ABELIAN p-GROUPS WITHOUT PROPER ISOMORPHIC PURE DENSE SUBGROUPS

BY

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Introduction

This paper is concerned with finding abelian p-groups G, without non-zero elements of infinite height, that have the property that if H is a pure dense subgroup of G which is isomorphic to G, then $H = G$. This, of course, is a generalization of the problem, first solved by Crawley [2] and later considered by Pierce [8] and Hill and Megibben [5], of finding p-groups G without any proper isomorphic subgroups whatsoever.

In §1 we show that the problem reduces completely to one on the socle of the group and that it can be solved in several cases. Furthermore, as a corollary to this work, we extend a result of Hill and Megibben on the existence of groups without proper isomorphic subgroups [5; Theorem 6.3]. In §3 we show that the class of groups we are interested in is contained in a class studied by Pierce in [8], namely, those groups G such that the Jacobson radical of the endomorphism ring of G is equal to the ideal of height increasing endomorphisms of G. Furthermore, these classes are identical for groups with bounded Ulm invariants but not, in general, otherwise.

Unless otherwise stated, all groups referred to will be p-groups (for a fixed p) without non-zero elements of infinite height. Topological statements will be with respect to the p-adic topology. The torsion subgroup of the completion with respect to the p-adic topology of a group G will be denoted by $\bar{G}$ and called the torsion completion of G. If $G = \bar{G}$, then G will be called *closed*. The height and order of an element g will be denoted by $h(g)$ and $o(g)$ respectively. If $o(g) = p^m$, then we will write $e(g) = m$. The endomorphism ring of a group G will be denoted by $E(G)$. Finally, the set theoretic difference of two groups A and B (such that $A \supseteq B$) will be denoted by $A \backslash B$ and the cardinal $2^{\aleph_0}$ will be written as c.

1. Groups without proper isomorphic pure dense subgroups

(1.1) DEFINITION. An endomorphism α on a group G will be called an *isometry* if α is one-to-one and $\alpha(G)$ is pure in G. The endomorphism α will be called *dense* if $\alpha(G)$ is dense in G.

(1.2) THEOREM. *A dense isometry on a closed group is onto.*

Proof. If α is a dense isometry on the closed group C, then $\alpha(C)$ is a direct summand of C and $C/\alpha(C)$ is divisible. Thus $C = \alpha(C) \oplus D$ for some divis-

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ible subgroup D. Inasmuch as a group without elements of infinite height can have no non-trivial divisible subgroups, $D = (0)$ and $\alpha(C) = C$.

To find other groups on which every dense isometry is onto we first show that the problem reduces to the socle.

(1.3) LEMMA. *The following conditions are equivalent for an endomorphism α on a group G.*

(i) $h(\alpha(x)) = h(x)$ *for every* $x \in G[p]$.

(ii) $h(\alpha(x)) = h(x)$ *for every* $x \in G$.

(iii) α *is an isometry on* G.

Proof. That (ii) implies (iii) and (iii) implies (i) is trivial. Assuming that (i) holds, we can show, by induction on $e(x)$, that (ii) holds. If $m = h(x) < h(\alpha(x))$, then $h(p\alpha(x)) \geq m + 2$, $h(px) \geq m + 2$ and $x = p^{m+1}y + z$ for some $z \in G[p]$. Inasmuch as $\alpha(z) = \alpha(x) - p^{m+1}\alpha(y)$, we infer that $h(\alpha(z)) \geq m + 1$, so that by (i), $h(z) \geq m + 1$. But then $h(x) \geq m + 1$, a contradiction.

(1.4) LEMMA. *An isometry α on a group G is dense if and only if $\alpha(G[p])$ is dense in $G[p]$.*

Proof. If $\alpha(G)$ is dense in G, $x \in G[p]$ and k is any natural number, there is an element y in G such that $h(x - \alpha(y)) \geq k$,

$$h(py) = h(p\alpha(y)) \geq k + 1$$

and $y = p^k w + z$ with $z \in G[p]$. Then $h(x - \alpha(z)) \geq k$ with $z \in G[p]$ as was to be shown.

Conversely, assuming that $\alpha(G[p])$ is dense in $G[p]$, we can show by induction on $e(x)$, that for any natural number k, there is an element $y \in G$ such that $h(x - \alpha(y)) \geq k$. For $x \in G$, there exists $w \in G$ such that

$$h(px - \alpha(w)) \geq k + 1.$$

But then $h(w) = h(\alpha(w)) \geq 1$, whence $w = pw'$ for some $w' \in G$, and

$$h(p(x - \alpha(w'))) \geq k + 1$$

so that $x - \alpha(w') = p^k u + z$ with $z \in G[p]$. Thus

$$h(x - \alpha(w') - z) \geq k$$

and since $z \in G[p]$, there is an element $v \in G[p]$ such that $h(z - \alpha(v)) \geq k$. Consequently,

$$h(x - \alpha(w' + v)) \geq k$$

and $w' + v$ is the desired element.

It is well known that a fixed endomorphism α on a vector space V over a field F induces an $F[X]$ module structure on V by

$$f(X)\cdot v = f(\alpha)(v)$$

for all $f(X) \in F[X]$. Given an endomorphism α on a group G, we can therefore speak of the $Z_p[X]$ module structure on $G[p]$ induced by α, or say that $G[p]$ is to be viewed as a $Z_p[X]$ module through α.

