A CHARACTERIZATION OF A CLASS OF RIGID ALGEBRAS

BY

David W. Knudson[1]

Introduction

Let A be an algebra over a field k. One of the principal problems of the deformation theory for algebras is to obtain a manageable necessary and sufficient condition for A to have only trivial deformations. Such an algebra is said to be rigid. Since the vanishing of the second Hochschild cohomology group of A, $H^2(A, A)$, is a sufficient condition for A to be a rigid k-algebra [7], it is of interest to determine when the converse is true. If A is an extension field of k or if A is a complete semi-local noetherian k-algebra such that A/m is a separable extension of k and depth $(A_m) = 0$ for each maximal ideal m of A, we shall show that the following conditions are equivalent (Corollary 3.8):

(1) $H^2(A, A) = 0$.

(2) A is a rigid k-algebra.

(3) $A \cong \prod_{1 \leq i \leq n} K_i$ where each factor K_i is a separable extension field of k and $[\Omega(K_i/k) : K_i] \leq 1$ where $\Omega(K_i/k)$ is the module of k-differentials of K_i.

We show that a deformation of a product of algebras (with 1) is equivalent to a "product of deformations" of the factors (Proposition 2.3). It follows that a product of algebras is rigid if and only if each of the factors is rigid. Thus since a complete semi-local noetherian k-algebra is isomorphic to a product of local k-algebras, we may reduce the above problem to the local case. The separability hypothesis assures us that a complete noetherian local k-algebra is isomorphic as a k-algebra to the semi-direct product of the residue field and the maximal ideal of the local algebra.

Notation. All rings will be assumed to have an identity and a ring homomorphism will preserve the identity. The expression "A is a k-algebra" will imply that k is a field.

1. Preliminary remarks

Let A be a k-algebra, M an A-bimodule, and $C^n(A, M)$ the k-module of all n-linear maps over k of A into M. As usual [9], we define the coboundary operator δ by

$$\delta_n f(a_1, \cdots, a_{n+1}) = a_1 f(a_2, \cdots, a_{n+1}) + \sum_{1 \leq i \leq n} (-1)^i f(a_1, \cdots, a_i a_{i+1}, \cdots, a_{n+1}) + (-1)^{n+1} f(a_1, \cdots, a_n) a_{n+1}$$

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where $f \in C^n(A, M)$. The n^{th} cohomology group of this complex is denoted by

$$H^n(A, M) = Z^n(A, M)/B^n(A, M)$$

and the elements of $Z^n(A, M) = \ker(\delta_n)$ and $B^n(A, M) = \operatorname{im}(\delta_{n-1})$ are called n-cocycles and n-coboundaries respectively.

Let $A[[t]]$ denote the formal power series ring in one variable over A. A *deformation* of the k-algebra A [7] is an associative $k[[t]]$-bilinear map f_t on $A[[t]]$ which is expressible in the form

$$f_t(a, b) = ab + tf_1(a, b) + t^2 f_2(a, b) + \cdots$$

where "ab" denotes the usual product in $A[[t]]$ and where each f_i is a k-bilinear map on A extended in the natural manner to a $k[[t]]$-bilinear map on $A[[t]]$.

The associativity condition on f_t is equivalent to the system of equations

$$(1) \qquad \textstyle\sum_{0<p<n} f_p(f_{n-p}(a, b), c) - f_p(a, f_{n-p}(b, c)) = \delta f_n(a, b, c)$$

for all $a, b, c \in A$ and each $n = 0, 1, 2, \cdots$. Following the notation of [5], we shall denote the 3-cochain on the left hand side of (1) by

$$\textstyle\sum_{0<p<n} f_p \circ f_{n-p} .$$

Hence if f is a 2-cocycle of A such that $f \circ f = 0$, then $f_t(a, b) = ab + tf(a, b)$ is a deformation of A. We will say that such a deformation is a *linear deformation* of A.

