ON THE VARIETY OF ORBITS

BY

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1. Statement of the main result

Let the algebraic group G have components $G_1, \cdots, G_s$ and let G operate *regularly* on the variety V, i.e., let the operation of G_i on V, for every i, be an everywhere defined rational map of $G_i \times V$ into V. (See 1 of §4, where the notes and remarks are assembled.) Let k be a field of definition for G, V, and the operation of G on V. Let $v \in V$ and let $g \in G_i$ be a generic point for G_i over $k(v)$. $k(v, g)$ is a regular extension of $k(v)$, so by [W., p. 18, Prop. 20] $k(v, gv)$ is also a regular extension of $k(v)$; thus gv is a generic point over $k(v)$ of a subvariety $\mathcal{O}_i(v)$ of V having $k(v)$ as a field of definition. On $(G_i \times v) \times \mathcal{O}_i(v)$, which has $k(v)$ as a field of definition, consider the subvariety W_i having $((g, v), gv)$ as generic point over $k(v)$. The (algebraic) projection of W_i on $\mathcal{O}_i(v)$ is $\mathcal{O}_i(v)$. The variety W_i consists of the points $((\bar{g}, v), \bar{g}v)$, where $\bar{g}$ varies over G_i, so $\mathcal{O}_i(v)$ contains the part of the orbit of v due to G_i; on the other hand, by [W, p. 169, Prop. 3], which also obviously applies in the abstract case, the set-theoretic projection of W_i on $\mathcal{O}_i(v)$ contains a non-empty $k(v)$-open subset of $\mathcal{O}_i(v)$. Thus $\mathcal{O}(v) = \mathcal{O}_1(v) \cup \cdots \cup \mathcal{O}_s(v)$ is the *union of the orbit of v and a $k(v)$-closed subset of dimension less than* $\dim \mathcal{O}(v)$.

We want to show that *there exists a proper G-invariant k-closed subset F such that on $V' = V - F$ the orbits consist of closed sets having at most s components; and that these closed sets are in one to one correspondence with the points of a variety W defined over k in such a way that the mapping which associates to each point of V' its orbit is an everywhere defined rational map over k of V' into W.*

The "variety of orbits" was defined in [R], by means of a generic point, and coincides to a large extent with our W, but its relation to the set of orbits was not considered, except for the remark that "[the variety of orbits] is a true variety of orbits only so far as *generic* orbits are concerned".

The motivation for the stated result lies in [E, Th. 5], or rather in its application to the classification of singular points of algebraic curves. This theorem says that *if G is a connected solvable algebraic group operating regularly on an abstract variety V, then there exists a constructable subset W of V such that for each v in V there is a unique w in W with v in Gw.* (See 2 of §4.) This rests on [R, Th. 10], which says that *if τ is the natural rational map from V to its variety of orbits T, and if G is connected and solvable, then there exists a cross-section, i.e., a rational map $\sigma : T \to V$ with $\tau\sigma = 1$.* The statement $\tau\sigma = 1$

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is in the sense of algebraic geometry—set-theoretically there may be exceptions; and the proof of [E, Th. 5] consists of a kind of spelling out of the exceptions and an inductive taking care of them. However, a prior consideration of the relation of the "variety of orbits" to the set of orbits appears to be needed for a complete proof. This matter is not touched upon in [E] or in its references.

The case that G and V are affine varieties suffices for the application mentioned, and a restriction to this case would yield many simplifications, both in detail and in conception. However, we thought it proper to consider matters with at least the generality occurring in [E, Th. 5]. Complete generality requires some, but not much, more detail.

Some background material on k-constructable sets, k-elementary formulae, and elimination theory is given in [S_3].

2. Preliminary theorems

Let U, V, and W be varieties, and let τ be a rational map of $U \times V$ into W, all defined over k. Let $v \in V$. If τ is defined at (u_0, v) for some u_0, then it is also defined at (u, v) for u a generic point of U over $k(v)$, and $\tau(u, v)$ is a generic point over $k(v)$ of a variety $\mathcal{O}(v)$; $\mathcal{O}(v)$ does not depend on which generic point u over $k(v)$ is chosen; if τ is not defined at (u_0, v) for any u_0, we place $\mathcal{O}(v) = \emptyset$. If τ is defined at u_0, v_0, we write $u_0 v_0$ for $\tau(u_0, v_0)$.

THEOREM 1. *Let U, V, W, τ, k be as just mentioned, and let U, V, W be affine. Then there is an integer N such that for every $v \in V$ there is a set of polynomials $f_i(\mathbf{V}, X)$ in $k[\mathbf{V}, X]$ of total degree $\leq N$ such that the $f_i(v, X)$ generate an ideal having $\mathcal{O}(v)$ as associated locus.*

Proof. Given a generic point $x = (x_1, \cdots, x_n)$ of an affine variety V^r over k, one knows how to compute a basis for an ideal having V as associated locus. One forms $r + 1$ linear combinations $t_i x$ with indeterminate coefficients $t_{ij} (i = 1, \cdots, r + 1; j = 1, \cdots, n)$; these are algebraically dependent over $k(t) = k(t_1, \cdots, t_{r+1})$; and the $-t_i x$ satisfy a polynomial

$$F(t; Z_1, \cdots, Z_{r+1}) \in k[t; Z_1, \cdots, Z_{r+1}] - 0,$$

which we may suppose is irreducible; into this one substitutes $-t_i X$ for Z_i; and then the coefficients of $F(t; -t_1 X, \cdots, -t_{r+1} X)$ considered as a polynomial in t yield the desired basis (see [v.d.W_1, Th. 6]).

