APPROXIMATION TO HARMONIC FUNCTIONS

BY
P. R. AHERN

Let K be a compact plane set. In what follows we use, for the most part, the notation and terminology of [4]. We let $D(K)$ be the set of functions continuous on K and harmonic on the interior of K, and let $H(K)$ be the set of functions harmonic in a neighborhood of K. $\overline{H(K)}$ will denote the uniform closure of $H(K)$ on K. It is known that the Choquet boundary of $D(K)$ is just the set R of regular points of K. We let P denote the Choquet boundary of $\overline{H(K)}$. It is known that $D(K) = \overline{H(K)}$ if and only if the capacity of $\partial K - P$ is zero. In a given case this condition may not be so easy to check since it requires some knowledge of the set P. In this note we give a sufficient condition that $D(K) = \overline{H(K)}$, which seems more "geometrical" in nature.

For each $p \in K$ there is a unique positive measure $d\mu_p$ carried on R such that $u(p) = \int u d\mu_p$ for all $u \in D(K)$. This is called the harmonic measure for p. There is also a unique positive measure $d\nu_p$ carried on P such that $u(p) = \int u d\nu_p$ for all $u \in \overline{H(K)}$. This measure is called the Keldysh measure for p. We let $\partial_0 K$ be the set of points in ∂K that lie on the boundary of some component of the complement of K; and let $\partial_i K$ be the set of points in ∂K that lie on the boundary of some component of the interior of K. In what follows dm will denote planar Lebesgue measure. We will show:

THEOREM. *If*

$$d\mu_p(\partial K - \partial_0 K) = 0 \quad \textit{for all} \quad p \in \operatorname{int} K \tag{1}$$

$$dm(\partial K - (\partial_0 K \cup \partial_i K)) = 0, \tag{2}$$

then $\overline{H(K)} = D(K)$.

(Note that (2) holds if int K is dense in K and has finitely many components.)

We will give examples to show that neither (1) nor (2) alone is sufficient to give the conclusion.

The method of proof is that used by Carleson in his proof of Walsh's Theorem [2]. See also [3].

Let α be a real measure carried on ∂K. We let

$$\hat{\alpha}(\xi) = \int \log \frac{1}{|z - \xi|}\, d\alpha(z).$$

We say $\hat{\alpha}(\xi_0)$ exists if

$$\int \log \frac{1}{|z - \xi_0|}\, d|\alpha|(z) < \infty.$$

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In the proof of Lemma 1 in [2], Carleson shows that if X a compact plane set that has a connected complement and if $d\alpha$ is a measure carried on X such that $\hat{\alpha}(\xi) = 0$ for $\xi \notin X$, then $\hat{\alpha}(\xi) = 0$ for all $\xi \in \partial X$ for which $\hat{\alpha}(\xi)$ exists. The very same proof will show the following: Let X be a compact plane set and let $d\alpha$ be a measure carried on X, suppose that $\xi_0 \in \partial_0 X$ and that $\hat{\alpha}(\xi_0)$ exists and that $\lim_{\xi \to \xi_0, \xi \notin X} \hat{\alpha}(\xi)$ exists and is equal to α_0 ; then $\hat{\alpha}(\xi_0) = \alpha_0$. In what follows we will take $X = \partial K$ so that $\partial_0 X = \partial K_0 \cup \partial_i K$.

LEMMA 1. *Suppose $\mathcal{G}$ is a component of* int K *and $p \in \mathcal{G}$ and ξ_0 is a regular point of $\partial\mathcal{G}$; then*

$$\int \log \frac{1}{|z - \xi_0|} \, d\mu_p(z) = \log \frac{1}{|p - \xi_0|}.$$

Proof. Note that if $\xi \notin \bar{\mathcal{G}}$ then

$$\hat{\mu}_p(\xi) = \int \log \frac{1}{|z - \xi|} \, d\mu_p(z) = \log \frac{1}{|p - \xi|};$$

since $\hat{\mu}_p$ is lower semi-continuous we have

$$\hat{\mu}_p(\xi_0) \leq \lim_{\xi \to \xi_0, \xi \notin \mathcal{G}} \hat{\mu}_p(\xi) = \log 1/|p - \xi_0|;$$

that is, $\hat{\mu}_p(\xi_0)$ exists.

If $\xi \in \mathcal{G}$, then

$$\hat{\mu}_p(\xi) = \int \log \frac{1}{|z - \xi|} \, d\mu_p(z) = g(p, \xi) + \log \frac{1}{|p - \xi|}$$

where $g(p, \xi)$ is the negative Green's function with pole at ξ. So,

$$\hat{\mu}_p(\xi) = g(\xi, p) + \log 1/|p - \xi| \to \log 1/|p - \xi_0|$$

as $\xi \to \xi_0$, because ξ_0 is regular. It follows from the remarks preceding the lemma that $\hat{\mu}_p(\xi_0) = \log 1/|p - \xi_0|$.