Let f_t and g_t be deformations of A. We say that f_t is *equivalent* to g_t if there is a $k[[t]]$-linear automorphism μ_t of $A[[t]]$ of the form

$$\mu_t(a) = a + t\mu_1(a) + t^2\mu_2(a) + \cdots$$

where each μ_i is a k-linear map on A extended in the natural manner to a $k[[t]]$-linear map on $A[[t]]$ such that

$$\mu_t(g_t(a, b)) = f_t(\mu_t(a), \mu_t(b)) \quad \text{for all} \quad a, b \in A[[t]].$$

We may easily check that $g_1 = f_1 + \delta\mu_1$ in this case. A deformation f_t of A is said to be *trivial* if f_t is equivalent to the deformation g_t of A defined $g_t(a, b) = ab$. Thus if the cocycle f_1 of the deformation f_t is not a coboundary, it follows that f_t is a non-trivial deformation. If every deformation of A is trivial, we say that A is a *rigid* k-algebra. Gerstenhaber proved that if $H^2(A, A) = 0$, then A is a rigid k-algebra [7, page 65]. In general, the converse is not known. We refer the reader to [7] for a detailed discussion of the deformation of an algebra.

2. Deformation of a product of algebras

We shall need the following lemma which is well known (see [6]) but a proof does not seem to be available.

LEMMA 2.1. *If f_t is a deformation of a k-algebra A (with 1), then the deformed* *algebra has an identity. Furthermore, f_t is equivalent to a deformation g_t such that 1 is the identity of the deformed algebra with multiplication g_t.*

Proof. The second statement implies the first statement by the definition of the equivalence relation on the set of deformations of A.

We shall define a map

$$\pi_t : A[[t]] \to A[[t]]$$

of the form $\pi_t(a) = a + t\pi_1(a) + \cdots$ by

$$\pi_n(a) = \mu_n(a) + \sum_{I_2} \mu_{i_1} \mu_{i_2}(a) + \cdots + \sum_{I_s} \mu_{i_1} \cdots \mu_{i_s}(a)$$

where s is such that

$$n = s(s+1)/2 \quad \text{or} \quad s(s+1)/2 < n < (s+1)(s+2)/2$$

and

$$I_m = \{(i_1, \cdots, i_m) | i_1 > \cdots > i_m > 0, i_1 + \cdots + i_m = n\}.$$

The μ_i are defined inductively as follows. Let μ_1 be such that

$$(f_1 + \delta\mu_1)(a, b) = 0$$

whenever a or b is 1. μ_1 always exists since f_1 is a cocycle [9]. Let

$$M_i(a) = a + t^i\mu_i(a) \quad \text{and} \quad M^i(a) = M_i M_{i-1} \cdots M_1(a).$$

Suppose we have ckosen μ_i, $i < n$, such that the deformation

$$(M^{n-1})^{-1} f_t(M^{n-1}(a), M^{n-1}(b)) = ab + tg_1(a, b) + \cdots$$

has the property that $g_i(a, b) = 0$ whenever a or b is 1 and $i < n$. Then

$$\begin{aligned}\delta g_n(a, 1, 1) &= \textstyle\sum_{1<i<n} g_i \circ g_{n-i}(a, 1, 1,) = 0 \\ &= ag_n(1, 1) - g_n(a, 1) + g_n(a, 1) - g_n(a, 1).\end{aligned}$$

Thus $ag_n(1, 1) = g_n(a, 1)$. Similarly, $g_n(1, 1)a = g_n(1, a)$. Define $\mu_n(a) = -ag_n(1, 1)$. We may easily check that $(g_n + \delta\mu_n)(a, b) = 0$ whenever a or b is 1. The deformation $\pi_t^{-1} f_t(\pi_t(a), \pi_t(b))$ clearly has the desired property.

DEFINITION 2.2. Let $A = \prod_{1 \le i \le n} A_i$ be a k-algebra and let f_t be a deformation of A. We say that f_t is a *product of deformations* of the factors A_i if $f_n(a, b) = 0$ for each $n \ge 0$ whenever $a \in A_i$ and $b \in A_j$ with $i \ne j$.