Let v be an arbitrary point of V and u a generic point of U over $k(v)$. If τ is not defined at (u, v), then τ is also not defined at (u, v_0) for any k-specialization v_0 of v. If τ is defined at (u, v), then, using the generic point uv, we compute a basis for $\mathcal{O}(v)$ over $k(v)$ in the way indicated. Let Y be the k-closure of v (so v is a "generic point" for Y over k). (See 3 of §4.) We now examine how uniform these computations are as v_0 varies over Y. Let $r = \dim \mathcal{O}(v)$. We first examine $\dim \mathcal{O}(v_0)$ as v_0 varies over Y. The coordinates w_i of uv may be written as rational functions in u, v. Let these be $P_i(u, v)/d(u, v)$, where P_i, d are polynomials over k. We may assume that only a k-algebraically independent subset of the coordinates of u occur in $d(u, v)$. Let v_0 be another point of Y. We may suppose u generic for U over $k(v, v_0)$: this does not change the computations for $\mathfrak{O}(v)$ in any way, but prepares them to be correct for v_0. In particular, elements of $k(u)$ algebraically independent over k remain such over $k(v_0)$. Let $c(v)$ be one of the coefficients of $d(\mathbf{U}, v)$, $c(v) \neq 0$. Making exception of the k-closed subset of Y defined by $c(\mathbf{V}) = 0$, we have $d(u, v_0) \neq 0$; and $P_i(u, v_0)/d(u, v_0)$ are the coordinates of uv_0, a generic point of $\mathfrak{O}(v_0)$ over $k(v_0)$. Any $r + 1$ of the w_i, say $w_1, \cdots, w_{r+1}$, are algebraically dependent over $k(v)$. Let

$$f(v; P_1(u, v)/d(u, v), \cdots, P_{r+1}(u, v)/d(u, v)) = 0$$

be a non-trivial polynomial relation over k. Making exception of a proper k-closed subset of Y, we get the non-trivial polynomial relation

$$f\left(v_0; \frac{P_1(u, v_0)}{d(u, v_0)}, \cdots, \frac{P_{r+1}(u, v_0)}{d(u, v_0)}\right) = 0.$$

The argument is repeated for every $(r + 1)$-tuple of the w_i. Thus, with exception of a k-closed subset of Y, $\dim \mathfrak{O}(v_0) \leq r$. Now let $w_1, \cdots, w_r$ (say) be algebraically independent over $k(v)$. Let $H_i(\mathbf{U}) = 0$ be a finite set of polynomial equations over k having U as associated locus; and let $K_j(\mathbf{V}) = 0$ be a similar set for Y. Consider the conjunction (for all i, j, and for $k = 1, \cdots, r$) of

$$(*) \quad H_i(\mathbf{U}) = 0, \quad K_j(\mathbf{V}) = 0, \quad P_k(\mathbf{U}, \mathbf{V})/d(\mathbf{U}, \mathbf{V}) = C_k, \quad d(\mathbf{U}, \mathbf{V}) \neq 0.$$

Eliminating $\mathbf{U}$ (see say [T, p. 39, Th. 1 and p. 54, note 16] or [S_1, p. 370, Th. 3 and p. 373, Remark (c)] or [S_2, p. 237, Th. 1] or [S_3]; see also [C]), we get a finite disjunction of finite conjunctions of polynomial equations and inequalities over k in $\mathbf{V}$, C. (See 4 of §4.) At least one of these conjunctions, which we may assume involves a sole inequality $e(\mathbf{V}, C) \neq 0$, is satisfied by $\mathbf{V} = v$, $C_k = P_k(u, v)/d(u, v)$; let us consider just this one. Let $f(\mathbf{V}, C) = 0$ be one of the equalities in it. Since $w_1, \cdots, w_r$ are algebraically independent over $k(v)$, we have $f(v, C) = 0$. Hence for any v_0, c satisfying $e(v_0, c) \neq 0$, one can solve $(*)$ for $\mathbf{U}$. We make exception of a k-closed subset of Y defined by a non-zero coefficient of $e(\mathbf{V}, C)$ regarded as a polynomial in C; and take for $c_1, \cdots, c_r$ quantities algebraically independent over $k(v_0)$. Let $\bar{u}$ be a solution of $(*)$ for $\mathbf{V} = v_0$, $C = c$. Then $P_k(\bar{u}, v_0)/d(\bar{u}, v_0)$, $k = 1, \cdots, r$ are algebraically independent over $k(v_0)$. Hence with exception of a proper k-closed subset of Y, $\dim \mathfrak{O}(v_0) = r$.

We now form $F(v, t, Z)$ as mentioned; we may assume $F(\mathbf{V}, t, Z) \in k[\mathbf{V}, t, Z]$, which we do. Making exception of a proper k-closed subset of Y, we will have $F(v_0, t, Z) \neq 0$. We note that $F(v_0, t, -t \cdot uv_0) = 0$, since v_0 is a specialization of v over $k(u)$; in fact, since u and v are independent over k and $k(u)$

is regular over k, $k(u)$ and $k(v)$ are linearly disjoint over k, by [W, p. 18, Th. 5], and by [W, p. 15, Th. 3], v_0 remains a specialization of v over $k(u)$. Hence if $F(v_0, t, Z)$ is irreducible over $k(v_0)$, then the coefficients of $F(v_0, t, -tX)$ yield the desired basis. The condition of (absolute) irreducibility places another polynomial condition on v_0. (See [v.d.W$_1$, Th. 3] and [v.d.W$_2$, p. 707]). Altogether, with exception of a proper k-closed subset of Y, the computations proceed uniformly. (See 5 of §4.) In a similar way, we take care of the exceptional set, and get a bound on the total degree of the $f_i(\mathbf{V}, X)$ at least for v_0 varying over Y (i.e., we take a k-component of the exceptional set, make a computation at a "generic point" thereof, find a smaller exceptional set; etc.). By taking $Y = V$, i.e., by taking v generic for V over k, we complete the proof.