(1.5) LEMMA. *If the group G has finite Ulm invariants, then every isometry α on G is dense.*

Proof. Given $x \in G[p]$ and a natural number k, observe that α induces an endomorphism α' on the finite dimensional vector space $(G[p])/((p^kG)[p])$, so that there is a polynomial $f(X)$ in $Z_p[X]$ such that

$$f(\alpha')((G[p])/((p^kG)[p])) = 0$$

or

$$f(\alpha)(G[p]) \subseteq (p^kG)[p].$$

Suppose, for our fixed x, that $g(X) \in Z_p[X]$ is of least degree such that

$$g(\alpha)(x) \in (p^kG)[p].$$

Then g must have a non-zero constant term, for otherwise

$$g(\alpha)(x) = \alpha g_1(\alpha)(x)$$

for some $g_1 \in Z_p[X]$ which, since

$$h(g_1(\alpha)(x)) = h(\alpha g_1(\alpha)(x)) \geq k$$

contradicts the minimality of the degree of g. Thus g can be written as

$$g(X) = \sum_{i=0}^{m} a_i X^i$$

with $a_0 \neq 0$, whence

$$x - \alpha(\sum_{i=1}^{m} (-a_i/a_0)\alpha^{i-1}(x)) \in (p^kG)[p]$$

and $\sum_{i=1}^{m} (-a_i/a_0)\alpha^{i-1}(x)$ is the desired element.

We recall that if α is an endomorphism on a group G, it has a unique extension $\bar{\alpha}$ to $\bar{G}$ which is given by

$$\bar{\alpha}(x) = \lim_i \alpha(x_i)$$

where $\{x_i\}_{i=1}^{\infty}$ is any sequence in G such that $x = \lim_i x_i$.

(1.6) LEMMA. *If α is a dense isometry on the group G, then its extension $\bar{\alpha}$ to $\bar{G}$ is a dense isometry on $\bar{G}$.*

Proof. Since $\bar{\alpha}(\bar{G}) \supseteq \alpha(G)$, $\alpha(G)$ is dense in G and G is dense in $\bar{G}$, it follows immediately that $\bar{\alpha}(\bar{G})$ is dense in $\bar{G}$. If $x \in \bar{G}[p]$ and $x = \lim_i g_i$ with $g_i \in G$ for every i, observe that there is a subsequence $\{g_i'\}_{i=0}^{\infty}$ of $\{g_i\}_{i=0}^{\infty}$ with $x = \lim_i g_i'$, $g_i' \in G[p]$ and $h(g_i') \leq h(x)$ for every i. Inasmuch as

$$h(\bar{\alpha}(x)) \leq \sup_i \{h(\alpha(g_i'))\} = \sup_i \{h(g_i')\} \leq h(x)$$

we conclude that $\bar{\alpha}$ is an isometry.

(1.7) THEOREM. *If the group G has the property that $(\bar{G}[p])/(G[p])$ is finite, then every dense isometry on G is an automorphism.*

Proof. If α is a dense isometry on G, then by 1.6 and 1.2, its extension $\bar{\alpha}$ to $\bar{G}$ is an automorphism of $\bar{G}$. The induced map $\bar{\alpha}_1$ on $\bar{G}[p]/G[p]$ is then onto and hence one-to-one, since this quotient group is finite. Now, given $x \in G[p]$, there is an element $y \in \bar{G}[p]$ such that $\bar{\alpha}(y) = x$. But since $\bar{\alpha}_1$ is one-to-one, y must also be in $G[p]$ and α is onto $G[p]$. Inasmuch as the purity of $\alpha(G)$ implies that $\alpha(G)$ is a maximal subgroup supported by $\alpha(G[p])$, we conclude that $\alpha(G) = G$.

(1.8) DEFINITION. Given a property (P) for endomorphisms on a vector space, we will say that the endomorphism α of the group G *eventually has* (P) if there is a natural number n such that $\alpha \mid ((p^n G)[p])$ has the property (P).

We recall the definitions, given by Kaplansky in [7], that an endomorphism α of a vector space V over a field F is algebraic (locally algebraic) if V is bounded (torsion) when viewed as an $F[X]$ module through α.

(1.9) LEMMA. *An eventually locally algebraic one-to-one endomorphism α on a group G is onto under either of the following two conditions*: (i) *α is a dense isometry*; (ii) *G has finite Ulm invariants.*

Proof. First observe that if α is a one-to-one map on a group G and for a given x in $G[p]$, $f(X) = \sum_{i=1}^{m} a_i X^i$ is a polynomial in $Z_p[X]$ of smallest degree such that $f(\alpha)(x) = 0$, then $a_0 \neq 0$ and

$$x = \alpha(\textstyle\sum_{i=1}^{m} (-a_i/a_0)\alpha^{i-1}(x)).$$

For the proof of (i), we infer from this observation that for some natural number k, $\alpha((p^k G)[p]) = (p^k G)[p]$. But then for $x \in G[p]$, there is an element $y \in G[p]$ such that

$$x - \alpha(y) \in (p^k G)[p] = \alpha((p^k G)[p])$$

so that, by letting $z \in G[p]$ be such that $x - \alpha(y) = \alpha(z)$, it follows that $x = \alpha(y + z)$. Thus $\alpha(G[p]) = G[p]$ and, as in the proof of (1.8), since α is an isometry, $\alpha(G) = G$.