PROPOSITION 2.3. *If f_t is a deformation of the k-algebra $A = \prod_{1 \le i \le n} A_i$, then f_t is equivalent to a deformation g_t which is a product of deformations of the A_i.*

Proof. We may assume that $n = 2$. We will use the notation "a_i" to indicate the i^{th} component of $a \in A$ except that we set $1 = (e_1, e_2)$. We shall define a map $\pi_t : A[[t]] \to A[[t]]$ by the same formula as in the proof of Lemma 2.1 where the μ_i are now defined inductively as follows.

By Lemma 2 of [9], there is a 1-cochain μ_1 such that

$$(f_1 + \delta\mu_1)(a, b) = 0$$

whenever a or b is e_1 or e_2. Thus suppose we have chosen μ_i, $i < n$, such that

$$(M^{n-1})^{-1}f_t(M^{n-1}(a), M^{n-1}(b)) = ab + tg_1(a, b) + \cdots$$

where $g_i(a, b) = 0$ whenever a or b is e_1 or e_2 for $i < n$. For then we have that

$$\begin{aligned}\delta g_m(a_i, e_i, b_j) &= \textstyle\sum_{0<p<m} g_p \circ g_{m-p}(a_i, e_i, b_j) = 0\\ &= a_i\, g_m(e_i, b_j) - g_m(a_i, b_j) + 0 - g_m(a_i, e_i)b_j\end{aligned}$$

if $i \neq j$ and $m \leq n$. Hence $g_m(a_i, b_j) = 0$ if $i \neq j$ and $m < n$. Thus it will suffice to define $\mu_n : A \to A$ such that

$$(g_n + \delta\mu_n)(a, b) = 0$$

whenever a or b is e_1 or e_2. We may assume that $g_n(a, b) = 0$ whenever a or b is 1 by Lemma 2.1 and so we need only consider e_1.
Define $\alpha: A \to A$ by

$$\alpha(a) = -(a_1\, g_n(e_1, e_1), a_2\, g_n(e_2, e_2)).$$

As in the proof of Proposition 2, by considering $g_n + \delta\alpha$ we may assume that $e_i g_n(e_i, b_i) = 0$ (consider the cochain $e_i g_n$: $A_i \times A_i \to A_i$). Hence we may assume that $e_i\, g_n(e_j, b_i) = 0$ since $e_i\, g_n(1, b_i) = 0$. Similarly, $e_i\, g_n(b_i, e_j) = 0$. Thus using this reduction, we have that

$$\begin{aligned}g_n(e_1, b) &= (e_1\, g_n(e_1, b_2), e_2\, g_n(e_1, b_1)),\\ \delta g_n(e_1, b_1, e_1) &= \textstyle\sum_{0<p<n} g_p \circ g_{n-p}(e_1, b_1, e_1) = 0\\ &= e_1\, g_n(b_1, e_1) - g_n(b_1, e_1) + g_n(e_1, b_1) - g_n(e_1, b_1)e_1.\end{aligned}$$

Hence $g_n(e_1, b_1) = g_n(b_1, e_1)$. Similarly, $g_n(e_2, b_2) = g_n(b_2, e_2)$ and so $g_n(e_i, b_j) = g_n(b_j, e_i)$. Thus

$$\delta g_n(e_1, b_1, e_1) = e_1\, g_n(b, e_1) - g_n(b_1, e_1) + g_n(e_1, b_1) - g_n(e_1, b)e_1 = 0$$

implies that $e_1\, g_n(b, e_1) = e_1\, g_n(e_1, b)$. Similarly $e_2\, g_n(b, e_2) = e_2\, g_n(e_2, b)$ and so $g_n(e_1, b) = g_n(b, e_1)$ since $g_n(1, b) = 0 = g_n(b, 1)$.

