SOME NON-SOLUBLE FACTORIZABLE GROUPS

BY

A. R. CAMINA AND T. M. GAGEN

1. Introduction

In this paper we prove the following theorem:

THEOREM. *Let G be a finite non-soluble group such that $G = AB$ where A is a cyclic group and B is a metacyclic group. Then $G/S(G) \cong PGL(2, p)$, where p is a prime greater than* 3.

Metacyclic group will mean throughout a finite group all of whose Sylow subgroups are cyclic. $S(G)$ is the maximal soluble normal subgroup of G and $PGL(2, p)$, $PSL(2, p)$ denotes the projective general linear and the projective special linear groups respectively of dimension 2 over a finite field of p elements.

It will be shown in Section 3 that $S(G)$ is not necessarily a direct factor of G.

For any subset T of a group G, $C(T)$, $N(T)$ and $|T|$ denote respectively the centralizer, normalizer and the number of elements in T. The subgroup generated by T will be written $\langle T\rangle$ and a Sylow p-subgroup of G will be called an S_p-subgroup of G. A subgroup H of a group G is called a $T.I.$ subgroup if from $x^{-1}Hx \cap H \neq 1$ it follows that $x \in N(H)$. All groups considered will be finite.

2. Proof of the theorem

We note some properties of a metacyclic group G, see for example [9]. G/G' and G' are cyclic groups of co-prime orders and $G' \cap Z(G) = 1$, where $Z(G)$ denotes the center of G.

We begin with two easy lemmas.

LEMMA 1. *Let G be a group which satisfies the following conditions*:

(i) *G contains a maximal subgroup B which is metacyclic.*

(ii) *G has no non-trivial normal soluble subgroup.*

(iii) *G has no normal subgroup of index prime to* $[G:B]$.

Then $Z(B) = 1$ and B' is a $T.I.$ subgroup.

Proof. Let $x \in B' \cap B'^{g}$, $g \in G$. If $x \neq 1$, we have $N(\langle x\rangle) \geqq B, B^{g}$ since $\langle x\rangle$ is a characteristic subgroup of B'. Since B is maximal, $N(\langle x\rangle) = B$ by (ii). Hence $B^{g} = B$ and so $g \in B$ by (i) and (ii). Note that only conditions (i) and (ii) are used so far.

Now let $x \in Z(B)$ have prime order p. Then $N(\langle x\rangle) = B$ by (i) and (ii). We have two cases:

(a) An S_p-subgroup of G is not contained in B. Let P be an S_p-subgroup

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of B and $P_1 \geqq P$ an S_p-subgroup of G. The normalizer of P in P_1 contains P properly if $P_1 > P$. But $\langle x\rangle$ is characteristic in the cyclic group P and so if $P_1 > P$, $N(\langle x\rangle) \cap P_1 > P$. This contradicts $N(\langle x\rangle) = B$.

(b) An S_p-subgroup of G is contained in B. Then since $\langle x\rangle$ is characteristic in P, $N\langle x\rangle \geqq N(P)$. But $N(\langle x\rangle) = B$ has a normal p-complement and so does $N(P) \leqq N(\langle x\rangle)$. By Burnside's Theorem [9, p. 137] G has a normal p-complement. But p does not divide $[G:B]$. This contradicts condition (iii). This completes the proof.

Lemma 2. *Let G be a group satisfying conditions* (i), (ii) *and* (iii) *of Lemma* (i) *which is doubly transitive as a permutation group on the cosets of B. If $C \leqq B$ is the stabilizer of two points, then $B'C = B$ and $B' \cap C = 1$.*

Proof. By Lemma 1, $B' \cap C = 1$. For if $x \in B' \cap C$ then there exists $g \in G\backslash B$ such that $Bgx = Bg$. It follows that $g \in N(\langle x\rangle) = B$, a contradiction.

Now let P be an S_p-subgroup of B which is not contained in B'. Assume that P fixes only one point. Then $N(P)$ fixes just one point. Thus $N(P) \leqq B$. But now $P \leqq Z(N(P))$ and P is an S_p-subgroup of G. By Burnside's Theorem [9, p. 137], G has a normal p-complement. But $p \nmid [G:B]$. This contradicts condition (iii).

Since G is doubly transitive, B is transitive on the cosets Bx, $x \notin B$. Thus the stabilizers of each of these cosets are conjugate. Hence C contains an S_p-subgroup of B for all p such that $p \nmid |B'|$. Thus $B'C = B$. This completes the proof.

Note that as G is doubly transitive, B has only two double cosets and so

$$|B| + |B|^2/|C| = [G:B]\,|B|$$

Thus

$$1 + |B'| = [G:B].$$

We begin the proof of the main theorem. Let G be a minimal counter example. We show that G has a unique non-abelian composition factor and it is isomorphic to $PSL(2, p)$, for some prime $p > 3$. It is easy to see that we are then done. For let $\bar{N}$ be a minimal normal subgroup of $\bar{G} = G/S(G)$. Then $\bar{N} \cong PSL(2, p)$ and $\bar{G}/\bar{N}$ induces a group of automorphisms on $\bar{N}$. Thus by [3], $|\bar{G}/\bar{N}| \leqq 2$. But $PSL(2, p)$ is not factorizable as a product of a metacyclic and a cyclic group and so $\bar{G} \neq \bar{N}$. Thus $|\bar{G}/\bar{N}| = 2$ and $\bar{G} \cong PGL(2, p)$ by [3].

