POINT-LIKE UPPER SEMI-CONTINUOUS DECOMPOSITIONS OF S^3

BY

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In this paper it is shown that point-like upper semi-continuous decompositions of S^3 which satisfy certain conditions on the distribution of their nondegenerate elements are topologically equivalent to S^3. In [3], J. F. Wardwell obtained similar results for arbitrary compact metric spaces but stronger hypotheses were necessary. The proof of Theorem 1 of [2] by R. H. Bing is used to obtain a stronger result for S^3. This proof shows that if for each arbitrary open set U containing the nondegenerate elements of a point-like upper semi-continuous countable decomposition G, if for each $\varepsilon > 0$ there exists a homeomorphism h of E^3 onto E^3 which shrinks each element of G into a set of diameter less than ε and which is fixed on $E^3 - U$, then $E^3/G = E^3$. It is easy to see the proof also applies to S^3.

A point-like set in S^n is one whose complement is topologically equivalent to the complement of a point. For a decomposition G we define

$$H_0(G) = \{g \in G : g \text{ is nondegenerate}\},$$

and define recursively

$$H_k(G) = \{g \in H_0(G) : g \cap \limsup H_j(G) \neq \emptyset, \forall j < k\}$$

This motivates a generalization for any ordinal number α:

$$H_\alpha(G) = \{g \in H_0(G) : g \cap \limsup H_\beta(G) \neq \emptyset, \forall \beta < \alpha\}.$$

(In the following the symbol "=" will also mean "is homeomorphic to". It should be clear from the context when this is meant and when strict equality is meant.)

Converting his results to the notation which I will use, J. F. Wardwell proved in [3]:

LEMMA. *If G is an upper semi-continuous decomposition of a compact metric space M into point-like sets and there exists a positive integer k such that $H_k(G) = \emptyset$, then $M/G = M$.*

THEOREM. *If G is an upper semi-continuous decomposition of a compact metric space M into point-like sets, if $\bigcap_{i=0}^{\infty}$ ($\limsup H_i(G)$) is zero-dimensional, and if for some countable ordinal α, $H_\alpha(G) = \emptyset$; then $M/G = M$.*

In Theorem 2 of this paper the above theorem is proved for S^3 with weakened hypotheses. Theorem 2 is applied in Theorem 3 to show how "bad" a point-

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like upper semi-continuous decomposition G of S^3 must be in order to have $S^3/G \neq S^3$.

The following Lemmas and Theorem lead up to Theorem 2.

DEFINITION. We will say that a decomposition G has $P(\alpha)$ when G has the property that for each open set U containing $H_0(G)$ and for each positive number ε there is a homeomorphism h of S^3 onto itself, fixed on $S^3 - U$, which shrinks each element of $H_\alpha(G)$ to diameter less than ε.

LEMMA 1. *If G is a point-like upper semi-continuous decomposition of S^3 with a countable number of nondegenerate elements and if there is a countable ordinal α such that G has $P(\alpha)$, then $S^3/G = S^3$.*

Proof of Lemma 1. If G has $P(0)$, the proof of the lemma is the same as the proof of Theorem 1 of [2]. The method of proof will be to show that if G has $P(\alpha)$ for some α, then G has $P(0)$. Let U be an arbitrary open set containing $H_0(G)$ and ε an arbitrary positive number. The attack is to show that if h_n is a homeomorphism fixed on $S^3 - U$ which shrinks each element of $H_{\alpha_n}(G)$ to diameter less than ε, then there is an ordinal α_{n+1} less than α_n and a homeomorphism h_{n+1}, fixed on $S^3 - U$, which shrinks each element of $H_{\alpha_{n+1}}(G)$ to diameter less than ε. We assume for the moment that we can do this, and we suppose that G has $P(\alpha)$. We let $\alpha = \alpha_0$; h_0 will be the homeomorphism guaranteed by $P(\alpha)$ for U and ε. We proceed as indicated above to find a strictly decreasing sequence $\{\alpha_i\}$ of ordinals. Since every strictly decreasing sequence of ordinals is finite there will be a positive integer m such that $\alpha_m = 0$; h_m will be the homeomorphism which is fixed on $S^3 - U$ and which shrinks each element of $H_0(G)$ to diameter less than ε. This will show that G has $P(0)$. It now remains to prove the following:

SUB-LEMMA. *Let U be an arbitrary open set containing $H_0(G)$ and ε an arbitrary positive number. If there is a homeomorphism h fixed on $S^3 - U$ which shrinks each element of $H_\beta(G)$ to size less than ε, then for some $\alpha < \beta$ there is a homeomorphism h^{**} fixed on $S^3 - U$ which shrinks each element of $H_\alpha(G)$ to diameter less than ε.*

Proof of Sub-Lemma. (If $H_\beta(G) = \emptyset$, the homeomorphism h may be assumed to be the identity map. If this happens, G' as defined below is equal to G.)

Case I. β is not a limit ordinal. Then for some α, $\beta = \alpha + 1$.

Claim 1. Only a finite number of elements of $H_\alpha(G)$ have diameter greater than or equal to ε.

Proof of Claim 1. Because h is a homeomorphism, the decomposition $G' = \{h(g) : g \in G\}$ of S^3 is also upper semi-continuous and for any ordinal γ,

$$H_\gamma(G') = \{h(g) : g \in H_\gamma(G)\}.$$

Thus if $g' \in H_\beta(G')$ the diameter of g' is less than ε. Now suppose there are more than a finite number of elements of $H_\alpha(G')$ with diameter greater than or equal to ε. Pick a countable set of these and label it $\{g_i'\}_{i=1}^{\infty}$. For each i, pick a point p_i belonging to g_i'. Since S^3 is compact, the set $\{p_i\}_{i=1}^{\infty}$ has at least one cluster point p_0. By upper semi-continuity of G', p_0 must belong to an element of G' with diameter greater than or equal to ε. Call this element g_0'. But since p_0 is a cluster point of the p_i, $g_0' \cap \limsup H_\alpha(G')$ is nonempty. This implies that $g_0' \in H_\beta(G')$. But the diameter of g_0' is not less than ε, a contradiction, Q.E.D. Claim 1.

Denote the elements of $H_\alpha(G')$ of diameter greater than or equal to ε by $\{g_i'\}_{i=1}^{n}$. We shall now define h^{**}. Since no g_i belongs to $H_\beta(G')$ we can find mutually disjoint open sets $\{V_i\}_{i=1}^{n}$ such that for all i, $g_i \subset V_i \subset U$ and if $g \in H_\alpha(G')$, $g \neq g_i$, then $g \cap V_i = \emptyset$. Because each g_i is point-like it is possible to find a homeomorphism h_i of S^3 onto S^3 fixed on $S^3 - V_i$ such that the diameter of $h_i(g_i)$ is less than ε.

Now define

$$\begin{aligned} h^*(x) &= x \qquad \text{if} \quad x \in S^3 - \textstyle\bigcup_{i=1}^{n} V_i \\ &= h_i(x) \quad \text{if} \quad x \in V_i . \end{aligned}$$

Since there are only a finite number of V_i, all are mutually disjoint, and each h_i is fixed on $S^3 - V_i$, h^* is a homeomorphism. If all elements of $H_\alpha(G')$ have diameter less than ε, h^* can be taken as the identity map.

Let $h^{**}(x) = h^*(h(x))$. Obviously, h^{**} is a homeomorphism. Let $g \in H_\alpha(G)$. If $g = h^{-1}(g_i')$, then

$$\text{diameter } (h^{**}(g)) = \text{diameter } (h^*(h(h^{-1}(g_i')))) = \text{diameter } (h_i(g_i')) < \varepsilon.$$

If $g \in H_\alpha(G) - \{h^{-1}(g_i)\}_{i=1}^{n}$, then $h(g) \in S^3 - \bigcup_{i=1}^{n} V_i$, so

$$\text{diameter } (h(g)) < \varepsilon.$$

But $h^{**}(g) = h^*(h(g)) = h(g)$ so diameter $(h^{**}(g)) < \varepsilon$. So h^{**} is a homeomorphism of S^3 onto S^3 shrinking every element of $H_\alpha(G)$ to diameter less than ε. Since both h and h^* are fixed on $S^3 - U$, h^{**} is fixed there also.