Let U, V, W be varieties, τ a rational map of $U \times V$ into W, and let U, V, W, τ be defined over k. Let $v \in V$ and u a generic point for U over $k(v)$. If τ is defined at (u_0, v), then (u_0, v) is a k-specialization of (u, v), τ is defined at (u, v), and $\tau(u_0, v)$ is a $k(v)$-specialization of $\tau(u, v)$. Hence $\mathcal{O}(v)$ *is the* $k(v)$*-closure of the set of points* $\tau(u_0, v)$, *where* u_0 *ranges over the points for which* $\tau(u_0, v)$ *is defined.*

THEOREM 2. *Let* U, V, W, τ, k *be as just mentioned. Then the set of points* (x, v) *in* $W \times V$ *such that* $x \in \mathcal{O}(v)$ *is* k*-constructable.*

Proof. Let U be defined via affine varieties U_α (and birational transformations $T_{\alpha\beta}$), let V be defined via affine varieties V_γ, and W via affine varieties W_δ. Let $\mathcal{O}_\delta(v)$ denote the representative of $\mathcal{O}(v)$ on W_δ if there is one, and otherwise place $\mathcal{O}_\delta(v) = \emptyset$; let δ' indicate an index such that $\mathcal{O}_{\delta'}(v) \neq \emptyset$ for some v. For every α, γ, δ', the mapping τ induces a rational mapping

$$\tau_{\alpha\gamma\delta'}: U_\alpha \times V_\gamma \rightarrow W_{\delta'} .$$

If v has a representative v_γ in V_γ, then $\mathcal{O}_{\delta'}(v)$ is just the same as $\mathcal{O}_{\delta'}(v_\gamma)$ as previously defined relative to $U_\alpha \times V_\gamma \rightarrow W_{\delta'}$. We may write $\mathcal{O}_{\delta'}(v) = \mathcal{O}_{\delta'}(v_\gamma)$ to indicate this.

Let u, v be independent generic points for U, V over k, and let u_α, v_γ be the representatives of u, v in U_α, V_γ. Let $w = uv$. Then $w_{\delta'}$ can be written in a finite number of ways as rational functions over k in u_α, v_γ, each time with a common denominator, in such a way that $U_\alpha \times V_\gamma \rightarrow W_{\delta'}$ is defined at $(u_{0\alpha}, v_{0\gamma})$ if and only if one of the denominators does not vanish at $(u_{0\alpha}, v_{0\gamma})$ (in this connection see [W, p. 171, Th. 2, Proof]). Thus the statement that $\tau_{\alpha\gamma\delta'}$ is defined at $(u_{0\alpha}, v_{0\gamma})$ can be written as a finite disjunction of finite conjunctions of polynomial equations and inequalities over k in $u_{0\alpha}$, $v_{0\gamma}$. (See 6 of §4.)

Let $x_{\delta'}$ be a point in $W_{\delta'}$ and v_γ a point of V_γ. Then $x_{\delta'}$ is in $\mathcal{O}_{\delta'}(v_\gamma)$ if and only if every polynomial $f(v_\gamma, X)$ in $k(v_\gamma)[X]$ which vanishes over $\mathcal{O}' = \{\tau_{\alpha\gamma\delta'}(u_{0\alpha}, v_\gamma)\}$ vanishes at $x_{\delta'}$; here $u_{0\alpha}$ varies over the points for which $\tau_{\alpha\gamma\delta'}$ is defined at $(u_{0\alpha}, v_\gamma)$. We may suppose $f(v_\gamma, X) \in k[v_\gamma, X]$ and, by

Theorem 1, can place a bound N on the total degree of $f(\mathbf{V}, X)$. Moreover, one can relax the condition that f have its coefficients in k, as any $f \in \Omega[X]$—Ω, the universal domain—which vanishes over $\mathcal{O}'$ vanishes over $\mathcal{O}_{\delta'}(v_\gamma)$; in fact, if k' is a field containing k, v_γ, and the coefficients of f, and if $u_{0\alpha}$ is a generic point of U over k', then, dismissing the case $\mathcal{O}_{\delta'}(v_\gamma) = \emptyset$ as trivial, f vanishes at $\tau_{\alpha\gamma\delta'}(u_{0\alpha}, v_\gamma)$ and hence over $\mathcal{O}_{\delta'}(v_\gamma)$. The set of polynomials $f(\mathbf{V}, X) \in \Omega[\mathbf{V}, X]$ of total degree $\leq N$ is parametrized by the points c of an affine space. Hence the statement $x_{\delta'} \in \mathcal{O}_{\delta'}(v_\gamma)$ can be written as a finite disjunction, properly quantified, of finite conjunctions of polynomial equations and inequalities over k in $x_{\delta'}$, v_γ, $u_{0\alpha}$, and c. (See 7 of §4.) Eliminating the parameters $u_{0\alpha}$, c (say by [S_1, pp. 370, 373] or [S_3]), we see that the set of points $(x_{\delta'}, v_\gamma)$ in $W_{\delta'} \times V_\gamma$ such that $x_{\delta'} \in \mathcal{O}_{\delta'}(v_\gamma)$ is k-constructable.

By [W, p. 188, Prop. 10], the V_γ, W_δ are k-open covers of V, W; and similarly the $W_\delta \times V_\gamma$ are a k-open cover of $W \times V$. As $x \in \mathcal{O}(v)$ if and only if for some γ, δ', x has a representative $x_{\delta'}$ in $W_{\delta'}$ and v has a representative v_γ in V_γ and $x_{\delta'} \in \mathcal{O}_{\delta'}(v_\gamma)$, the set of (x, v) in $W \times V$ for which $x \in \mathcal{O}(v)$ is the union of the sets $(x_{\delta'}, v_\gamma)$ for which $x_{\delta'} \in \mathcal{O}_{\delta'}(v_\gamma)$. Hence the set of (x, v) in $W \times V$ for which $x \in \mathcal{O}(v)$ is k-constructable.