(1) $S(G) = 1$.

For if $S(G) \neq 1$, let $\bar{G} = G/S(G)$. Then since $\bar{G} = \overline{AB}$, where $\bar{A} = AS(G)/S(G)$, $\bar{B} = BS(G)/S(G)$ by the minimality of G we have the result.

(2) $A \cap B = 1$.

Let $N = \langle (A \cap B)^x : x \in G\rangle$. Then $N \trianglelefteq G$ and if $x \in G$, $x = ab$ where $a \in A$, $b \in B$.

$$(A \cap B)^x = (A \cap B)^b \leqq B.$$

Hence N is a soluble normal subgroup of G and so $N = 1$.

(3) G has at least 2 classes of involutions.

If either $|A|$ is odd or $|B|$ is odd, an S_2-subgroup of G is cyclic, whence G has a normal 2-complement M. If for example $M \geqq A$ then $M = A(B \cap M)$ is factorizable as a product of a metacyclic group and a cyclic group of odd order.

Then M is soluble by [7, Satz 5]. This is a contradiction.

Thus we may assume that both $|A|$ and $|B|$ are even. Now an involution of A is never conjugate to an involution of B by (2).

(4) G is not 2-nilpotent.

For let M be a normal 2-complement of G, S an S_2-subgroup of G which contains an S_2-subgroup Y of A. Let $S_1 \trianglelefteq S$ be a proper normal subgroup of S containing Y. Then $MS_1 \trianglelefteq MS = G$ and $MS_1 \geqq A$. Thus $MS_1 = (MS_1 \cap B)A$ and by induction MS_1 is soluble. Then G is soluble, a contradiction.

(5) $|A|$ and $|B|$ are both divisible by 4.

If $|A|$ or $|B|$ is exactly divisible by 2, then an S_2-subgroup S of G has a cyclic subgroup of index 2. But then S is either dihedral, semi-dihedral, semi-abelian or abelian.

If S is abelian or semi-abelian, G is 2-nilpotent by Burnside's Theorem if S is abelian, since G has 2 classes of involutions, and by [11, Theorem 1], if S is semi-abelian. This contradicts (4). If S is semi-dihedral, $S(G) \neq 1$ by [11] Theorem 2 because G has 2 classes of involutions. This contradicts (1).

If S is dihedral, by [5], G contains a normal subgroup $H \cong PGL(2, q)$, where $q = p^n$, $n \geqq 1$, p an odd prime, and $|G/H|$ is an odd divisor of n or $G \cong A_7$. Remember G has 2 classes of involutions. Now A_7 is not factorizable as a product of a metacyclic and a cyclic group. Since the S_p-subgroups of G are extensions of a cyclic group by a cyclic group, by [6], $n \leqq 2$ if $H \cong PGL(2, q)$. But then $G \cong PGL(2, q)$. Now $PGL(2, p^2)$ is not factorizable as a product of a metacyclic and a cyclic group. Hence $G \cong PGL(2, p)$, a contradiction.

(6) There exists a unique minimal normal subgroup M of G.

For if $M_1, M_2 \trianglelefteq G$, $M_1 \cap M_2 = 1$, are minimal normal subgroups of G, induction on G/M_1, G/M_2 shows that G has precisely two non-abelian composition factors so that M_1, M_2 are simple and isomorphic to $PSL(2, p_1)$, $PSL(2, p_2)$ respectively, for some primes p_1, p_2. Now $M_1 M_2 \trianglelefteq G$ and so $C(M_1 M_2) \trianglelefteq G$. It is clear that $C(M_1 M_2) \cap M_1 M_2 = 1$ and so $C(M_1 M_2) = 1$ because it is solvable. Now $|G/M_1 C(M_1)| \leqq 2$ since $M_1 \cong PSL(2, p_1)$; see for example [3]. Let $D = C(M_1)$ and consider $C(M_2) \cap D$. Then $C(M_2) \cap D$

$= C(M_1 M_2) = 1$. Thus $|D/M_2| \leqq 2$ and so $|G/M_1 M_2| \leqq 4$. Now $G/M_1 M_2$ is not cyclic of order 4 because if $x \in G \backslash M_1 M_2$, then

$$x^2 \in M_1 C(M_1) \cap M_2 C(M_2) = M_1 M_2.$$

Now let S be an S_2-subgroup of G. It follows that S is a product of two cyclic groups. By Satz 2, Huppert [6], the Frattini subgroup $\phi(S)$ of such a group S is itself a product of two cyclic groups and in particular is 2-generated. It follows that $|\phi(S)/\phi(\phi(S))| \leqq 4$. Since S is 2-generated, $|S/\phi(S)| \leqq 4$. Now $S/S \cap M_1 M_2$ is an elementary abelian group and so

$$\phi(S) \leqq S \cap M_1 M_2 \quad \text{and} \quad [S \cap M_1 M_2 : \phi(S)] \leqq 2.$$

Put $T = S \cap M_1 M_2$. Then T is an S_2-subgroup of $M_1 M_2$ and $T/\phi(T) = 16$. Hence $\phi(S)/\phi(T) \cap \phi(S)$ is elementary abelian of order at least 8, a contradiction. Now let M be the unique minimal normal subgroup of G. Then $C(M) \trianglelefteq G$, $C(M) \cap M = 1$ and so $C(M) = 1$.