Proof of the Theorem. By Choquet theory it is enough to show that if $d\alpha$ is a real measure carried on R that annihilates $H(K)$ then $d\alpha \equiv 0$. If $d\alpha$ is such a measure and $p \in \text{int } K$ then by the lemma we have

$$\hat{\mu}_p(\xi) = \int \log \frac{1}{|z - \xi|} \, d\mu_p(z) = \log \frac{1}{|\xi - p|}$$

almost everywhere $(d\alpha)$. Since $d\alpha$ annihilates $H(K)$,

$$\hat{\alpha}(\xi) = \int \log \frac{1}{|z - \xi|} \, d\alpha(z) = 0 \quad \text{if} \quad \xi \notin K$$

and so $\hat{\alpha}(\xi) = 0$ for all $\xi \in \partial_0 K$ for which $\hat{\alpha}(\xi)$ exists. We calculate for

$p \in \operatorname{int} K$,

$$\iint \log \frac{1}{|z-\xi|}\, d|\alpha|(z)\, d\mu_p(\xi) = \iint \log \frac{1}{|z-\xi|}\, d\mu_p(\xi)\, d|\alpha|(z)$$

$$= \int \log \frac{1}{|z-p|}\, d|\alpha|(z) < \infty.$$

So $\alpha(\xi)$ exists almost everywhere $(d\mu_p)$. By assumption (1) $d\mu_p$ is carried on on $\partial_0 K$ and so we have

$$0 = \int \alpha(z)\, d\mu_p(z) = \iint \log \frac{1}{|z-\xi|}\, d\alpha(\xi)\, d\mu_p(z)$$

$$= \iint \log \frac{1}{|z-\xi|}\, d\mu_p(z)\, d\alpha(z)$$

$$= \int \log \frac{1}{|\xi-p|}\, d\alpha(\xi) = \hat{\alpha}(p),$$

where the use of Fubini's theorem was justified by the previous calculation and we again used the fact that $\hat{\mu}_p(\xi) = \log 1/|\xi - p|$, almost everywhere $(d|\alpha|)$. So we have that $\hat{\alpha}(\xi) = 0$ in $\operatorname{int} K$ and hence $\hat{\alpha}(\xi) = 0$ for all $\xi \in \partial_i K$ for which $\hat{\alpha}(\xi)$ exists. Since $\partial K - (\partial_0 K \cup \partial_i K)$ has planar measure zero it follows that $\hat{\alpha} = 0$ almost everywhere (dm), hence that $d\alpha \equiv 0$ [2].

If we delete from the closed unit disc, disjoint open discs with centers along the x-axis in an appropriate way we get an example where $D(K) \neq \overline{H(K)}$. In this case (2) holds but not (1) [4].

In finding an example where $D(K) \neq \overline{H(K)}$ and (1) holds we will show a little more. We find a compact set K for which $\operatorname{int} K$ is dense in K, each component of $\operatorname{int} K$ is simply connected, $R = \partial K$, (1) holds and $\overline{H(K)} \neq D(K)$. Note that because (1) holds we have $d\mu_p = d\nu_p$ for all $p \in \operatorname{int} K$ [4, Lemma 9.2]. Hence if $q \in \partial K - P$ and $p_n \in \operatorname{int} K$ with $p_n \to q$ we have $d\nu_{p_n} = d\mu_{p_n} \to \delta_q$ weak* since q is a regular point; here δ_q denotes point mass at q. This is in contrast to the situation for the regular points, i.e. if $q \in \partial K$ and if for every sequence $p_n \in \operatorname{int} K$, with $p_n \to q$ we have $\mu_{p_n} \to \delta_q$ weak* then q is regular.

To get the example let J be a compact subset of the unit disc obtained by removing disjoint open discs D_n, so that $D(J) = C(J) \neq \overline{H(J)}$. Let K be the compact subset of the unit disc obtained by removing a tubular subset T_n of each D_n which spirals to accumulate on the boundary of D_n. Since no point of $D_n - T_n$ can be joined to the boundary of D_n by an arc it follows that harmonic measure for any point of $D_n - T_n$ is carried on $\partial T_n - \partial D_n$ [1], so (1) holds. Clearly $\operatorname{int} K$ is dense in K, each component of $\operatorname{int} K$ is simply connected and $R = \partial K$ since ∂K is connected. Let $u \in C(J) - \overline{H(J)}$, since J is a closed subset of ∂K we can extend u to be continuous on ∂K and then since $R = \partial K$ to be an element of $D(K)$. Clearly the extended function is not in $\overline{H(K)}$.

Remark. It is easy to see that the condition $\hat{\mu}_p(\xi_0) = \log 1/|p - \xi_0|$ of Lemma 1 is sufficient as well as necessary for ξ_0 to be a regular point.

References

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The University of Wisconsin
Madison, Wisconsin