For (ii), suppose that α is locally algebraic on $(p^k G)[p]$. Since G has finite Ulm invariants, it follows, as in the proof of (1.5), that there is a polynomial $f(X) \in Z_p[X]$ such that $f(\alpha)(G[p]) \subseteq (p^k G)[p]$. Therefore, α is locally algebraic on $G[p]$. Thus, by our observation above, for each $x \in G[p]$,

$$x = \alpha(\textstyle\sum_{i=1}^{m} (-a_i/a_0)\alpha^{i-1}(x))$$

for some $\{a_i\}_{i=0}^{m}$ in Z_p. Hence α maps each $(p^n G)[p]$ onto itself and we infer by a theorem of Pierce [8; 13.1] that α is an automorphism.

Thus we can produce groups without proper isomorphic pure dense subgroups or even without any proper isomorphic subgroups at all by finding groups all of whose endomorphisms are eventually locally algebraic. Following closely the technique of Hill and Megibben [5], we will show that there are many such groups.

In fact, we begin by citing a result in [5, Corollary 6.2].

(1.10) LEMMA. *If α is an endomorphism of a closed group C and if S is a subsocle of C such that $|(C[p])/S| < c$ and $|\alpha(S)| < c$, then α is eventually 0 on C.*

The following lemma is essentially a part of the proof of Theorem 6.3 in [5]. For the sake of completeness, we give its proof here.

(1.11) LEMMA. *Let W be a subspace of the vector space V over a field F and let α be an endomorphism of V such that $\alpha(W) \subseteq W$. Then either $(\alpha - t)(V) \subseteq W$ for some $t \in F$, or there is an element $v \in V$ such that $\{W, (v), (\alpha(v))\} \perp$.*

Proof. Denoting by α_1, the endomorphism of V/W induced by α, we see that the lemma is equivalent with the assertion that either $(\alpha_1 - t)(V/W) = 0$ for some $t \in F$, or there is an element $\bar{v} \in V/W$ such that $\{(\bar{v}, (\alpha_1(\bar{v}))\} \perp$. However if there does not exist $\bar{v} \in V/W$ such that $\{(\bar{v}), (\alpha_1(\bar{v}))\} \perp$, then every element $\bar{v} \in V/W$ is an eigenvector for α_1, whence α_1 is constant and there is an element t in F such that $(\alpha_1 - t)(V/W) = 0$.

(1.12) THEOREM. *Let C be a closed unbounded group with a countable basic subgroup. Then there is a dense subsocle S of C, of infinite index in $C[p]$, such that for every pure dense subgroup G supported by S, every endomorphism of G is algebraic.*

Proof. Since the basic subgroup B of C is countable, there are at most c endomorphisms of C which are not eventually algebraic, and we can index this set by $\{\varphi_\alpha\}_{\alpha<\mu}$ where μ is an ordinal which does not exceed the first ordinal of cardinality c. We now choose, inductively, a collection of elements $\{x_\alpha\}_{\alpha<\mu}$ in $C[p]$ such that

$$\{B[p], (x_\alpha), (\varphi_\alpha(x_\alpha))\}_{\alpha<\mu} \perp.$$

If $\{x_\alpha\}_{\alpha<\lambda}$ has been chosen for $\lambda < \mu$, view $C[p]$ as a $Z_p[X]$ module through φ_λ and let W be the $Z_p[X]$ submodule of $C[p]$ generated by $\{B[p], x_\alpha, \varphi_\alpha(x_\alpha)\}_{\alpha<\lambda}$. Then, by 1.11, either there is an element $x_\lambda \in C[p]$ such that

$$\{B[p], (x_\alpha), (\varphi_\alpha(x_\alpha))\}_{\alpha<\lambda} \perp \tag{1}$$

or

$$(\varphi_\lambda - t)(C[p]) \subseteq W \tag{2}$$

for some $t \in Z_p$. But since $B[p]$ is countable, λ has fewer than c predecessors, and $Z_p[X]$ is a countable ring, W is of cardinality less than c. In view of

(1.10) then, (2) implies that φ_λ is eventually algebraic, which is a contradiction and (1) holds, as was to be shown.

Thus, if we take $S = B[p] \oplus (\oplus \sum_{\alpha<\mu} (x_\alpha)$ and G is a pure dense subgroup such that $G[p] \subseteq S$ and φ is an endomorphism of G, it cannot be that $\varphi(x_\alpha) = \varphi_\alpha(x_\alpha)$ for any $\alpha < \mu$. Consequently, $\varphi \neq \varphi_\alpha$ for all $\alpha < \mu$ and φ is eventually algebraic.

The following theorem, when taken with 1.9, contains an extension of Theorem 6.3 in [5].

(1.13) THEOREM. *Let C be a closed group with a countable basic subgroup and let S be a proper dense subsocle of C of cardinality c. Then S supports a pure dense subgroup G of C such that every endomorphism of G is eventually algebraic.*

Proof. Index the set of endomorphisms of C that leave S invariant but are not eventually algebraic by $\{\varphi_\alpha\}_{\alpha<\mu}$ where μ does not exceed the first ordinal of cardinality c. We will inductively construct an ascending chain of subgroups. $\{T_\alpha\}_{\alpha<\mu}$ of C such that for every $\alpha < \mu$, (i) $|T_\alpha| \leq \aleph_0 + |\alpha|$, (ii) $T_\alpha[p] \subseteq pC \cap S$, (iii) If G is a subgroup of C such that $T_\alpha \subseteq G$ and $G[p] \subseteq S$, then $\varphi_\alpha(G) \nsubseteq G$.

