COMPACTNESS AND CONVERGENCE IN THE SPACE OF MEASURES

BY
F. DENNIS SENTILLES

I. Introduction

Let S be a locally compact Hausdorff space and T a topological space. A kernel on T into the space $M(S)$ of bounded regular Borel measures on S with variation norm is a mapping $\lambda: T \to M(S)$ which is weak* bounded and continuous. Equivalently,

$$\lambda(f)(x) = \int_S f(y)\lambda(x)(dy) = \int_S f(y)\lambda(x, dy)$$

is a bounded continuous function on T for each $f \in C_0(S)$, the set of continuous functions on S which vanish at infinity. Such kernels have been used in the integral representation of operators on $C_0(S)$ and the space $C(S)$ of bounded continuous functions on S and less general kernels have been extensively studied in probability and potential theory.

A study will be made of mappings λ which are (weakly) continuous into $M(S)$. A consideration of several diverse examples will show that the results obtained include (and improve) as special cases most of the important results on (weak) compactness and convergence in $M(S)$. Furthermore, the results herein are consequences of but [3] and [7] along with certain results of a general nature.

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II. Definitions and notation

If $\lambda: T \to M(S)$ we denote by $\lambda(x, E) = \lambda(x)(E)$ the value of the measure $\lambda(x)$ at the Borel set E and set $|\lambda|(x, E) = |\lambda(x)|(E)$, the variation of $\lambda(x)$ on E. We will call λ a (weakly) continuous kernel if λ is continuous in the (weak) strong topology on $M(S)$. Notice that for any kernel λ,

$$\|\lambda\| = \sup\{\|\lambda(x)\| : x \in T\} < \infty.$$

If T is a locally compact Hausdorff space and $\mu \in M(T)$ the formula $\lambda(\mu)(E) = \int_T \lambda(x, E)\mu(dx)$ defines an element of $M(S)$ (see [7]). If $H \subset M(T)$ let

$$\lambda(H) = \{\lambda(\mu) : \mu \in H\}$$

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and if $K \subset T$ let

$$\lambda(K) = \{\lambda(x): x \in K\}.$$

Let $\mathring{x}(E) = 1$ if $x \in E$, 0 if $x \notin E$ and notice that $\lambda(x) = \lambda(\mathring{x})$.

A set $N \subset M(S)$ is uniformly inner regular (uniformly outer regular) if given $\varepsilon > 0$ and U an open subset of S (P a compact subset of S) there is a compact set $Q \subset U$ (an open set $V \supset P$) such that

$$|\mu|(U\backslash Q) < \varepsilon(|\mu|(V\backslash P) < \varepsilon)$$

for all $\mu \in N$.

The β or strict topology on $C(S)$ is that locally convex topology on $C(S)$ defined by the seminorms

$$P_\phi(f) = \sup\{|f(x)\phi(x)|: x \in S\} \quad \text{for} \quad \phi \in C_0(S).$$

The space $C(S)_\beta$ is studied in [1], [2], [3] and [7] and is known to have dual space $M(S)$. In particular, Conway [3, Theorem 2] has shown that a bounded subset H in $M(S)$ is β-equicontinuous if and only if given $\varepsilon > 0$ there is a compact set $R \subset S$ such that $|\mu|(S\backslash R) < \varepsilon$ for all $\mu \in H$. It is easy to see that a bounded set $N \subset M(S)$ is uniformly inner regular if and only if N is uniformly outer regular and β-equicontinuous. Both of these statements are equivalent to the weak relative compactness of N as was shown by Grothendieck [6] and since obtained by Conway using his work in [3]. Further terminology and notation can be found in [7].

Finally, a Borel set E will be called a strict G_δ if E is closed and its complement is σ-compact.

III. Examples

(1) Any kernel λ defines a bounded linear operator $f \to \lambda(f)$ from $C_0(S)$ into $C(T)$. Conversely, any such operator A has a kernel representation λ such that $Af = \lambda(f)$. Furthermore, the mapping $\mu \to \lambda(\mu)$ defines a bounded linear map of $M(T)$ into $M(S)$ by [7], and under the conditions given in [7] the converse is valid.