Define $\mu_n : A \to A$ by

$$\begin{aligned}\mu_n(a) &= -e_1\, g_n(e_1, a_2) + e_2\, g_n(e_1, a_1).\\ \delta\mu_n(a, b) &= -a_1\, g_n(e_1, b_2) + a_2\, g_n(e_1, b_1) + e_1\, g_n(e_1, a_2 b_2) - e_2\, g_n(e_1, a_1 b_1)\\ &\quad - g_n(e_1, a_2)b_1 + g_n(e_1, a_1)b_2\\ \delta\mu_n(e_1, b) &= -e_1\, g_n(e_1\, g_n(e_1, b_2) - e_2\, g_n(e_1, b_1)\\ &= -g_n(e_1, b)\\ \delta\mu_n(a, e_1) &= -e_2\, g_n(e_1, a_1) - e_1\, g_n(e_1, a_2)\\ &= -e_1\, g_n(a_2, e_1) - e_2\, g_n(a_1, e_1)\\ &= -g_n(a, e_1).\end{aligned}$$

Hence μ_n is the required cochain.

COROLLARY 2.4. *The k-algebra* $A = \prod_{1 \leq i \leq n} A_i$ *is a rigid k-algebra if and only if each factor* A_i *is a rigid k-algebra.*

Proof. Let f_t be a deformation of A. We may assume that f_t is a product of deformations of the A_i. It is clear that a product of deformations is trivial if and only if each of the deformations of the product is trivial.

3. Deformation of semi-local algebras

Let M be a module over a commutative ring A and let $T_A(M)$ and $\Lambda_A(M)$ denote the tensor algebra and the exterior algebra on M respectively. We recall that $\Lambda_A(M) \cong T_A(M)/I_A(M)$ where $I_A(M)$ is the ideal generated by elements of the form $a \otimes a$ where $a \in M$ [4]. We refer the reader to [1] for the elementary properties of the direct limit of modules.

LEMMA 3.1. *Let* $\{(A_\alpha), (M_\alpha)\}_{\alpha \in S}$ *be a filtered direct system of modules over a filtered direct system of commutative rings. If* $A = \text{inj lim } A_\alpha$ *and* $M = \text{inj lim } M_\alpha$, *then* $\Lambda_A(M) \cong \text{inj lim } \Lambda_{A_\alpha}(M_\alpha)$.

Proof. We have that $T_A(M) \cong \text{inj lim } T_{A_\alpha}(M_\alpha)$ since the corresponding statement for the direct limit of tensor products of modules is true.

Let $\Lambda = \Lambda_A(M) \cong T_A(M)/I_A(M) = T/I$ and similarly for the pair (A_α, M_α). Thus we have the following commutative diagram with exact rows and columns since inj lim is an exact functor.

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & I & \longrightarrow & T & \longrightarrow & \Lambda & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \omega} & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \text{inj lim } I_\alpha & \longrightarrow & \text{inj lim } T_\alpha & \longrightarrow & \text{inj lim } \Lambda_\alpha & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

It will suffice to show that ω is surjective. But since $T \cong \text{inj lim } T_\alpha$, every element of T can be represented in the direct limit by an element of T_β for some $\beta \in S$. It follows immediately that ω is surjective.

DEFINITION 3.2. Let A be a commutative k-algebra (k need not be a field). The *module of k-differentials* of A is an A-module $\Omega(A/k)$ and a k-derivation $d: A \rightarrow \Omega(A/k)$ which is universal with respect to k-derivations of A into A-modules. Hence we have a natural isomorphism

$$\text{Hom}_A(\Omega(A/k), M) \cong \text{Der}_k(A, M)$$

where M is an A-module [8].

DEFINITION 3.3. An extension field L of k is said to be a *separable extension* if every finitely generated subfield of L is separable generated over k. We refer the reader to [3] for the properties of separable extensions. One may show that an extension field L of k is a rigid k-algebra in the commutative deformation theory if and only if L is a separable extension of k [12].