Assuming that T_β has been given for every ordinal $\beta < \alpha$, view $C[p]$ as a $Z_p[X]$ module through φ_α and let W be the submodule of $pC \cap S$ generated by $\bigcup_{\beta<\alpha} (T_\beta[p])$. Note that routine arguments yield that $|\bigcup_{\beta<\alpha} T_\beta| \leq \aleph_0 + |\alpha|$ and $|W| \leq \aleph_0 + |\alpha|$. To form T_α we apply 1.11 to the Z_p-space $pC \cap S$, its subspace W and the endomorphism φ_α to consider two cases.

Case 1. There is an element $x \in (pC) \cap S$ such that

$$\{W, (x), (\varphi_\alpha(x))\} \perp.$$

Choose $z \in C[p] \backslash S$ and $x' \in C$ such that $px' = x$. Letting

$$T_\alpha = (\textstyle\bigcup_{\beta<\alpha} T_\beta) + (x') + (\varphi_\alpha(x') - z),$$

it is easily seen that $|T_\alpha| \leq \aleph_0 + |\alpha|$, and $T_\alpha[p] \subseteq (pC) \cap S$. Further, if G is a subgroup containing T_α such that $G[p] \subseteq S$ and $\varphi_\alpha(G) \subseteq G$, then x', $\varphi_\alpha(x')$ and $\varphi_\alpha(x') - z$ are in G, whence $z \in G[p] \subseteq S$, a contradiction. Thus T_α is the desired subgroup.

Case 2. There is an element $t \in Z_p$ such that

$$(\varphi_\alpha - t)((pC) \cap S) \subseteq W. \tag{3}$$

Inasmuch as $|S| = c$, $|(C[p])/((pC)[p])| = \aleph_0$, and

$$\begin{aligned} S/((pC) \cap S) &= S/((pC)[p] \cap S) \cong (S + (pC)[p])/((pC)[p]) \\ &= (C[p])/((pC)[p]), \end{aligned}$$

it must be that $|pC \cap S| = c$. If we let K be the kernel of $\varphi_\alpha - t$ restricted to $(pC) \cap S$ we then have, in view of (1) and the fact that $|W| < c$, that

$|K| = c$. Thus we can choose $x \in K \backslash (K \cap W)$ and, letting x' be such that $px' = x$, we have an element with the properties

$$(4) \qquad e(x') = 2, \qquad px' \in S \backslash W, \qquad (\varphi_\alpha - t)(x') \in C[p].$$

We wish to show next that there is an element $y \in C[p]$ such that

$$(5) \qquad (\varphi_\alpha - t)(x' + y) \in C[p] \backslash S.$$

If there is no such element, then, for every $y \in C[p]$,

$$(\varphi_\alpha - t)(x' + y) \in S,$$

whence

$$(\varphi_\alpha - t)C[p] \subseteq ((\varphi_\alpha - t)(x')) + S,$$

and

$$(6) \qquad (\varphi_\alpha - t)^2 C[p] \subseteq ((\varphi_\alpha - t)^2(x')) + (\varphi_\alpha - t)(S).$$

Writing the subgroup S as $S = U \oplus ((pC) \cap S)$, with U countable, we see that

$$\begin{aligned}(\varphi_\alpha - t)(S) &\subseteq (\varphi_\alpha - t)(U) + (\varphi_\alpha - t)((pC) \cap S) \\ &\subseteq (\varphi_\alpha - t)(U) + W.\end{aligned}$$

We infer, then, from (6) that

$$(\varphi_\alpha - t)^2 C[p] \subseteq ((\varphi_\alpha - t)^2(x')) + (\varphi_\alpha - t)(U) + W.$$

Inasmuch as the cardinality of the right hand side of this formula is less than c, we conclude from 1.10 that φ_α is eventually algebraic, a contradiction. Thus, an element y with the property (5) exists. Let

$$T_\alpha = (\textstyle\bigcup_{\beta<\alpha} T_\beta) + (x' + y).$$

Clearly, $|T_\alpha| \leq |\alpha| + \aleph_0$. It is readily checked, using (4), that

$$T_\alpha[p] \subseteq pC \cap S.$$

Now, if G is a subgroup of C such that $G[p] \subseteq S$, $T_\alpha \subseteq G$ and $\varphi_\alpha(G) \subseteq G$, then

$$x' + y \in G, \qquad (\varphi_\alpha - t)(x' + y) \in G[p] \subseteq S,$$

and the property (2) is contradicted.

We have therefore constructed the desired chain of subgroups. Now choose a maximal subgroup G of C such that $G[p] = S$ and $T_\alpha \subseteq G$, for every $\alpha < \mu$. Then G is neat with a dense subsocle, whence it is pure and dense in C (see [4; Theorem 1]). Since G does not admit any of the endomorphisms $\{\varphi_\alpha\}_{\alpha<\mu}$, every endomorphism of G is eventually algebraic, as was to be shown.