(2) Let $\{\mu_n\}$ be a sequence in $M(S)$ which is weak* convergent to a measurement $\mu \in M(S)$. Let T denote the one point compactification of the integers with the discrete topology and ω the point at infinity. Let $\lambda(n, E) = \mu_n(E)$, $\lambda(\omega, E) = \mu(E)$ for Borel sets E. Since $\lambda(f)(n) = \int_S f\, d\mu_n$, λ is a kernel on T.

(3) Let $\{\mu_\alpha\}$ be a bounded net in $M(S)$ and $\mu \in M(S)$. Let

$$T = \{\mu\} \cup \{\mu_\alpha\}$$

and let $\mathfrak{F}$ be a collection of bounded Borel functions on S. Give T the discrete topology at points $\nu \neq \mu$ and let the collection of sets

$$\omega(f, \varepsilon) = \left\{\nu \in T: \left|\int_S f\, d\nu - \int_S f\, d\mu\right| < \varepsilon\right\} \quad \text{for } f \in \mathfrak{F} \text{ and } \varepsilon > 0$$

be a subbase for the neighborhood system at μ. Define $\lambda(\nu, E) = \nu(E)$ for $\nu \in T$. Then $\lambda(f)(\nu) = \int_S f\, d\nu$ and λ is a kernel when $\mu_\alpha \to \mu$ weak* and $\mathfrak{F} \supset C_0(S)$.

(4) Let N be a bounded subset of $M(S)$ and let T denote the set N with the relative weak* topology. Setting $\lambda(\nu, E) = \nu(E)$ for $\nu \in T$ defines a kernel with range N.

(5) Let G be a locally compact group and set $\lambda(x, E) = \mu(Ex^{-1})$ for $\mu \in M(G)$ and $x \in G$. Then

$$\lambda(f)(x) = \int_G f(yx)\mu\,(dy) \quad \text{and} \quad \lambda(\nu)(E) = (\mu * \nu)(E)$$

since λ is a kernel.

(6) Let μ be a regular Borel measure on S and let P be a bounded subset of $L^1(\mu)$. Let T be the set P with the weak topology and define $\lambda(g, E) = \int_E g\, d\mu$ for $g \in T$. Then for $f \in L^\infty(\mu)$, $\lambda(f)(g) = \int_S f \cdot g\, d\mu$ and λ is a kernel.

IV. Main Results

We take note of a lemma needed to prove

THEOREM 1. *Let T be a topological space, $\lambda: T \to M(S)$ with $\|\lambda\| < \infty$. If $\lambda(\cdot, C)$ is continuous on T for each strict G_δ set C, then λ is a kernel on T and the set $\lambda(K)$ is weakly compact in $M(S)$ for each compact set $K \subset T$.*

LEMMA 1. *Let λ be as in Theorem 1. Then $\lambda(\cdot, U)$ is continuous for each open set U if and only if $\lambda(f) \in C(T)$ for all bounded lower semicontinuous functions f on S.*

The proof is standard and we omit it.

Proof of Theorem 1. We begin by showing that λ is a kernel on T.

Suppose $f \in C_0(S)$ and $0 \le f \le 1$, and let

$$A_{k,n} = \{s \in S : f(s) \le k/n\} \quad \text{for} \quad k = 0, 1, \cdots, n.$$

Each set $A_{k,n}$ is closed and

$$S \backslash A_{k,n} = \textstyle\bigcup_{i=1}^{\infty} \{s \in S : k/n + 1/i \le f(s)\}$$

so that $A_{k,n}$ is a strict G_δ. Consequently, $\lambda(\cdot, A_{k,n})$ is continuous on T, and $\lambda(f)$ is the uniform limit of continuous functions

$$\textstyle\sum_{k=1}^{n} k/n\lambda(\cdot, A_{k,n} \backslash A_{k-1,n}) \quad \text{for} \quad n = 1, 2, \cdots.$$

Let K be a fixed compact subset of T. Since λ is a kernel the set $\lambda(K)$ is weak* compact and hence weakly closed. By Eberlein's theorem the set $\lambda(K)$ will be weakly compact provided a sequence $\{\lambda(x_n)\} \subset \lambda(K)$ is weakly relatively compact and by [6, Theorem 2] and our remarks above, it suffices to show that the set $\{\lambda(x_n)\}$ is uniformly inner regular. This is what we will prove.