THEOREM 3. *Let U, V, τ, k be as in Theorem 2. Let s be an integer ≥ -1. Then the set S of points v such that* $\dim \mathcal{O}(v) \neq s$ *is k-constructable.*

Proof. Let γ, δ' be two indices (with δ' as in the proof of Theorem 2). Suppose we know that $\dim \mathcal{O}_{\delta'}(v_\gamma) \neq s$ on a k-constructable set W^* and a subset of a k-constructable set W (which we do for $W^* = \emptyset$ and $W = V_\gamma$). W (unless empty) is the finite union of sets each of which is a k-irreducible algebraic set minus a proper relatively k-closed subset. Let W_1 be one of these k-irreducible sets and W_1' the associated relatively k-closed subset. Then W is the union of a k-constructable set W' disjoint from W_1, $W_1 - W_1'$, and a k-constructable subset of W_1'. Let v_1 be a "generic point" for W_1 over k. If $\mathcal{O}_{\delta'}(v_1) \neq \emptyset$, then by note 5 of §4, $\dim \mathcal{O}_{\delta'}(v_2) = \dim \mathcal{O}_{\delta'}(v_1)$ for $v_2 \in W_1$ with possible exception of the points v_2 in a proper k-closed subset W_2; and the same is true if $\mathcal{O}_{\delta'}(v_1) = \emptyset$, as then $\mathcal{O}_{\delta'}(v_2) = \emptyset$ for every $v_2 \in W_1$. If $\dim \mathcal{O}_{\delta'}(v_1) = s$, we throw away $W_1 - W_2$, and otherwise keep it (i.e., adjoin it to W^*). Then we examine $W' \cup W_2 \cup W_1'$, etc. In this way we come to the desired conclusion. (See 8 of §4.)

By a *k-atomic formula* we mean a formula of the form $(x_1, \cdots, x_s) \in F$, where x_i is a free variable ranging over a variety V_i defined over k and F is a k-closed subset of $V_1 \times \cdots \times V_s$. By a *k-elementary formula* we mean a formula built up in a finite number of steps from k-atomic formulae by negation, conjunction, disjunction, and quantification of the form $\exists x_i(\cdots)$. One checks easily that the set of points satisfying a k-elementary formula is k-constructable, and conversely. A k-elementary formula involving only bound variables is called a *k-elementary sentence.*

For example, let U and V be varieties, and τ a rational map of U into V, all defined over k. Then the expression "τ is defined at u" is, or can be written as, a k-elementary formula; in fact, we saw this in the case U, V are affine, and the extension to arbitrary varieties offers no difficulty. The expression

$$\tau \text{ is defined at } u \text{ and has there the value } v$$

is also a k-elementary formula. In fact, let Γ_τ be the graph of τ. Then the mentioned expression can be written as

$$\tau \text{ is defined at } u \text{ and } (u, v) \in \Gamma_\tau .$$

THEOREM 4. *Let U, V, τ, k be as in Theorem 2. Let $\mathcal{O}'(v) = \{\tau(u, v)\}$, where u varies over the points such that τ is defined at (u, v). Then the set S of points x such that for some v, $x \in \mathcal{O}(v)$ but $x \notin \mathcal{O}'(v)$ is k-constructable.*

Proof. The expression $x \in \mathcal{O}'(v)$ is a k-elementary formula, as it can be written as

$$\exists u(\tau \text{ is defined at } (u, v) \text{ and } x = \tau(u, v)).$$

The formula $x \in \mathcal{O}(v)$ is also k-elementary, by Theorem 2. Hence the formula

$$\exists v(x \in \mathcal{O}(v) \text{ and } x \notin \mathcal{O}'(v))$$

is k-elementary.

THEOREM 5. *Let U, V, W, τ, k be as in Theorem 2, and let F be a k-closed subset of W. Then the set S of points v such that $\mathcal{O}(v) \subset F$ is k-constructable.*

Proof. Let $\mathcal{O}'(v)$ be as in Theorem 4. As $\mathcal{O}(v)$ is the $k(v)$-closure of $\mathcal{O}'(v)$, we have $\mathcal{O}(v) \subset F$ if and only if $\mathcal{O}'(v) \subset F$. The set of v satisfying $\mathcal{O}'(v) \subset F$ is the same as the set satisfying $\forall u \exists y$ (τ is not defined at (u, v) or τ is defined at (u, v) and has there the value y and $y \in F$). Hence S is k-constructable. (See 9 of §4.)

3. The case $U = G_i$ (and $W = V$)

To get the picture of the $\mathcal{O}_i(v)$ (§1) clear, we recall some facts about G. Let G_1 be a component of G containing the identity e. As a generic point of G_i cannot lie in any other component, one has $G_1 G_i \subset G_i$ and $G_i G_1 \subset G_i$; hence, in particular, there is only one component containing e. Hence if $G_i G_j$ contains e, then $G_i G_j \subset G_1$, whence G_1 is a normal subgroup of G and the G_i are its cosets.

Let $\mathcal{O}'_i(v) = \{gv \mid g \in G_i\}$. Then $\mathcal{O}_i(v)$ is the $k(v)$-closure of $\mathcal{O}'_i(v)$. If $g_i \in G_i$, then $G_i = G_1 g_i$, so $\mathcal{O}'_i(v) = \{gg_i v \mid g \in G_1\} = \mathcal{O}'_i(g_i v)$; so *an orbit under G is made up of s (or fewer) orbits under G_1*. Let $v \in V$ and g a generic point of G_1 over $k(v)$; then gv is a generic point over $k(v)$ of $\mathcal{O}_1(v)$. Let g_i be a point of G_i algebraic over k; then $g_i gv \in \mathcal{O}_i(v)$. As $k(v, g_i, g_i gv) = k(v, g_i, gv)$, $\dim \mathcal{O}_i(v) \geq \dim \mathcal{O}_1(v)$; and similarly $\dim \mathcal{O}_1(v) \geq \dim \mathcal{O}_i(v)$. Thus *for every v, all the $\mathcal{O}_i(v)$ have the same dimension*. Let $g_i \in G_i$, $g \in G$. Then

$$[\mathcal{O}'_1(v), \cdots, \mathcal{O}'_s(v)] = [\mathcal{O}'_1(g_1 v), \cdots, \mathcal{O}'_1(g_s v)]$$

and

$$[\mathfrak{O}_1'(gv), \cdots, \mathfrak{O}_s'(gv)] = [\mathfrak{O}_1'(g_1 gv), \cdots, \mathfrak{O}_1'(g_s gv)].$$

As $g_i g$, $g_j g$ are in different cosets if g_i, g_j are, $[\mathfrak{O}_1'(gv), \cdots, \mathfrak{O}_s'(gv)]$ *is a permutation of* $[\mathfrak{O}_1'(v), \cdots, \mathfrak{O}_s'(v)]$; and similarly for the $k(v, g)$-closures $\mathfrak{O}_i(v)$, $\mathfrak{O}_i(gv)$.