We close this section with the remark that the subsocle S in (1.13) cannot be taken to be countable because a group with a countable socle is itself countable and as we shall see in §3, such a group always has a dense isometry which is not onto. It seems (to this author) doubtful that the conclusion of 1.13 could be obtained in case the cardinality of S is uncountable but less than c.

2. A counterexample

We show in this section the (not very surprising) fact that the condition that $(\bar{G}[p])/(G[p])$ be finite in (1.7) cannot be weakened. We include this counterexample not only for its own sake but because, as we shall show, its construction sheds some light on a problem posed by Pierce.

We fix for this section the following notation:

$B = \oplus \sum_{i=0}^{\infty} (b_i)$ such that $e(b_i) = i + 1$,

$\alpha \in \bar{B}$ given by $\alpha(b_i) = b_i + pb_{i+1}$,

$\bar{B}[p]$ is a $Z_p[X]$ module through α.

(2.1) LEMMA. *Given $f \in Z_p[X]$ such that $f(1) \neq 0$, then $f(\alpha)$ is an isometry on $\bar{B}[p]$.*

Proof. If $f(X) = \sum_{i=1}^{m} a_i X^i$ and $\beta = \alpha - 1$, then

$$f(\alpha) = \sum_{i=0}^{m} a_i + \beta g(\beta) = f(1) + \beta g(\beta)$$

where $g \in Z_p[X]$. Then, for $x \in G[p]$,

$$h(f(\alpha)(x)) = h(f(1)x + \beta g(\beta)(x)) = h(x),$$

since

$$h(\beta g(\beta)(x)) > h(g(\beta)(x)) \geq h(x).$$

(2.2) THEOREM. *There is a dense subsocle S of $\bar{B}$ such that $\alpha(S) \subseteq S$, α is not onto S and $(\bar{B}[p])/S$ is a countable injective $Z_p[X]$ module.*

Proof. We first note that $\bar{B}[p]$ is a torsion-free $Z_p[X]$ module. To see this, observe that it suffices to show that $f(\alpha)$ is one-to-one on $\bar{B}[p]$ for every prime $f \in Z_p[X]$. If $f(1) \neq 0$, then by (2.1), $f(\alpha)$ is an isometry and hence is one-to-one. If f is prime and $f(1) = 0$, then $f(\alpha) = r(\alpha - 1)$ with $r \in Z_p$, so that if, for $r_i \in Z$ $(i = 0, 1, 2, \cdots)$,

$$f(\alpha)(\textstyle\sum_0^{\infty} r_i b_i) = 0$$

then

$$r(\textstyle\sum_0^{\infty} r_i p b_{i+1}) = 0$$

and $p^{i+1} \mid rr_i$ for every i. Therefore, either $r = 0$ or $\sum_{i=0}^{\infty} r_i b_i = 0$.

Since $\bar{B}[p]$ is an uncountable torsion-free module and $Z_p[X]$ is countable, $\bar{B}[p]$ is of uncountable torsion free rank. Further, since $B[p]$ is of countable rank, the module $M = (\bar{B}[p])/(B[p])$ is of infinite rank. We have, therefore, a $Z_p[X]$ monomorphism ρ

$$0 \to (\oplus \textstyle\sum_{i=0}^{\infty} Z_p[X]) \xrightarrow{\rho} M.$$

On the other hand, since $Z_p(X)$, the ring of rational functions, is countable and $\oplus \sum_{i=0}^{\infty} Z_p[X]$ is a free $Z_p[X]$ module of countable rank, there is an epimorphism σ

$$(\oplus \textstyle\sum_{i=0}^{\infty} Z_p[X]) \xrightarrow{\sigma} Z_p(X) \to 0.$$

However, since $Z_p(X)$ is injective, the epimorphism σ can be lifted to an epimorphism τ of M onto $Z_p(X)$, so that we have the commuting diagram

$$\begin{tikzcd} \left(\oplus \sum_{i=0}^{\infty} Z_p[X]\right) \arrow[r, two heads, "\rho"] \arrow[d, two heads, "\sigma"'] & M. \arrow[dl, two heads, "\tau"] \\ Z_p(X) & \end{tikzcd}$$

Viewing $Z_p[X]$ as a submodule of $Z_p(X)$, let

$$N = \{m \in M \mid \tau(m) \in Z_p[X]\},$$

and let S be the subsocle of $\bar{B}[p]$ containing $B[p]$ such that $S/(B[p]) = N$. Clearly S is dense in $\bar{B}[p]$ since it contains $B[p]$. Inasmuch as

$$(\bar{B}[p])/S \cong \frac{(\bar{B}[p])/(B[p])}{S/(B[p])} = M/N \cong (Z_p(X))/(Z_p[X]),$$

$(\bar{B}[p])/S$ is countable and injective.

Since τ is an epimorphism, there is an element $g \in \bar{B}[p]$ such that

$$\tau(g + B[p]) = X^{-1}.$$

Clearly, $g \notin S$, but $\alpha(g) \in S$, because

$$\tau(\alpha(g) + B[p]) = \tau(X \cdot g + B[p]) = X \cdot \tau(g + B[p]) = 1.$$

In view of the fact that α is a one-to-one map on $\bar{B}[p]$, we then conclude that α does not map S onto S.