The sequence $\{x_n\}$ has a cluster point $x \in K$ and a convergent subnet $\{x_\alpha\} \subset K$. Notice that $\{x_\alpha\} \subset \{x_n\}$. Set $\mu_0 = \lambda(x)$ and $\mu_n = \lambda(x_n)$. We will show that $\lambda(x_\alpha, U) \to \lambda(x, U)$ for an open set U.

For a fixed open set U we construct by induction and the inner regularity of the measures $\{\mu_n : n = 0, 1, 2, \cdots\}$ a sequence of compact sets $Q_n \subset U$ such that $|\mu_k|(U \backslash Q_n) \leq 1/n$ for $k = 0, 1, \cdots, n$ and such that Q_n is a subset of the interior of Q_{n+1}. The set $V = \bigcup_{k=1}^{\infty} Q_k$ is then open and its complement C is a strict G_δ. Furthermore, $|\mu_k|(U \backslash V) = 0$ for $k = 0, 1, \cdots$.

Since $\lambda(\cdot, S)$ and $\lambda(\cdot, C)$ are continuous so is $\lambda(\cdot, V)$. Hence

$$\lambda(x_\alpha, U) = \lambda(x_\alpha, V) \to \lambda(x, V) = \lambda(x, U)$$

since $\{x_\alpha\} \subset \{x_n\}$. By Lemma 1, $\lambda(f)(x_\alpha) \to \lambda(f)(x)$ for all bounded lower semicontinuous functions f on S.

To show that $\{\lambda(x_n)\}$ is uniformly inner regular with respect to open sets, again let U be a fixed open set and construct V as above.

Let ν_n be the restriction of μ_n to V so that $\nu_n \in M(V)$. By [3, Theorem 2] one obtains uniform inner regularity on V by showing that $\{\nu_n : n = 1, 2, \cdots\}$ is β-equicontinuous as a subset of the dual of $C(V)_\beta$. Since V is σ-compact it suffices to show that $\{\nu_n\}$ is β-weak* compact, referring to [3, Theorem 4]. We obtain this by showing that ν_0 is a β-weak* cluster point of $\{\nu_n\}$.

If $g \in C(V)$ and $g \geq 0$, set $f(s) = g(s)$ for $s \in V$ and $f(s) = 0$ for $s \notin V$. Since f is a lower semicontinuous function on S,

$$\lambda(f)(x_\alpha) \to \lambda(f)(x) \quad \text{and} \quad \lambda(f)(x_n) = \int_S f \, d\mu_n = \int_V g \, d\nu_n$$

so that a subnet of $\{\int_V g \, d\nu_n\}$ converges to $\int_V g \, d\nu_0$. This completes the proof.

Remark 1. We point out that the result taken from [6] which we use above can be obtained from the work in our other reference [3] as is noted therein. Consequently, Theorem 1 relies only on [3].

Our next result completely characterizes weakly continuous kernels on a locally compact space T. Its proof relies on [3], [7] and a basic result on weakly compact operators.

THEOREM 2. *Let $\lambda: T \to M(S)$ be a kernel and suppose that T is a locally compact Hausdorff space. The following are equivalent.*

(1) *λ is a weakly continuous kernel.*

(2) *For each compact set $K \subset T$ the set $\lambda(K)$ is weakly compact in $M(S)$.*

(3) *The set $\lambda(K)$ is β-equicontinuous and uniformly outer regular for each compact subset K of T.*

(4) *The mapping $\mu \to \lambda(\mu)$ maps β-equicontinuous sets in $M(T)$ into weakly relatively compact subsets of $M(S)$.*

(5) *The set* $\{\lambda(f): f \in C(S), \|f\| \leq 1\}$ *is weakly relatively compact in* $C(T)_\beta$.

(6) *The set* $\{\lambda(f): f \in C_0(S), \|f\| \leq 1\}$ *is a weakly relatively compact subset of* $C(T)_\beta$.

(7) *For all bounded Borel functions* f, $\lambda(f) \in C(T)$.

(8) $\lambda(\cdot, C)$ *is continuous on* T *for all strict* G_δ *sets* C.