THEOREM 6. *If* $v_1 \in \mathfrak{O}(v)$, *then* $\mathfrak{O}(v_1) \subset \mathfrak{O}(v)$.

Proof. As $\mathfrak{O}(v)$ is closed and $\mathfrak{O}(v_1)$ is the closure of the orbit of v_1, it suffices to show that the orbit of v_1 is in $\mathfrak{O}(v)$. Let, then, $g' \in G$. Then $g'gv \in \mathfrak{O}(v)$ for every $g \in G$. Let $v_1 \in \mathfrak{O}_i(v)$ and let g be a generic point of G_i over $k(v, g')$. Then v_1 is a specialization of gv over $k(v, g')$, whence $g'v_1$ is a specialization of $g'gv$ over $k(v, g')$. Hence $g'v_1 \in \mathfrak{O}(v)$, Q.E.D.

COROLLARY. *The set* S *of points* v_1 *such that for some* v, v_1 *is in* $\mathfrak{O}(v)$ *but not in the orbit of* v, *is a* G*-invariant* k*-constructable set.*

The expressions $v_1 \in \mathfrak{O}_i(v)$, $v_1 \in \mathfrak{O}_i'(v)$ are k-elementary formulae, so S is k-constructable; it is G-invariant by the theorem.

THEOREM 7. *Let* F *be a* k*-closed subset of* V. *The set* S *of points* v *such that for some* i, $\mathfrak{O}_i(v) \subset F$ *is* k*-constructable and (obviously)* G*-invariant. The set* S_1 *of points* v *such that for some* i $\dim \mathfrak{O}_i(v) \neq r$, *where* $r = \dim \mathfrak{O}_1(v)$ *for a generic point* v *of* V *over* k, *is* k*-constructable and (obviously)* G*-invariant.*

This follows at once from our previous theorems. Here too (cf. Theorem 5) S is k-closed.

THEOREM 8. *The* k*-closure* $\bar{S}$ *of a* k*-constructable* G*-invariant set* S *containing no generic point of* V/k *is proper and* G*-invariant.*

Proof. S (unless empty) is the union of a finite number of sets, each a k-irreducible set F_i minus a proper k-closed subset. $\bar{S} = \bigcup F_i$ and is proper if no F_i equals V. If $\bar{P} \in \bar{S}$, then there is a $P \in S$ with $P \to \bar{P}$ over k. Let g_i be a generic point of G_i over $k(P, \bar{P})$. Then $P \to \bar{P}$ also over $k(g_i)$. Hence $g_i P \to g_i \bar{P}$ over k. $g_i P$ is in some F_j, say F_1. Then $g_i \bar{P} \subset F_1$; and, as $g_i \bar{P}$ is a generic point for $\mathfrak{O}_i(\bar{P})$ over $k(\bar{P})$, $\mathfrak{O}_i(\bar{P}) \subset F_1$. Hence the orbit of $\bar{P}$ is in $\bar{S}$.

In what follows we fix an index γ and speak of *the Chow form of an* $\mathfrak{O}_i(v)$ if $\mathfrak{O}_{i\gamma}(v)$ is not empty, and then mean thereby the Chow form of $\mathfrak{O}_{i\gamma}(v)$. We may write $F_i'(v, t, Z)$ for this form; the point v need not have a representative in V_γ. We speak of *the Chow form of* $\mathfrak{O}(v)$ if each $\mathfrak{O}_i(v)$ has a Chow form, and then mean thereby $\prod_{i=1}^{s} F_i'(v, t, Z)$. Each $\mathfrak{O}_i(v)$ occurs the same number of times amongst $\mathfrak{O}_1(v), \cdots, \mathfrak{O}_s(v)$, so every irreducible factor in the Chow form occurs with the same multiplicity. Hence the Chow form depends only on the locus $\mathfrak{O}(v)$, not on v.

If v is generic for V over k, then v has a representative in V_γ. Hence $\mathcal{O}_\gamma(v) \neq \emptyset$, as $v \in \mathcal{O}_1(v)$. We have $\mathcal{O}_i(v) = \mathcal{O}_1(g_i v)$ for $g_i \in G_i$. Since $k(g_i, g_i v) = k(g_i, v)$, by taking g_i independent from v over k, $g_i v$ remains generic for V over k. Hence $\mathcal{O}_{i\gamma}(v) \neq \emptyset$. Thus we may speak of the Chow form of $\mathcal{O}(v)$ if v is generic for V over k.

Let $F(v, t, Z)$ be the Chow form of $\mathcal{O}(v)$ with v generic for V over k. The coefficients of F (considered as a polynomial in t, Z) are the coordinates of a generic point P over k of a variety in projective space, the "variety of orbits". Let τ be the rational map defined by the generic point (v, P) over k. τ is defined at v_1 if, F having been normalized by making some coefficient $= 1$, the coefficients are defined at v_1. Let $g \in G$. Take v generic for V over $k(g)$; then gv is also generic for V over k and $\mathcal{O}(gv) = \mathcal{O}(v)$. Hence $\tau(gv) = \tau(v)$, whence τ and τg are the same rational map on V. Now let τ be defined at v_1. Then τg is defined at $g^{-1}v_1$, so τ is defined at $g^{-1}v_1$. (See 10 of §4.) Thus *the set S of points v_1 at which τ is defined is G-invariant*; it is also k-constructable. Hence

THEOREM 9. *The set S of points v_1 at which τ is defined is k-constructable and G-invariant.*