The following lemma removes the finite generation hypothesis of [10, Theorem 5.3].

LEMMA 3.4. *If A is a separable extension field of k and if M is an A-module, then*

$$H^*(A, M) \cong \mathrm{Hom}_A(\Lambda_A(\Omega(A/k)), M).$$

Proof. We recall that $H^*(A, M) \cong \mathrm{Hom}_A(\mathrm{Tor}_*^{A^e}(A, A), M)$ [10, Lemma 4.1]. By [10], $\mathrm{Tor}_*^{L^e}(L, L) \cong \Lambda_L(\Omega(L/k))$ if L is a finitely generated separable extension of k. Since A is the direct limit of such subfields L, we may apply Lemma 3.1 since $\Omega(A/k) \cong \mathrm{inj\ lim}\, \Omega(L/k)$ [8].

Remark 3.5. Let A be a commutative k-algebra with two distinct commuting k-derivations D and E. If char $(k) = 0$, Gerstenhaber has shown that the $k[[t]]$-bilinear map f_t on $A[[t]]$ defined by

$$f_t(a, b) = ab + tD(a)E(b) + t^2D^2(a)E^2(b)/2! + \cdots$$

is a non-trivial deformation of A [6]. If char $(k) = p \neq 0$ and if in addition $D^p = 0 = E^p$, then the map g_t defined by

$$g_t(a, b) = ab + tD(a)E(b) + \cdots + t^{p-1}D^{p-1}(a)E^{p-1}(b)/(p-1)!$$

is a non-trivial deformation of A [6].

If $\Omega(A/k)$ is a free A-module such that $[\Omega(A/k){:}A] > 1$, then such derivations always exist. We recall that if A is an extension field of k with char $(k) = 0$, then the cardinality of a transcendence base for A over k is $[\Omega(A/k){:}A]$. If char $(k) = p \neq 0$, then the cardinality of a p-basis for A over k is $[\Omega(A/k){:}A]$.

DEFINITION 3.6. Let A be a noetherian local ring with maximal ideal m. We say that A has *depth n*, depth $(A) = n$, if there is an A-sequence of elements of m of length n but no such sequence of length $n + 1$. For details, we refer the reader to [8, 16.4].

Note that depth $(A) = 0$ if and only if m consists only of zero divisors. We may also show that depth $(A) = 0$ if and only if the annihilator of m is non-zero [11, page 21].

THEOREM 3.7. *Let A be a noetherian local k-algebra such that* depth $(A) = 0$. *Assume that A is k-isomorphic to the semi-direct product $A/m \oplus m$ where m is the maximal ideal of A. The following conditions are equivalent:*

(i) $H^2(A, A) = 0$.
(ii) *A is a rigid k-algebra.*
(iii) *A is a separable extension of k and* $[\Omega(A/k): A] \leq 1$.

Proof. By [7], (i) implies (ii), and (iii) implies (i) by Lemma 3.4. Thus assume that A is a rigid k-algebra. We first show that m must be zero.

If m is generated by one element, then A is a local complete intersection. By applying the results of [12], we see that A is not rigid since A is not a regular local ring if $m \neq 0$. Thus we may assume that there are at least two elements in a minimal set of generators for m. We shall construct a k-bilinear map f on A such that the $k[[t]]$-bilinear map f_t on $A[[t]]$ defined by $f_t(a, b) = ab + tf(a, b)$ is a nontrivial linear deformation of A. Thus we must show that $f \circ f = 0$, $\delta f = 0$, and $f \neq \delta g$ where g is a k-linear map on A. It will clearly suffice to define f on a basis for the L-module $L \oplus m$.

Let $(0:m) = \{a \in A \mid am = 0\}$ be the annihilator of m and assume that $m^2 \neq 0$. Let $X = (x_i)_{i \in I}$ be a basis for the L-module $L \oplus m$ such that

$$1 \in X, \quad x_\alpha \in (0:m) \cap m^2 \quad \text{and} \quad x_\beta, x_\lambda \in m - m^2.$$

Assume that the remaining elements of X belong to m. Define $f(x_\beta, x_\lambda) = x_\alpha$ and $f(x_i, x_j) = 0$ if $(x_i, x_j) \neq (x_\beta, x_\lambda)$.