Case II. β is a limit ordinal.

Claim 2. There is an $\alpha < \beta$ such that if $g \in H_\alpha(G)$, then the diameter of $h(g)$ is less than ε.

Proof of Claim 2. Suppose that for all $\alpha < \beta$ there is a $g_\alpha \in H_\alpha(G)$ such that diameter $(h(g_\alpha)) \geq \varepsilon$. Pick a sequence of $\alpha_i < \beta$ such that $\lim_{i\to\infty} \alpha_i = \beta$. Pick $g_i \in H_{\alpha_i}(G)$ such that diameter $(h(g_i)) \geq \varepsilon$. For each i, pick $p_i \in h(g_i) \equiv g_i'$. The $\{p_i\}_{i=1}^{\infty}$ have an accumulation point p_0 since S^3 is compact and p_0 will belong to some g_0' in G'. By upper semi-continuity of G', diameter $(g_0') \geq \varepsilon$. Observe that if $g \in H_\alpha(G)$ then $g \in H_\gamma(G)$ for all $\gamma < \alpha$. So, since $\lim_{i\to\infty} \alpha_i = \beta$,

$$g_0' \cap \limsup H_\alpha(G') \neq \emptyset \quad \text{for all} \quad \alpha < \beta.$$

Therefore $g_0' \in H_\beta(G')$ implying that $g_0 = h^{-1}(g_0') \in H_\beta(G)$. This contradicts the hypothesis that diameter $(h(g)) < \varepsilon$ for all $g \in H_\beta(G)$, Q.E.D. Claim 2.

In this case h^{**} can be taken to be the identity map, Q.E.D. sub-lemma and lemma.

The lemma enables us to prove the following theorem, which is just Wardwell's theorem for S^3 with weakened hypothesis. Notice that the hypothesis that $H_\alpha(G)$ be empty in the following theorem necessarily limits G to at most a countable number of nondegenerate elements.

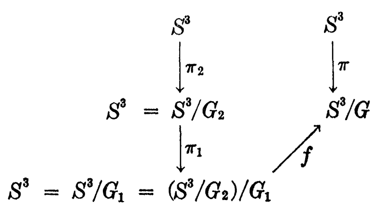
THEOREM 1. *If G is an upper semi-continuous decomposition of S^3 into point-like sets and if there is a countable ordinal α such that $H_\alpha(G) = \emptyset$ then $S^3/G = S^3$.*

Proof. G has $P(\alpha)$; the theorem follows from Lemma 1.

Theorem 1 may be generalized by using the following lemma:

LEMMA 2. *Let G and G_2 be upper semi-continuous decompositions of S^3 into point-like elements, and suppose the set of nondegenerate elements of G_2 is a subset of the set of nondegenerate elements of G. Let π_2 be the natural map of S^3 onto S^3/G_2 and let $G_1 = \{\pi_2(g) \mid g \in G\}$, a decomposition of S^3/G_2. If $S^3/G_2 = S^3$ and $S^3/G_1 = S^3$, then $S^3/G = S^3$.*

Proof of Lemma 2. The following situation arises:



π, π_1, and π_2 are natural maps as indicated. We define f taking $(S^3/G_2)/G_1$ onto S^3/G by $f(x) = \pi \circ \pi_2^{-1} \circ \pi_1^{-1}(x)$. Notice that upper semi-continuity of G and G_2 implies that of G_1.

Claim. f is a homeomorphism.

Proof of claim. The proof consists essentially of using the definition of open sets in a decomposition space.

1. *f is a function.* Let x belong to $(S^3/G_2)/G_1$, $\pi_1^{-1}(x)$ is a single element of G_1, the decomposition of S^3/G_2. By definition of G_1, $\pi_1^{-1}(x) = \pi_2(g)$ for some $g \in G$. Therefore $\pi_2^{-1} \circ \pi_1^{-1}(x) = g$, for some $g \in G$. So $\pi \circ \pi_2^{-1} \circ \pi_1^{-1}(x)$ is a single point of S^3/G and f is a well-defined function.