THEOREM 10. *Let v be generic for V over k and let $F(v, t, Z)$ be the Chow form of $\mathcal{O}(v)$. Then the set S of points v_1 at which $F(v, t, Z)$, after a suitable normalization, is defined and such that then $F(v_1, t, Z)$ yields the Chow form of $\mathcal{O}(v_1)$ is k-constructable and G-invariant.*

Proof. We first confine ourselves to the points v_1 for which τ is defined, for which $\mathcal{O}(v_1)$ has a Chow form, and for which $\dim \mathcal{O}(v_1) = \dim \mathcal{O}(v)$ (so the Chow forms of $\mathcal{O}(v_1)$, $\mathcal{O}(v)$ involve the same t and Z); this is a k-constructable G-invariant set. Let v_1 be such that $F(v_1, t, Z)$, i.e., $F(v, t, Z)\mid_{v=v_1}$ after a suitable normalization of F, is the Chow form of $\mathcal{O}(v_1)$. Then by note 4 of §4, $F(\bar{v}_1, t, Z)$ is the Chow form of $\mathcal{O}(\bar{v}_1)$ for almost all k-specializations $\bar{v}_1$ of v_1. Now let v_1 be such that $F(v_1, t, Z)$ is not the Chow form of $\mathcal{O}(v_1)$; let $G(v_1, t, Z)$ be the Chow form of $\mathcal{O}(v_1)$. Let a_i, b_i be corresponding coefficients of $F(v_1, t, Z)$, $G(v_1, t, Z)$; then $d(v_1) = a_j b_k - a_k b_j \neq 0$ for some j, k. For almost all k-specializations $\bar{v}_1$ of v_1, $F(\bar{v}_1, t, Z)$ remains defined, $G(\bar{v}_1, t, Z)$ is the Chow form of $\mathcal{O}(\bar{v}_1)$, and $d(\bar{v}_1) \neq 0$, so $F(\bar{v}_1, t, Z)$ is not the Chow form of $\mathcal{O}(\bar{v}_1)$. By note 8 of §4, S is k-constructable. It is also obviously G-invariant.

The main result (§1) now follows quickly. Let S be the set of points v_1 such that for some v_1' with $\dim \mathcal{O}(v_1') = \dim \mathcal{O}(v)$ for v a generic point of V/k, $v_1 \in \mathcal{O}(v_1')$ but v_1 is not in the orbit of v_1'; or $\dim \mathcal{O}(v_1) \neq \dim \mathcal{O}(v)$, where v is generic for V over k; or $\mathcal{O}(v_1)$ does not have a Chow form; or τ is not defined; or τ is defined but does not yield the Chow form of $\mathcal{O}(v_1)$. (See 11 of §4.) Then S is a k-constructable G-invariant subset of V containing no generic point of V/k, and so is its k-closure $\bar{S}$. The image under τ of $V - \bar{S}$

contains a non-empty k-open subset W of the "variety of orbits"; as τ induces a k-continuous map of $V - \bar{S}$ (cf. [W, p. 171, Th. 2]), the counterimage of W on $V - \bar{S}$ is a k-open G-invariant subset $V - F$ of V. Then $V' = V - F$ and W (viewed as a variety) satisfy the statement of §1.

The proof of [E, Th. 5] can now also be quickly completed. Before doing so, we prefix a remark which will give a somewhat stronger version of that theorem: *Let V, W be (say) affine varieties defined over an algebraically closed field k, let τ be a rational map of V into W defined over k, and assume there exists a rational map σ of W into V such that $\tau\sigma = 1$; then there also exists a rational map $\bar{\sigma}$ of W into V defined over k and such that $\tau\bar{\sigma} = 1$.* In fact, let σ be defined over a field k' containing k. Let y be a generic point of W over k' and $\sigma(y) = (\sigma_1(y), \cdots, \sigma_t(y)) \in V$. Write $\tau\sigma(y) = y$. More explicitly one can write $\tau_i(x) = P_i(x)/Q(x)$, where x is a generic point of V over k, P_i, Q are polynomials over k, $Q(\sigma_1(y), \cdots, \sigma_t(y)) \neq 0$, and

$$P_i(\sigma_1(y), \cdots, \sigma_t(y))/Q(\sigma_1(y), \cdots, \sigma_t(y)) = y_i .$$

The $\sigma_j(y)$ are rational functions over k'. The σ_j, having been written out in some explicit way with a common denominator in $k'[y]$, involve only a finite number of coefficients in k'; let these, arranged in some order, be designated σ. Let $d(\sigma, y) \in k[\sigma, y]$ be the denominator mentioned. Then $d^\rho Q(\sigma_1(y), \cdots, \sigma_t(y)) = Q_1(\sigma, y)$ is a polynomial over k in σ, y for some ρ. Now specialize (σ, y) over k to a k-rational point $(\bar{\sigma}, \bar{y})$ in such way that $d(\bar{\sigma}, \bar{y})Q_1(\bar{\sigma}, \bar{y}) \neq 0$. A fortiori $d(\bar{\sigma}, y)Q_1(\bar{\sigma}, y) \neq 0$. Let $\sigma_i(y) = n_i(\sigma, y)/d(\sigma, y)$, $\bar{\sigma}_i(y) = n_i(\bar{\sigma}, y)/d(\bar{\sigma}, y)$ and let $\bar{\sigma}$ be defined by

$$\bar{\sigma}(y) = (\bar{\sigma}_1(y), \cdots, \bar{\sigma}_t(y)).$$

As $(\bar{\sigma}, y)$ is also a specialization of (σ, y) over k, we have $\bar{\sigma}(y) \in V$ and $\tau\bar{\sigma}(y) = y$. Thus $\bar{\sigma}$ is a desired map.