Proof. The equivalence of (2) and (3) follows from [6, Theorem 2] or [3] (see Remark 1). Clearly (1) implies (2) since λ is weakly continuous. To see that (2) implies (1), let K be a compact set and notice that the weak and weak* topologies agree on the weakly compact set $\lambda(K)$ and λ, being a kernel, is weak* continuous on K. To obtain continuity at a point $x \in T$ let K be the closure of a neighborhood of x having compact closure.

Let us show that (3) implies (4). We again use Grothendieck's characterization of weak compactness in $M(S)$. If H is a β-equicontinuous set in $M(T)$ then there is a compact set $K \subset T$ such that

$$|\mu|(T\backslash K) < \varepsilon(2\|\lambda\|)^{-1} \qquad \text{for all} \quad \mu \in H.$$

Given a compact set $Q \subset S$, there is, by (3) and our earlier remarks, an open set $U \supset Q$ such that

$$|\lambda|(x, U\backslash Q) < \varepsilon/2\alpha \quad \text{where} \quad \alpha = \sup\{\|\mu\|: \mu \in H\} \qquad \text{for all} \quad x \in K\cdot$$

Hence for $\mu \in H$ one has

$$\begin{aligned} &|\lambda(\mu)|(U\backslash Q) \\ &\leq \int_K |\lambda|(x, U\backslash Q)|\mu|(dx) + \int_{T\backslash K} |\lambda|(x, U\backslash Q)|\mu|(dx) \\ &\leq \varepsilon/2\alpha|\mu|(K) + \|\lambda\|\,|\mu|(T\backslash K) < \varepsilon. \end{aligned}$$

Hence $\lambda(H)$ is uniformly outer regular and by [6] weakly relatively compact.

Assume that (4) holds. The set $H = \{\mathring{x}: x \in K\}$ is β-equicontinuous for K a compact subset of T and consequently $\lambda(H) = \lambda(K)$ is weakly relatively compact and hence β-equicontinuous. It follows from [7, Remark 5] that the formula $Af = \lambda(f)$ for $f \in C(S)$ defines a continuous linear operator A from $C(S)_\beta$ into $C(T)_\beta$ with $A^*\mu = \lambda(\mu)$ for $\mu \in M(T)$. Hence by (4), A^* maps equicontinuous sets into weakly relatively compact sets and consequently by [5, Theorem 9.3.1] A maps the β-bounded set $\{f \in C(S): \|f\| \leq 1\}$ into a weakly relatively compact set in $C(T)_\beta$ proving (5).

Clearly (5) implies (6). Given (6) define A as above for $f \in C_0(S)$. Then A is weakly compact and hence $A^{**}(C_0(S)^{**}) \subset C(T)$. But by [7, Theorem 3], $A^{**}f = \lambda(f)$ for f a bounded Borel function on S proving (7).

Since (7) readily implies (8) and (8) implies (2) by Theorem 1, this completes the proof.

Remark 2. The overriding hypothesis that λ be a kernel may be dropped and instead be required as part of the hypothesis in (2), (3) and (4).

Several interesting questions arise from Theorem 2. First, can condition (8) be formally weakened? Given a particular kernel λ what properties must the collection $\Sigma = \{ : \lambda(\cdot, E)$ is continuous$\}$ have in order that λ be weakly continuous? This question is related to the work of Dieudonne [4] and Conway [3] on quarrable sets. Finally, Dieudonne [4] gives conditions under which convergence of sequences of measures on certain classes of Borel sets implies that the sequence is uniformly bounded. We have found it necessary to assume that $\| \lambda \| < \infty$.

Notice that condition (3) involves only the variation $|\lambda(x)|$ of the measures $\lambda(x)$, save that λ be a kernel. One is led to ask then whether (say) condition (3) is equivalent to the continuity of $x \to |\lambda|(x, C)$ for strict G_δ sets C? The answer is no, essentially because either λ or $|\lambda|$ can be a weakly continuous kernel without the other being even a kernel.

At this point we consider only measures $\lambda(x)$ which are real-valued. Let T be the interval $[0, 1]$ and S a two point space $\{a, b\}$ with the discrete topology. For $x \neq 0$ let

$$\lambda(x, \{a\}) = 1, \quad \lambda(x, \{b\}) = -1 \quad \text{and} \quad \lambda(0, \{a\}) = \lambda(0, \{b\}) = 1.$$

Then $\lambda(x, S) = 0$, $|\lambda|(x, S) = 2$ and $|\lambda|(\cdot, C)$ is continuous for strict G_δ sets C while λ is not a kernel.