2. *f^{-1} is a function from S^3/G to $(S^3/G_2)/G_1$.* Notice that

$$\pi_1 \circ \pi_2 \circ \pi^{-1} \circ f = f \circ \pi_1 \circ \pi_2 \circ \pi^{-1} = \text{identity map},$$

so $f^{-1}(x) = \pi_1 \circ \pi_2 \circ \pi^{-1}(x)$. If x belongs to S^3/G, then $\pi^{-1}(x) = g$, a unique element of G. Thus $\pi_2(g)$ belongs to G_1, so $\pi_1 \circ \pi_2(g) = \pi_1 \circ \pi_2 \circ \pi^{-1}(x)$ is a unique point of $(S^3/G_2)/G_1$. Thus f^{-1} is a well-defined function. This shows directly that f is one-to-one and onto.

In parts three and four g will denote an element of a decomposition, g' the corresponding point of the decomposition space.

3. *f is open.* Let W be an open set in $(S^3/G_2)/G_1$. Since π_1 and π_2 are continuous, $U = \pi_2^{-1} \circ \pi_1^{-1}(W)$ is open.

Sub-claim. $\pi(U) = \{g' \in S^3/G : g \subseteq U\}$.

Proof of sub-claim. (1) It is clear that

$$\pi(U) \supseteq \{g' \in S^3/G : g \subseteq U\}$$

(2) Let $g' \in \pi(U)$. This implies $g \cap U \neq \emptyset$ and $\pi_1 \circ \pi_2(g) \in W$. So

$$U = \pi_2^{-1} \circ \pi_1^{-1}(W) \supseteq \pi_2^{-1} \circ \pi_1^{-1}(\pi_1 \circ \pi_2(g)) = g.$$

It follows that $\pi(U) \subseteq \{g' \in S^3/G : g \subseteq U\}$, Q.E.D. sub-claim.

That $\pi(U)$ is open follows directly from the sub-claim and the definition of open sets in an upper semi-continuous decomposition space. So

$$f(W) = \pi \circ \pi_2^{-1} \circ \pi_1^{-1}(W) = \pi(U)$$

and f is open.

4. *f is continuous.* Let U be an open set in S^3/G. Continuity of π implies that $\pi^{-1}(U)$ is open. An application of the proof of the sub-claim shows that $\pi_2 \circ \pi^{-1}(U)$ is open. Another application shows that $\pi_1 \circ \pi_2 \circ \pi^{-1}(U) = f^{-1}(U)$ is open, proving that f is continuous, Q.E.D. claim.

Since f is a homeomorphism of $(S^3/G_2)/G_1$ onto S^3/G and $(S^3/G_2)/G_1 = S^3$, it follows that $S^3/G = S^3$, Q.E.D. Lemma 2.

The following theorem is a corollary of Theorem 1 and Lemma 2; it strengthens Theorem 1.

THEOREM 2. *If G is an upper semi-continuous decomposition of S^3 into point-like sets and if for some countable ordinal α, G_2 is a decomposition of S^3 such that $H_0(G_2) = H_\alpha(G)$ and $S^3/G_2 = S^3$ then $S^3/G = S^3$.*

Proof of Theorem 2. Let $G_1 = \{\pi_2(g) : g \in G\}$ be a decomposition of S^3/G_2, where π_2 is the natural map of S^3 onto S^3/G_2. Since $H_0(G_2) = H_\alpha(G)$ the upper semi-continuity of G implies that of G_1 and G_2.

Claim. $H_\alpha(G_1) = \emptyset$.