Let now G be a connected algebraic group operating regularly on a variety V and let k be an algebraically closed field of definition for G, V, and the operation of G on V. Let W be the "variety of orbits" and τ the natural map of V into W; W and τ are also defined over k. *Assume that for every V there exists a rational map σ of W into V such that $\tau\sigma = 1$* (which by [R, Th. 10] will be the case if G is solvable). *Then we will show there exists a k-constructable* (and not merely constructable) *subset C of V such that every orbit meets the set C in precisely one point.* In fact, let F and W be as stated in the main result (§1), and let $\tau: V - F \to W$. By the last paragraph we may assume σ is defined over k. σ is defined except on a k-closed subset G of W. Let $V - F'$ be the inverse image of $W - G$ (under τ). The image of $W - G$ under σ is a k-constructable subset C' of V contained in $V - F'$; and every orbit in $V - F'$ meets C' in precisely one point. Replacing V by F', we would be through by induction on dim V, except that V is replaced not by a variety but by a bunch of varieties (of smaller dimension).

To meet this last point, let V_1 be a component of F'. Let $g \in G$ and v

a generic point of V_1 over $k(g)$. If $\bar{v} \in V_1$, then $(g, \bar{v})$ is a k-specialization of (g, v) and $g\bar{v}$ is a k-specialization of gv, so gV_1 is in the k-component of F' which contains gv. Thus every element of G carries every component of F' into, and hence also onto, another component; and the set $\{gV_1 \mid g \in G\}$ is finite. Let H be the subset of G leaving V_1 invariant; H is obviously a subgroup of G. Let Γ be the graph of the operation of G on V. Then the expression $hv \in V_1$ can be written in the form

$$\exists(y)(h \in G, v \in V, (h, v, y) \in \Gamma \text{ and } y \in V_1),$$

and hence is k-elementary. Then $\forall(v)$ $(v \in V_1 \Rightarrow hv \in V_1)$ is k-elementary, so H is k-constructable. Now one proves that H is k-closed (cf. note 6 of §4). Then G is a finite union of k-closed sets of the form $g^{-1}Hg$. As G is connected, $G = g^{-1}Hg$ for some g, whence $G = H$. Thus *V_1 is invariant under G.* Let V_2 be another component of F' and $v \in V_1 \cap V_2$; then the orbit of v is contained in $V_1 \cap V_2$. Hence if K is the set of points P in V_1 and in another component of F', then K is k-closed and for every $v \in V_1 - K$, the orbit of v is in $V_1 - K$. By induction on dim V, we take care of $V_1 - K$; and then similarly the rest of F'. In this way we complete the proof.

4. Notes and remarks

1. Our terminology is mainly that of [W]. From the definition of *algebraic group*, we recall that the product $g_i\, g_j$ of $g_i \in G_i$, $g_j \in G_j$ is given by an everywhere defined rational map of $G_i \times G_j$ into one of the components G_k; and similarly for g_i^{-1}. From the definition of *operate regularly*, $g_1(g_2(v)) = (g_1\, g_2)(v)$ and $e(v) = v$ for e the identity of G. (See [R].)

2. G is said to be *connected* if $s = 1$, i.e., if the underlying set is a variety. A subset W of V is said to be *constructable* if it is the finite union of sets each of which is the intersection of a closed set and an open set; the k-constructable subsets of a variety V defined over k are similarly defined. The complement in V of a k-constructable set and the finite union and finite intersection of k-constructable sets are k-constructable; and the set-theoretic projection of a k-constructable subset of a product $V \times W$ on a factor is k-constructable. (See [S_3]; see also [C, p. 38, Cor. to Th. 3].) The notion of *solvability* does not enter into our considerations.

3. The quotation marks indicate a deviation from the terminology of [W].

4. By a finite conjunction of polynomial equations and inequalities (or inequations) over k we mean a finite conjunction $f_1(x_1, \cdots, x_n) = f_1'(x_1, \cdots, x_n)$ and $\cdots$ and $f_s(x_1, \cdots, x_n) = f_s'(x_1, \cdots, x_n)$ and $g_1(x_1, \cdots, x_n) \neq g_1'(x_1, \cdots, x_n)$ and $\cdots$ and $g_t(x_1, \cdots, x_n) \neq g_t'(x_1, \cdots, x_n)$, where the f_i, f_i', g_j, g_j' are polynomials over k and the x_i are free variables (ranging over the universal domain Ω). Usually a conjunction of this kind can be replaced without loss of generality by an *equivalent* one,

i.e., one having the same solutions; and this is frequently tacitly done. Thus we may suppose all the f_i', g_j' to be zero. Adjoining $0 = 0$ and $1 \neq 0$, we may suppose $s > 0$ and $t > 0$. With $t > 0$ and the $g_j' = 0$, we may suppose $t = 1$ as we replace $g_1 \neq 0$ and $\cdots$ and $g_t \neq 0$ by $g_1 \cdots g_t \neq 0$. The theorem being used here (above) amounts to this: the projection of a k-constructable set is k-constructable.

5. $F(v, t, Z)$ is the so-called *Chow form* of $\mathfrak{O}(v)$, except that the Chow form is understood to be defined only up to a constant factor $\rho \neq 0$. Dropping the condition $F(V, t, Z) \in k[V, t, Z]$ (i.e., allowing it to be in $k(V)[t, Z]$), we have just proved that *if $F(v, t, Z)$ is the Chow form of $\mathfrak{O}(v)$, then for almost all k-specializations $\bar{v}$ of v (i.e., for all $\bar{v} \in Y$ except perhaps those lying in a proper k-closed subset) the coefficients of $F(v, t, Z)$ are defined at $\bar{v}$, $\mathfrak{O}(\bar{v}) \neq \emptyset$, and $F(\bar{v}, t, Z)$ is the Chow form of $\mathfrak{O}(\bar{v})$.*

