\beta$. That is, if A is continuous and weakly compact and

$$\sup \{|(Af)(x)| : \|f\| \leq 1\} = |\lambda|(x, S)$$

is continuous on T, then A is the difference $A^+ - A^-$ of positive continuous weakly compact operators on $C(S)_\beta$ into $C(T)_\beta$ such that

$$\sup \{|(Af)(x)| : \|f\| \leq 1\} = (A^+1)(x) + (A^-1)(x).$$

It is easily seen that the converse statement holds.

We now turn to a consideration of compact operators on $C(S)_\beta$ and prove the analogue of Theorem 1.

THEOREM 4. *Let A be a linear mapping of $C(S)_\beta$ into $C(T)_\beta$. Then A is β-compact if and only if A is compact with the β-topology on $C(T)$ and continuous with the norm topology on $C(T)$.*

Proof. One implication is clear. For the converse, suppose A is compact and continuous with the norm topology on $C(T)$. Let λ denote the kernel of A. As in the proof of Theorem 2, $\lambda(T)$ is a β-equicontinuous set. We define σ from the function ϕ obtained as before and again set

$$\mu(x, E) = \int_E \frac{1}{\sigma(y)} \lambda(x, dy).$$

Because λ here satisfies stronger conditions than in the proof of Theorem 2, μ is a kernel by that proof. We will show that the mapping $x \to \mu(x)$ is continuous with the norm topology on $M(S)$.

Define the sets W_n and W as before and set

$$\nu_B(x, E) = \int_{E \cap B} \frac{1}{\sigma(y)} \lambda(x, dy)$$

for B a Borel set. Then, $\mu(x, E) = \nu_W(x, E) + \sum_{n=1}^{\infty} \nu_{W_n}(x, E)$ for each $x \varepsilon T$ and Borel set E. If B is one of the sets W or W_{n+1} then

$$\| \nu_B(x) - \nu_B(x_0) \| \leq a_B \| \lambda(x) - \lambda(x_0) \|$$

where $a_B = 1$ if $B = W$ and $n + 1$ if $B = W_n$. Since A is a compact operator, the kernel λ satisfies condition (3) in [9, Theorem 3] so that $x \to \lambda(x)$ is continuous with the norm topology on $M(S)$ and from our above inequality, so is $x \to \nu_B(x)$.

We now show that the convergence of the above series is uniform on T. We have

$$\begin{aligned} \| \nu_{W_n}(x) \| &= \sup \{ | \textstyle\int_S f(y) \nu_{W_n}(x, dy) | : \| f \| \leq 1 \} \\ &= \sup \{ | \textstyle\int_{W_n} f(y)/\sigma(y) \lambda(x, dy) | : \| f \| \leq 1 \} \\ &\leq (n+1) | \lambda | (x, w_n) \leq 1/n^2 + 1/n^3. \end{aligned}$$

Hence μ is the uniform limit of continuous functions on T and is continuous. Hence by (3), Theorem 1, $f \to \mu(f)$ is a compact operator on $C(S)_\beta$ into $C(T)_\beta$. Hence $C = \{\mu(g) : \| g \| \leq 1\}$ is relatively compact in $C(T)_\beta$ and $\| f\sigma \| \leq 1$ implies $Af = \lambda(f) = \mu(f\sigma) \in C$ so that A maps a β-neighborhood in $C(S)_\beta$ into a relatively compact set in $C(T)_\beta$.

A slight modification of the proof of Corollary 2 yields its analogue for compact operators.

COROLLARY 5. *A linear mapping of $C(S)_\beta$ into a Banach space X is β-compact if and only if it is continuous and compact.*

Finally, one may apply [9, Theorem 3] to obtain analogues of Theorem 3 and Corollary 4 for the extension of compact operators on $C_0(S)$ to $C(S)_\beta$.

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