Thus to find a pure dense subgroup G of $\bar{B}$ such that α is a dense isometry on G but is not onto, it would suffice to find a pure subgroup G of $\bar{B}$ such that $\alpha(G) \subseteq G$ and $G[p] = S$. That is, we must give an affirmative answer in this case to the following problem posed by Pierce [6; p. 367]:

Let B be a basic group and P a subsocle of $\bar{B}$ such that $B[p] \subseteq P$. Let φ be an endomorphism of $\bar{B}$ such that $\varphi(P) \subseteq P$. Does there exist a pure subgroup G of $\bar{B}$ such that $B \subseteq G$, $G[p] = P$, and $\varphi(G) \subseteq G$?

Stringall shows in [9] that the answer to Pierce's question is, in general, no. However, it is not clear from the example he gives when, if ever, the answer would be yes. We show next therefore:

(2.3) THEOREM. *Let P be a dense subsocle of a closed group C and let φ be an endomorphism of C such that $\varphi(P) \subseteq P$. Then there is a pure dense subgroup G of C such that $\varphi(G) \subseteq G$ and $G[p] = P$ if and only if there is a sequence of pairs of subgroups $\{H_i, G_i\}_{i=0}^{\infty}$ such that for $i = 1, 2, 3, \cdots$,*

(a) $H_1 = C[p]$, $G_1 = P$,
(b) $H_{i+1} = \{g \in C \mid pg \in G_i\}$,
(c) $H_i \cap G_{i+1} = G_i$,
(d) $H_i + G_{i+1} = H_{i+1}$,
(e) $\varphi(G_i) \subseteq G_i$, $\varphi(H_i) \subseteq H_i$.

Proof. For necessity, observe that if such a group G exists, then $G_i = G[p^i]$ $(i = 1, 2, \cdots)$, $H_1 = C[p]$ and

$$H_{i+1} = \{g \in C \mid pg \in G_i\}, \qquad i = 1, 2, \cdots,$$

satisfy the conditions (a)–(e).

On the other hand, given such a sequence of subgroups, take $G = \bigcup_{i=1}^{\infty} G_i$. Inasmuch as $\{G_i\}_{i=1}^{\infty}$ forms an ascending sequence, G is a subgroup of C such that $\varphi(G) \subseteq G$. Next note that repeated applications of (a), (c) and (d) yield

(f) $G_i \cap C[p] = P,\ G_i + C[p] = H_i$, for $i = 1, 2, 3, \cdots$.

We infer from the second of these formulas that $pH_i \subseteq pG_i$. Consequently,

$$G[p] = (\bigcup_{i=1}^{\infty} G_i) \cap C[p] = \bigcup_{i=1}^{\infty} (G_i \cap C[p]) = P,$$

$$G \cap pC = (\bigcup_{i=1}^{\infty} G_i) \cap pC = \bigcup_{i=1}^{\infty} (G_i \cap pC) \subseteq \bigcup_{i=1}^{\infty} pH_{i+1} \subseteq \bigcup_{i=1}^{\infty} pG_{i+1} \subseteq pG.$$

Thus G is a neat subgroup with P as its socle and G is pure and dense.

Remark. We see, then, that in order to find the group G in Pierce's problem, it is necessary and sufficient to be able to carry out an inductive process. Assuming the subgroups $\{H_i, G_i\}_{i=1}^{k}$ satisfying (a)–(e) are given, we obtain H_{k+1} from (b) and the problem is to find G_{k+1}. Notice that by (b), H_{k+1}/G_k is a Z_p vector space and by (e), it is a module over $Z_p[X]$ through $\bar{\varphi}$, the map induced on H_{k+1}/G_k by φ. The conditions (c)–(e) then require that H_k/G_k be a direct summand of the $Z_p[X]$ module H_{k+1}/G_k. It is easy to see, then, how one could define endomorphisms so that even G_2 could not be found. Further, this gives a hint of why we demanded that $\bar{B}[p]/S$ be an injective $Z_p[X]$ module in (2.2).

2.4) THEOREM. *There is a subgroup G of $\bar{B}$ such that $G[p] = S$, $\alpha(G) \subseteq G$ but α is not onto G.*

Proof. Inasmuch as H_1 and G_1 are given by the condition (a) in (2.3), we must show that for $\{H_i, G_i\}_{i=1}^{k}$ satisfying (a)–(e), we can find H_{k+1} and G_{k+1}. As noted above, this reduces to showing that H_k/G_k is a direct summand of the $Z_p[X]$ module H_{k+1}/G_k. It follows from condition (f) in the proof of 2.3 that as abelian groups,

$$H_k/G_k \cong (\bar{B}[p])/S.$$

But this shows that H_k/G_k is a direct summand of H_{k+1}/G_k, since this isomorphism is clearly a $Z_p[X]$ isomorphism and by (2.2), $(\bar{B}[p])/S$ is an injective $Z_p[X]$ module.

3. The Jacobson radical of $E(G)$ and the height increasing endomorphisms of G

In [8; §14] Pierce shows that for a group G, there is a ring homomorphism of $E(G)$ into a complete direct sum of endomorphism rings of vector spaces over Z_p which has as its kernel

$$H(G) = \{\varphi \in E(G) \mid h(\varphi(x)) > h(x) \quad \text{for} \quad x \in G[p] \quad \text{and} \quad h(x) < \infty\}.$$

He notes that the Jacobson radical $J(G)$ of $E(G)$ is contained in $H(G)$ and raises the question of when these two ideals are equal. We will show that this question is closely related to those we have been considering.