6. Let $d_1, \cdots, d_s$ be the denominators mentioned—they are polynomials over k—let $g_1 = 0, \cdots, g_t = 0$ be a finite set of polynomial equations over k for the locus U_α, and let $h_1 = 0, \cdots, h_u = 0$ be a set for V_γ. If $u_{0\alpha}$, $v_{0\gamma}$ are understood to vary over U_α, V_γ, as will be the case later, the condition mentioned can be written as $d_1(u_{0\gamma}, v_{0\gamma}) \neq 0$ or $\cdots$ or $d_s(u_{0\alpha}, v_{0\gamma}) \neq 0$. For the present we write $(g_1(u_{0\alpha}) = 0$ and $\cdots$ and $g_t(u_{0\alpha}) = 0$ and $h_1(v_{0\gamma}) = 0$ and $\cdots$ and $h_u(v_{0\gamma}) = 0$ and $d_1(u_{0\alpha}, v_{0\gamma}) \neq 0)$ or $\cdots$ or $(g_1(u_{0\alpha}) = 0$ and $\cdots$ and $g_t(u_{0\alpha}) = 0$ and $h_1(v_{0\gamma}) = 0$ and $\cdots$ and $h_u(v_{0\gamma}) = 0$ and $d_s(u_{0\alpha}, v_{0\gamma}) \neq 0)$. This is a desired disjunction.

7. Let $d_1, \cdots, d_s$ be the denominators mentioned in the last paragraph and let $d_1\tau^{(1)}_{\alpha\gamma\delta'}, \cdots, d_s\tau^{(s)}_{\alpha\gamma\delta'}$ be the corresponding numerators. Let $g_1 = 0, \cdots, g_t = 0$ be a finite set of polynomial equations over k defining the locus U_α, and let $h_1 = 0, \cdots, h_u = 0$ be a set for V_γ. Let $f(c; \mathbf{V}, X)$ be the "general" polynomial of total degree N in $\mathbf{V}$, X with coefficients c. Then the statement that $x_{\delta'} \in \mathfrak{O}_{\delta'}(v_\gamma)$ can be written as the following disjunction, properly quantified, for $i = 1, \cdots, s$: $(g_1(u_{0\alpha}) \neq 0$ or $\cdots$ or $g_t(u_{0\alpha}) \neq 0)$ or $(g_1(u_{0\alpha}) = 0$ and $\cdots$ and $g_t(u_{0\alpha}) = 0$ and $d_i(u_{0\alpha}, v_\gamma) = 0)$ or $[(g_1(u_{0\alpha}) = 0$ and $\cdots$ and $g_t(u_{0\alpha}) = 0$ and $h_1(v_\gamma) = 0$ and $\cdots$ and $h_u(v_\gamma) = 0$ and $d_i(u_{0\alpha}, v_\gamma) \neq 0)$ and $(d_i(u_{0\alpha}, v_\gamma))^N f(c; v_\gamma, \tau^{(i)}_{\alpha\gamma\delta'}(u_{0\alpha}, v_\gamma)) = 0 \Rightarrow f(c, v_\gamma, x_{\delta'}) = 0]$. We write these, with obvious abbreviations, as $(g_1(u_{0\alpha}) \neq 0$ or $\cdots$ or $g_t(u_{0\alpha}) \neq 0)$ or A_i or $[B_i$ and $(C_i \Rightarrow D_i)]$, where A_i, B_i are finite conjunctions of polynomial equations and inequalities over k in $x_{\delta'}$, v_γ, $u_{0\alpha}$, and c; and C_i, D_i are polynomial equations over k. We rewrite A_i or $[B_i$ and $(C_i \Rightarrow D_i)]$ as A_i or $[B_i$ and $(D_i$ or not $C_i)]$ and then as A_i or $[(B_i$ and $D_i)$ or $(B_i$ and not $C_i)]$. Then $g_1(u_{0\alpha}) \neq 0$ or $\cdots$ or $g_t(u_{0\alpha}) \neq 0$ or A_1 or $(B_1$ and $D_1)$ or $(B_1$ and not $C_1)$ or $\cdots$ or A_s or $(B_s$ and $D_s)$ or $(B_s$ and not $C_s)$, is a desired disjunction. Of course, this disjunction is to be quantified for all $u_{0\alpha}$ over the ambient space of U_α and over all c.

8. The proof shows that *a set S is k-constructable if and only if for every P (in V) if P is not in S then almost all k-specializations of P are not in S and if P is in S then almost all k-specializations of P are in S.* On the basis of this characterization one may give a simple proof that the set-theoretic projection of a k-constructable set is k-constructable. (See [S3].)

9. The set S is even k-closed. To show that a k-constructable set S is k-closed it suffices to show that every k-specialization of every P in S is in S. Let, then, v be in S and let $\bar{v}$ be a k-specialization of v. Setting aside trivial cases, let $\mathcal{O}(v)$, $\mathcal{O}(\bar{v})$ be $\neq \emptyset$. Let u be generic for U over $k(v, \bar{v})$. Then uv, $u\bar{v}$ are generic for $\mathcal{O}(v)$, $\mathcal{O}(\bar{v})$ over $k(v, \bar{v})$; and $uv \in F$. As $(u, \bar{v})$ is a k-specialization of (u, v), $u\bar{v}$ is a k-specialization of uv, and $u\bar{v} \in F$. Hence $\mathcal{O}(\bar{v}) \subset F$. This illustrates a useful technique for proving that a closed set is closed.

10. Compare this part of the argument with [E, p. 461].

11. The first condition, along with the second, assures us that the orbits in the G-invariant set $V - S$ are (relatively) closed. However, this follows also from the second condition alone (deleting the first condition). In fact, if v_1 is in the closure $\mathcal{O}(v)$ of the orbit $\mathcal{O}'(v)$ of v but not in $\mathcal{O}'(v)$, then the orbit $\mathcal{O}'(v_1)$ cannot meet $\mathcal{O}'(v)$, hence lies in $\mathcal{O}(v) - \mathcal{O}'(v)$, which is contained in a closed set K of dimension less than $\dim \mathcal{O}(v)$. Then the closure $\mathcal{O}(v_1)$ of $\mathcal{O}'(v_1)$ is contained in K. This is impossible, as $\dim \mathcal{O}(v_1) = \dim \mathcal{O}(v)$.

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