3-MANIFOLDS WITH DISJOINT SPINES ARE PRODUCTS

BY

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It is the purpose of this note to show that those 3-manifolds which are products of 2-manifolds and the unit interval, are characterized by the property of having a pair of disjoint spines.

Definitions and Terminology

The statement that M is an *n-manifold* means that M is a compact, connected metric space, each of whose points has a neighborhood which is homeomorphic with E^n, euclidean n-dimensional space, or with E^n_+, the closed upper half of euclidean n-dimensional space. As is usual, the *boundary of* M, denoted ∂M is the set of all points of M which do not have neighborhoods homeomorphic with E^n, and the *interior of* M, denoted int M, is $M - \partial M$. If M is an n-manifold, and $S \subset M$, then S is a *spine* of M if and only if (i) $S \subset \text{int}\, M$, and (ii) $M - S$ is homeomorphic with $\partial M \times [0, 1)$. We note for future reference that if S is a spine of M and h is a homeomorphism of $\partial M \times [0, 1)$ onto $M - S$, then $h(\partial M) = \partial M$, and also that S, as the intersection of a decreasing sequence of compact connected sets, is connected.

THEOREM. *Suppose that M is a 3-manifold which has two disjoint spines. Then there is a 2-manifold N such that M is homeomorphic with $N \times [0, 1]$.*

Before proceeding with the proof of the theorem, we collect some lemmas. In what follow it is assumed that M is a 3-manifold with two disjoint spines.

LEMMA 1. *M has at most two boundary components.*

Proof. Let S_1 and S_2 be disjoint spines of M. Let $C_1, C_2, \cdots C_n$ be the boundary components of M. Then $M - S_1$ is homeomorphic with $\bigcup_{i=1}^n [C_i \times [0, 1)]$. We assume that the notation is chosen so that

$$S_2 \subset C_1 \times [0, 1).$$

Now $S_1 \cup [\bigcup_{i=2}^n C_i \times [0, 1)]$ is a connected set in the complement of S_2 which contains each of $C_2, C_3, \cdots C_n$. This set is connected because S_1 is connected (as the intersection of a decreasing sequence of compact connected sets), and each of $C_i \times [0, 1)$ $(i = 2, \cdots n)$ has a limit point on S_1. But since S_2 is a spine, S_2 separates each pair of boundary components of M. This implies $n \leq 2$ and establishes Lemma 1.

LEMMA 2. *If C is a boundary component of M, and U is an open set containing C, then U contains a spine of M.*

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Proof. Let C and U be as in the hypothesis, and let S_1 and S_2 be disjoint spines of M.

Case 1. Suppose $\partial M = C$. Then there is a homeomorphism h from $C \times [0, 1)$ onto $M - S_1$. Let t_1 be a number such that $h(C \times [0, t_1]) \subset U$, and let t_2 be a number such that $S_2 \subset h(C \times [0, t_2])$. Now there is a homeomorphism g of $C \times [0, 1)$ onto itself which carries $C \times [0, t_2]$ onto $C \times [0, t_1]$ and which is the identity on $C \times [S, 1)$ for some $S < 1$. Then $h(g(S_2))$ is a spine of M, and lies in U.

Case 2. Suppose that $\partial M = C \cup K$. Then there is a homeomorphism h from $C \times [0, 1) \cup K \times [0, 1)$ onto $M - S_1$. If $S_2 \subset h(K \times [0, 1))$ then the component of $M - S_2$ containing C contains S_1, and so, without loss of generality, we may assume that $S_2 \subset h(C \times [0, 1))$. As in Case 1 we choose numbers t_1 and t_2 such that

$$h(C \times [0, t_1]) \subset U \quad \text{and} \quad S_2 \subset C \times [0, t_2).$$

Now let g be a homeomorphism of $C \times [0, 1)$ onto itself which carries $C \times [0, t_2)$ onto $C \times [0, t_1)$ and which is the identity on $C \times [S, 1)$ for some $S < 1$. Then $h(g(S_2))$ is a spine of M and is contained in U. This establishes Lemma 2.

LEMMA 3. *Suppose that S is a spine of M and U is an open set in M containing S. Then M can be embedded in U.*

Proof. Let h be a homeomorphism of $\partial M \times [0, 1)$ onto $M - S_1$. Now there is a number t such that $h(\partial M \times [t, 1))$ lies in U. Then $M - h(\partial M \times [0, t))$ is homeomorphic with M and lies in U.