(3.1) THEOREM. *If the group G has the property that every dense isometry on G is onto, then $H(G) = J(G)$. If, further, there is a finite bound on the Ulm invariants of G, then the converse also holds.*

Proof. Since $H(G)$ is an ideal in $E(G)$, the condition that $H(G) = J(G)$ is equivalent with the condition that $1 - \alpha$ be an automorphism for every α in $H(G)$. Thus the first part of the theorem would be established if we could show that for $\alpha \in H(G)$, $1 - \alpha$ is a dense isometry.

Given $\alpha \in H(G)$ and $x \in G[p]$, since $h(\alpha(x)) > h(x)$, we infer that

$$h((1 - \alpha)(x)) = h(x - \alpha(x)) = h(x),$$

whence by (1.3), $1 - \alpha$ is an isometry. For any $x \in G[p]$ and any natural number k,

$$x - (1 - \alpha)\Big(\sum_{i=0}^{k-1} \alpha^i(x)\Big) = \alpha^k(x) \in p^k G,$$

so that by (1.4), $1 - \alpha$ is dense.

On the other hand, suppose G has a finite bound on its Ulm invariants and $H(G) = J(G)$. Let α be a dense isometry on G. Then the endomorphism induced by α on $((p^k G)[p])/((p^{k+1}G)[p])$ is one-to-one and hence an automorphism. Thus, for each k, there is an integer $m(k)$ (depending only on the dimension of $((p^k G)[p])/((p^{k+1}G)[p])$ such that $\alpha_k^{m(k)} = 1_k$, where 1_k is the identity map on $((p^k G)[p])/((p^{k+1}G)[p])$ induced by the identity map 1 on G. Since the number of distinct dimensions among the vector spaces

$$((p^k G)[p])/((p^{k+1}G)[p])$$

is finite, there is a common multiple m of the integers $\{m(k) \mid k = 0, 1, 2, \cdots\}$, so that for every k, $1_k - (\alpha_k)^m = 0$. But clearly $(\alpha_k)^m = (\alpha^m)_k$ and $1_k - (\alpha^m)_k = (1 - \alpha^m)_k$, so that $(1 - \alpha^m)_k = 0$. Thus $1 - \alpha^m \in H(G)$ and $1 - (1 - \alpha^m) = \alpha^m$ is an automorphism. But then α must also be an automorphism and the proof is complete.

(3.2) COROLLARY. *If G is a countable group of unbounded order, then G has a dense isometry which is not onto.*

Proof. Inasmuch as, in our case, such a group must be the direct sum of cyclic groups of unbounded order, we have by [8, Corollary 14.7] that $J(G) \subset H(G)$, whence by (3.1), G has a dense isometry which is not onto.

We show next that there is a group G with finite but unbounded Ulm invariants such that $H(G) = J(G)$, while G has a dense isometry which is not onto. For this we need the following theorem of A. L. S. Corner [1]:

(3.3) THEOREM (Corner). *Let C be a closed p-group with unbounded countable basic subgroup B and let Φ be a topologically separable closed subring of $E(C)$ such that $\Phi(B) \subseteq B$ and such that for all positive integers m, Φ satisfies the* *following*:

(C_m) *if* $\varphi \in \Phi$ *is such that* $\varphi((p^nC)[p^m]) = 0$, *for some* n, *then* $\varphi \in p^m\Phi$. *Then there is a pure subgroup* G *of* C *such that* $B \subseteq G$ *and*

$$E(G) = \Phi \oplus E_s(G),$$

where $\alpha \in E_s(G)$ *if and only if for each* k, *there is an* n *such that* $\alpha((p^nG)[p^k]) = 0$.

Remark. The topology on $E(C)$ referred to in this theorem is the p-adic topology given by a neighborhood basis at 0 consisting of $\{p^kE(C)\}_{k=0}^{\infty}$.

With Corner, we note first that

(3.4) LEMMA. *Let* R *be a subring of* $E(C)$ *that satisfies* (C_1). *Then* R *satisfies* (C_m) *for every natural number* m.

We fix for the remainder of this section the following notation:
$B = \oplus \sum_{n=0}^{\infty} \sum_{i=0}^{n} (b_{ni})$, $e(b_{ni}) = n + 1$, $i = 0, 1, \cdots, n$;
α is the endomorphism on $\bar{B}$ given by

$$\alpha(b_{ij}) = b_{i,j+1}, \qquad j < i$$
$$\alpha(b_{ii}) = b_{i0}.$$

(3.5) THEOREM. *The set*

$$R = \left\{\sum_{i=0}^{\infty} p^i f_i(\alpha) \mid f_i \in Z[X]\right\}$$

is a topologically separable closed subring of $E(\bar{B})$ *that satisfies* (C_m) *for every* m *and is such that* $R(B) \subseteq B$.