Proof of claim. Suppose $g \in H_\alpha(G_1)$. Then $g \cap \limsup H_\beta(G_1) \neq \emptyset$ for all $\beta < \alpha$. So $\pi_2^{-1}(g) \cap \pi_2^{-1}(\limsup H_\beta(G_1)) \neq \emptyset$ for all $\beta < \alpha$. Now

$$\pi_2^{-1}(\limsup H_\beta(G_1)) = \limsup H_\beta(\{\pi_2^{-1}(g) : g \in G_1\})$$
$$= \limsup H_\beta(\{g : g \in H_0(G) - H_\alpha(G)\}) = \limsup H_\beta(G),$$

so

$$\pi_2^{-1}(g) \cap \limsup H_\beta(G) \neq \emptyset \quad \text{for all } \beta < \alpha.$$

This implies $\pi_2^{-1}(g) \in H_\alpha(G)$; but $\pi_2^{-1}(g) \in \pi_2^{-1}(H_0(G_1)) = H_0(G) - H_\alpha(G)$, a contradiction, Q.E.D. claim.

By Theorem 1, $S^3/G_1 = S^3$; so, by Lemma 2, $S^3/G = S^3$, Q.E.D. Theorem 2.

The preceding theorem may be applied in a negative sense to show how "bad" a decomposition G must be to have $S^3/G \neq S^3$. The motivation for this comes from consideration of the open question: Is there an upper semi-continuous decomposition G of S^3 whose non-degenerate elements are a countable number of tame cells and such that $S^3/G \neq S^3$?

THEOREM 3. *Let G be an upper semi-continuous decomposition of S^3 with a countable number of point-like sets as nondegenerate elements such that $S^3/G \neq S^3$. There is a subset F of $H_0(G)$ such that the decomposition G_2 of S^3 with $H_0(G_2) = F$ is upper semi-continuous, $S^3/G_2 \neq S^3$, and $H_1(G_2) = H_0(G_2)$.*

Proof of Theorem 3. *Claim* 1. There is a countable ordinal α_0 such that $H_{\alpha_0+1}(G) = H_{\alpha_0}(G)$.

Proof of Claim 1. Suppose no such α_0 exists. Then for all countable α, there is a set $g_\alpha \in H_{\alpha+1}(G) - H_\alpha(G)$. Since the sets $H_\alpha(G)$ are nested, if $\alpha \neq \beta$ then $g_\alpha \neq g_\beta$. So $\{g_\alpha\}_{\alpha\in\Omega}$ is an uncountable set of distinct nondegenerate elements of G. A contradiction, Q.E.D. Claim 1.

Let $H_{\alpha_0}(G) = F = H_0(G_2)$. Then $H_1(G_2) = H_1(H_{\alpha_0}(G)) = H_{\alpha_0+1}(G) = H_{\alpha_0}(G) = H_0(G_2)$.

Claim 2. G_2 is upper semi-continuous.

Proof of Claim 2. Let $\{g_i\}_{i=1}^{\infty}$ be any subset of G_2 such that $p_i \in g_i$ and $\{p_i\}_{i=1}^{\infty}$ converges to p_0. Each g_i belongs to G so upper semi-continuity of G implies existence of $g_0 \in G$ such that $\limsup \{g_i\}_{i=1}^{\infty} \subseteq g_0$. Since each g_i belongs to $H_{\alpha_0}(G)$, if g_0 is nondegenerate it belongs to $H_{\alpha_0+1}(G)$. But $H_{\alpha_0+1}(G) = H_{\alpha_0}(G) = H_0(G_2)$ so $g_0 \in H_0(G_2) \subseteq G_2$ and G_2 is upper semi-continuous, Q.E.D. Claim 2.

Claim 3. $S^3/G_2 \neq S^3$.

Proof of Claim 3. Suppose $S^3/G_2 = S^3$. Then by Theorem 2, $S^3/G = S^3$, a contradiction, Q.E.D. Claim 3 and Theorem 2.

In [1], Bing describes an upper semi-continuous decomposition G of E^3 such that $E^3/G \neq E^3$ which has a countable number of point-like sets (in this case indecomposable continua) as nondegenerate elements. This decomposition has the same properties in S^3 and $H_0(G) = H_1(G)$ so it serves as an illustration of Theorem 3 with $G_2 = G$.

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