LEMMA 4. *If C is a compact contractible 3-manifold in M, then C is a 3-cell.*

Proof. It follows from Lemmas 2 and 3 that C can be embedded in the product of a 2-manifold and the unit interval. Since C is contractible, it can be embedded in the universal covering space of the product of a 2-manifold and the unit interval. Since these spaces are all embeddable in E^3, it follows that C can be embedded in E^3. Then C is a 3-cell. This establishes Lemma 4.

LEMMA 5. *Let C be a boundary component of M. Then the homomorphism*

$$i_* : \pi_1(C) \rightarrow \pi_1(M),$$

induced by inclusion, is onto. If M has two boundary components, then i_* is one-to-one.

Proof. We first consider the case where M has two boundary components C and K. Let $p \in C$, and let l be a loop in M based at p. Then since K has a product neighborhood in M, l is homotopic to a loop l_1 in $M - K$. Now by Lemma 2, there exists a spine S of M such that the image of l_1 misses S. Then the image of l_1 lies in a subset of M that is homeomorphic with $C \times [0, 1]$

and hence l_1 is homotopic in M to a loop l_2 in C. This shows that $i_* : \pi_1(C) \to \pi_1(M)$ is onto. A similar argument shows that in this case i_* is one-to-one.

Now suppose that C is the only boundary component of M. Let X be a product neighborhood of C in M, and using Lemma 2 let S be a spine of M in int X. Now let $p \in C$, and let l be a p-based loop in M. Now l is homotopic in M to a loop l_1 whose image intersects X only in the fiber of X from $p \times \{0\} = p$ to $p \times \{1\}$. Now since S does not separate M, S does not separate the boundary components of X and so there is a path f in X from $p \times \{0\}$ to $p \times \{1\}$, whose image misses S. Now let g be the projection of the path f onto the boundary component of X distinct from C. Now the loop obtained by traversing f then g^{-1} then the part of l in $M - X$, then g, then f^{-1}, is homotopic to l_1 and misses the spine S. Since this loop misses S, it is homotopic, in M, to a loop in C. Hence l is homotopic to a loop in C. This shows that i_* is onto.

Proof of the theorem. We first assume that M has one boundary component C. Now since $i_* : \pi_1(C) \to \pi_1(M)$ is onto, we may use the loop theorem and Dehn's lemma [3] to find a disk D in M such that $D \cap \partial M = \partial D$ and ∂D is not homotopic to 0 in ∂M. We then cut M along D. If ∂D does not separate ∂M we obtain a manifold M' with connected boundary C' such that the genus of C' is less than the genus of C. An application of the Tietze extension theorem shows that $i_* : \pi_1(C') \to \pi_1(M')$ is onto. If ∂D separates ∂M we obtain manifolds M_1 and M_2 with boundaries C_1 and C_2 respectively such that the genus of C_i is less than the genus of C and $i_* : \pi_1(C_i) \to \pi_1(M_i)$ is onto. A repeated application of this argument and the fact that each compact contractible 3-manifold in M is a 3-cell shows that M is homeomorphic to a 3-cell with solid handles (some of these handles may be non-orientable). Hence in this case M is homeomorphic to the product of a 2-manifold and the unit interval. It is well known that the factorization is not unique.

If M has two boundary components C and D, then it follows from Lemma 5 that these boundary components are homeomorphic. Now if these boundary components are not projective planes, then it follows from Lemma 4, Lemma 5, and Theorem 3.1 of [1] or [4] that M is homeomorphic with $C \times I$. In the case that C is a projective plane we must make a special argument. Let $\tilde{M}$ be the universal covering space of M and let $P : \tilde{M} \to M$ be the covering map. Since $i_* : \pi_1(C) \to \pi_1(M)$ is onto, $\tilde{M}$ has two boundary components, each of which is a 2-sphere. Let S be a spine of M. The $M - S$ is homeomorphic with $C \times [0, 1) \cup D \times [0, 1)$. Then $P^{-1}(C \times [0, 1))$ is the universal covering space of $C \times [0, 1)$ and $P^{-1}(D \times [0, 1))$ is the universal covering space of $D \times [0, 1)$. Hence $P^{-1}(S)$ is a spine of $\tilde{M}$. It follows that $\tilde{M}$ has two disjoint spines and by what we have already shown that $\tilde{M}$ is homeomorphic with $S^2 \times [0, 1]$. Now the covering translation

$$\tau : S^2 \times [0, 1] \to S^2 \times [0, 1]$$

is a fixed point free involution which leaves boundary components invariant and it follows from [2] that M is homeomorphic with $C \times [0, 1]$. This completes the proof of the theorem.

It should be noted that the product of a 2-manifold and an interval does have two disjoint spines, and also that the assumption of the connectivity of M was not necessary since M has two disjoint spines if and only if each component of M does.

References

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