We first show that

$$f \circ f(x_i, x_j, x_k) = f(f(x_i, x_j), x_k(- f(x_i, f(x_j, x_k)) = 0.$$

Since f vanishes on the element x_α, we certainly have that $f \circ f = 0$. Since $f(x_\beta, x_\lambda) \neq f(x_\lambda, x_\beta)$, f is not a coboundary. We now consider

$$\delta f(x_i, x_j, x_k) = x_i f(x_j, x_k) - f(x_i x_j, x_k) + f(x_i, x_j x_k) - f(x_i, x_j)x_k.$$

If x_i, x_j or x_k is 1, we certainly obtain 0 for this expression. Hence we may assume that the basis elements belong to m. But f vanishes on elements in m^2 and takes values in $(0:m)$ and so we easily check that $\delta f = 0$.

Thus we are reduced to the case $m^2 = 0$. We now let X be a basis for the L-module $L \oplus m$ such that $1 \in X$ and such that the remaining elements of X belong to m. Let $x_\alpha \in X \cap m$. Define $f(x_\alpha, x_\alpha) = x_\alpha$ and $f(x_i, x_j) = 0$ if $(x_i, x_j) \neq (x_\alpha, x_\alpha)$. The same reasoning as above shows that $\delta f = 0$ and that $f \circ f = 0$.

Suppose that $f = \delta g$ where g is a k-linear map on A. Then $f(x_\alpha, x_\alpha) = x_\alpha = \delta g(x_\alpha, x_\alpha) = 2x_\alpha g(x_\alpha)$ since $x_\alpha^2 = 0$. We may assume that char $(k) \neq 2$. It will suffice to show that $g(x_\alpha) \in m$. Let $x_\beta \in X \cap m$ with $x_\beta \neq x_\alpha$. Then

$$f(x_\beta, x_\beta) = 0 = \delta g(x_\beta, x_\beta) = 2x_\beta g(x_\beta)$$

and so $g(x_\beta) \in m$. But then

$$f(x_\alpha, x_\beta) = 0 = \delta g(x_\alpha, x_\beta) = x_\alpha g(x_\beta) + x_\beta g(x_\alpha) = x_\beta g(x_\alpha).$$

Hence $g(x_\alpha) \in m$ and so $f \neq \delta g$.

Thus assume that A is a field which is a rigid k-algebra. Since A is rigid in the commutative deformation theory, we have that A is a separable extension of k [12]. Suppose that $[\Omega(A/k):A] > 1$. By Remark 3.5, we see that A has a non-trivial deformation and so we must have $[\Omega(A/k):A] \leq 1$.

COROLLARY 3.8. *Let A be a complete noetherian semi-local k-algebra such that A/m is a separable extension of k and* depth $(A_m) = 0$ *for each maximal ideal m of A. The following conditions are equivalent:*

(i) $H^2(A, A) = 0$.

(ii) *A is a rigid k-algebra*

(iii) $A \cong \prod_{1\leq i\leq n} K_i$ *where each factor K_i is an extension field of k (necessarily separable) such that* $[\Omega(K_i/k):K_i] \leq 1$.

Proof. It will suffice to show that (ii) implies (iii). Since A is complete, $A \cong \prod_m A_m$ where the product is over the set of maximal ideals m of A. Thus by Corollary 2.4, we may assume that A is local. Since A/m is a separable extension of k, A is k-isomorphic to the semi-direct product $A/m \oplus m$ and so we may apply Theorem 3.7.

The reader should note that the hypotheses of Corollary 3.8 are satisfied if A is a commutative artinian k-algebra with k a perfect field.

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UNIVERSITY OF WASHINGTON
SEATTLE, WASHINGTON