Proof. It is clear that R is a subring of $E(C)$ which takes B into B. Further, R is separable since it has

$$R_0 = \{f_i(\alpha) \mid f_i \in Z[X]\}$$

as a dense subset. We show next that R_0 has the property (C_1). If

$$f(\alpha) = \sum_{i=0}^{k} r_i \alpha^i \in R_0$$

is such that

$$f(\alpha)((p^n\bar{B})[p]) = 0$$

for some n, then choosing $m > k, n$,

$$\sum_{i=0}^{k} r_i p^m b_{mi} = \sum_{i=0}^{k} r_i \alpha^i (p^m b_{m1}) = 0,$$

which implies that for $i = 0, 1, \cdots, k$, $pr_i' = r_i$ for some r_i' in Z, and

$$f(\alpha) = p \sum_{i=0}^{k} r_i' \alpha^i \in pR_0.$$

From this we conclude that R satisfies (C_1) since if

$$\left(\sum_{i=0}^{\infty} p^i f_i(\alpha)\right)(p^n\bar{B})[p] = 0,$$

then

$$f_0(\alpha)((p^n\bar{B})[p]) = 0,$$

whence $f_0(\alpha) = pf_0'(\alpha)$ for some $f_0'(X) \in Z[X]$ and

$$\sum_{i=0}^{\infty} p^i f_i(\alpha) = p(f_0'(\alpha) + \sum_{i=1}^{\infty} p^{i-1} f_i(\alpha)).$$

Thus, by (3.4), R satisfies (C_m) for every m.

Finally, suppose that $\beta \in R^-$ (the closure of R). Then $\beta \in R_0^-$. We will form a sequence of polynomials $\{f_i(X)\}_{i=0}^{\infty}$ in $Z[X]$ such that for each k

$$(\beta - \sum_{i=0}^{k} p^i f_i(\alpha)) \in p^{k+1}E(\bar{B}).$$

Clearly, then $\beta = \sum_{i=0}^{k} p^i f_i(\alpha)$ which is in R. First note that $\beta \in R_0^-$ implies that there is an $f_0(X) \in Z[X]$ such that

$$\beta - f_0(\alpha) \in pE(\bar{B}).$$

Assuming that we have $\{f_i(X)\}_{i=0}^{k}$ with the property

$$(1) \qquad \beta - \sum_{i=0}^{k} p^i f_i(\alpha) = p^{k+1}\gamma \in p^{k+1}E(\bar{B}),$$

note that for some $g(X) \in Z[X]$

$$(2) \qquad \beta - g(\alpha) = p^{k+2}\delta \in p^{k+2}E(\bar{B}).$$

Combining (1) and (2), we infer that

$$g(\alpha) - \sum_{i=0}^{k} p^i f_i(\alpha) = p^{k+1}(\gamma - p\delta),$$

whence

$$(g(\alpha) - \sum_{i=0}^{k} p^i f_i(\alpha))(\bar{B}[p^{k+1}]) = 0.$$

Since R_0 has the property (C_{k+1}), it follows that

$$g(\alpha) - \sum_{i=0}^{k} p^i f_i(\alpha) = p^{k+1} f_{k+1}(\alpha),$$

with $f_{k+1}(X) \in Z[X]$. Thus

$$g(\alpha) = \sum_{i=0}^{k+1} p^i f_i(\alpha),$$

and, combining this with (2), we get the desired result.

Now, applying Corner's theorem, we obtain a pure dense subgroup G of $\bar{B}$ such that $B \subseteq G$ and

$$E(G) = E_s(G) \oplus R.$$

(3.6) THEOREM. *The group G has the property that $H(G) = J(G)$, but also has a dense isometry which is not onto.*

Proof. First observe that for any endomorphism φ of G, $\varphi = \beta + \gamma$ with $\beta \in E_s(G)$ and $\gamma \in R$. But since each $\gamma \in R$, when applied to $\bar{B}[p]$, can be written as $f(\alpha)$ with $f \in Z_p[X]$, we infer that every endomorphism φ of G is eventually equal to $f(\alpha)$ for some $f(X) \in Z_p[X]$.

Now, if $\varphi \in H(G)$, there is an integer k such that

$$\varphi = \sum_{i=0}^{m} a_i \alpha^i \quad \text{on} \quad (p^k G)[p],$$

with $a_i \in Z_p$, $i = 0, 1, 2, \cdots, m$. Then choosing $n > m, k$,

$$\varphi(p^n b_{nj}) = \sum_{i=0}^{n-j} a_i p^n b_{n,j+1} + \sum_{i=n-j+1}^{m} a_i p^n b_{n,j+i+n-1}.$$

On the other hand, since $h(\varphi(p^n b_{n,j})) > n$, it follows that $\varphi(p^n b_{n,j}) = 0$ for $j = 0, \cdots, n$, and $1 - \varphi$ is eventually algebraic. However, as was pointed out in the proof of (3.1), for φ in $H(G)$, $1 - \varphi$ is an isometry. Thus, by (1.9), $1 - \varphi$ is an automorphism and $H(G) = J(G)$.

Finally, note that $\alpha \in R$, so that α is an endomorphism of G. Further, since α is clearly an isometry on B, we have by (1.5) and (1.6) that α is a dense isometry on $\bar{B}$. If α is onto, then $\alpha^{-1} \in E(G)$ and there is an integer k such that

$$\alpha^{-1} = \sum_{i=0}^{m} a_i \alpha^i \quad \text{on} \quad (p^k G)[p],$$

with $a_i \in Z_p$, $i = 0, 1, 2, \cdots, m$. But then

$$1 = \sum_{i=0}^{m} a_i \alpha^{i+1} \quad \text{on} \quad (p^k G)[p]$$

which leads to a contradiction by applying this map to $p^n b_{n,0}$ where $n > k$, $m + 1$. Thus α is not onto and is the desired dense isometry.

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