As a second example we define a weakly continuous kernel λ such that $|\lambda|$ is not a kernel. Let $T = [0, 1]$, $S = [0, 2\pi]$ and define

$$\lambda(x, E) = \int_E \cos t/x \, dt \quad \text{for} \quad x \neq 0$$

and $\lambda(0, E) = 0$. If $E = [a, b]$,

$$\lambda(x, E) = x[\sin b/x - \sin a/x] \quad \text{and} \quad |\lambda|(2^{-n}, S) = 4$$

for $n = 1, 2, \cdots$ so that $|\lambda|(\cdot, S)$ is not continuous on T. It is easy to see that $\lambda(\cdot, U)$ is continuous for all open sets U so that λ is a weakly continuous kernel by Theorems 1 and 2.

Finally, condition (3) of Theorem 2 does tell us that if both λ and $|\lambda|$ are kernels then λ is weakly continuous if and only if $|\lambda|$ is weakly continuous.

Before considering strongly continuous kernels we see how the above theorems may be applied to our earlier examples.

COROLLARY 1. *A bounded subset of $M(S)$ is weakly relatively compact if and only if it is relatively compact for the weak topology τ on $M(S)$ defined by the collection of characteristic functions of strict G_δ sets.*

Proof. Let N be such a set and define λ on T to $M(S)$ as in example (4) where T is the τ closure of N with topology τ. Clearly λ satisfies the hypothesis of Theorem 1 and consequently $\lambda(T)$ is weakly compact and contains N when N is τ relatively compact. The converse is clear.

The corollary is an improvement on the result in [5, p. 284] and can also be used to generalize results on weakly convergent sequences to weakly relatively compact nets $\{\mu_\alpha\} \subset M(S)$. One can also use examples (6) and (3) of Theorem 2 to obtain a result for weak compactness in a space $L^1(\mu)$ similar to the Dunford-Pettis theorem. Other similar observations can be made.

We now turn to a brief study of strongly continuous kernels and prove a result similar to Theorem 2.

THEOREM 3. *Let* $\lambda: T \to M(S)$ *be a kernel and* T *a locally compact Hausdorff space. The following are equivalent.*

(1) λ *is a strongly continuous kernel.*

(2) *For each compact set* $K \subset T$ *the set* $\lambda(K)$ *is compact in* $M(S)$.

(3) $\{\lambda(f): f \in C(S), \|f\| \leq 1\}$ *is* β*-relatively compact in* $C(T)$.

(4) $\{\lambda(f): f \in C_0(S), \|f\| \leq 1\}$ *is* β*-relatively compact in* $C(T)$.

(5) *The mapping* $\mu \to \lambda(\mu)$ *takes* β*-equicontinuous subsets of* $M(T)$ *into relatively compact subsets of* $M(S)$.

Proof. Clearly (1) implies (2). Given (2) it follows that $\lambda(f) \in C(T)$ for $f \in C(S)$ as in the proof of Theorem 2. Since λ is weak* continuous the norm and weak* topologies agree on sets $\lambda(K)$ for K compact in T. Consequently, given a neighborhood U with compact closure, λ is norm continuous on U. This allows one to show that

$$\{\lambda(f): f \in C(S), \|f\| \leq 1\}$$

is equicontinuous and being uniformly bounded is then compact in the compact-open and hence β-topology on $C(T)$ by Ascoli's theorem.

Evidently (3) implies (4) and (4) implies that the mapping $f \to \lambda(f)$ defines a compact operator on $C_0(S)$ and so that the adjoint mapping $\mu \to \lambda(\mu)$ satisfies (5) by [5, Theorem 9.2.1]. Finally, (5) implies (2), for if K is compact in T, the set $\{\mathring{x}: x \in K\}$ is β-equicontinuous and hence $\lambda(K) = \{\lambda(x): x \in K\}$ is relatively compact and weak* closed and hence compact. A review of the argument that (2) implies (3) yields a proof that (2) implies (1). This completes the proof.

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UNIVERSITY OF MISSOURI
COLUMBIA, MISSOURI