Now if $M \cong PSL(2, p)$ we are done because $|G/M| \leqq 2$ by [3] and as before $G \cong PGL(2, p)$. Also $G = MA = MB$ for if $MA < G$, $MA = (MA \cap B)A$ and by the minimality of G, $M \cong PSL(2, p)$. Note that, if K is any proper normal subgroup of G which is factorizable into a product of a cyclic and a metacyclic group, we are done since $K \geqq M$.

(7) B is maximal in G.

Let $R \geqq B$ be a maximal subgroup of G. Then $R = B(R \cap A)$ and by the minimality of G, R has at most one non-abelian composition factor and this is isomorphic to $PSL(2, p)$. But

$$U = \langle (R \cap A)^x : x \in G \rangle \lhd G.$$

Now $$(R \cap A)^x = (R \cap A)^{ab} = (R \cap A)^b \leqq R, \qquad a \in A, b \in B$$

Thus R contains a normal subgroup U of G and so $M \leqq R$. Then $M \cong PSL(2, p)$ and we are done. Hence B is maximal in G.

(8) G satisfies the conditions of Lemmas 1 and 2. We consider G as a permutation group on the cosets of B. Since B is maximal, G is primitive.

We have already verified conditions (i) and (ii) of Lemma 1. Let $|A| = \alpha$. As $B \cap A = 1$, $[G:B] = \alpha$. Let K be a normal subgroup with $[G:K]$ prime to α. Then $A \leqq K$. But $K \geqq M$ and $MA = G$ by (6). Hence $K = G$ and we have condition (iii) of Lemma 1.

Now α is not prime by (5) and so G is doubly transitive by Theorem 25.4 of [10].

(9) If $B = B'C$, $B' \cap C = 1$, $|N(C)| = 2|C|$.

For $N(C) \cap B = C$ by Lemma 1. The result follows from the double transitivity of G and Theorem 9.4 of [10].

Let $t \in A$ be the involution in A. We may suppose without loss that $t \in N(C)$.

(10) Any subgroup K of G containing A satisfies $K \cap B' = 1$.

For $(K \cap B')^{ba} = (K \cap B')^{a} \leqq K$, $a \in A$, $b \in B$, and so K contains a proper normal subgroup of G, the normal closure of $K \cap B'$. But $K = A(K \cap B)$ is of known type and so $K \geqq M$ and $M \cong PSL(2, p)$.

Hence any maximal subgroup of G containing A is a product of two cyclic groups and so is supersoluble by [6] and metabelian by [9], 13.3.2.

(11) $C(A) = A$.

$$L = \langle (C(A) \cap B)^{x} : x \in G \rangle \lhd G.$$

$$(C(A) \cap B)^{ab} = (C(A) \cap B)^{b} \leqq B.$$

Thus L is soluble normal subgroup of G and $L = 1$.

(12) An S_2-subgroup of $C(t)$ is an extension of a cyclic group by a cyclic group.

Let $T = C(t) \geqq A$. Then T is supersoluble and metabelian. Consider AT'. Since T' is abelian by Fitting's Lemma [9, 4.5.6], $T' = T_1 \oplus T_2$ where AT_1 is nilpotent and $[A, T_2] = T_2$.

Suppose $AT_1 > A$. Then $AT_1 \cap B \neq 1$ and we may choose

$$x \in AT_1 \cap B \cap N(A)$$

of order p, a prime. Note that $N(A) \cap AT_1 = A(N_{AT_1}(A) \cap B)$. Then if $y \in A$ has order prime to p, $[x, y] = 1$ since AT_1 is nilpotent. Now $[x, A] \neq 1$, because by (11), $C(A) = A$. Let an S_p-subgroup of A be $\langle z \rangle$, where z has order p^n. If p is odd, $z^x = z^{1+p^{n-1}}$ and so $| C(x) \cap \langle z \rangle | = p^{n-1}$. Thus in the permutation representation of G, x fixes $| A |/p = \gamma = | C(x) \cap A |$ points.

Now $\gamma \mid | A |$ and $\gamma - 1 \mid | A | - 1 = | B' | \geqq 5$. Thus $\gamma^2 \leqq | A |$. Hence $| A | \leqq p^2$. If $| A | = p^2$, we have a contradiction since p is odd. If $| A | < p^2$, $[x, A] = 1$, again impossible.

If $p = 2$, $z^x = z^{-1}$ or $z^x = z^{2^{n-1}-1}$ since if $z^x = z^{1+2^{n-1}}$, $n \geqq 2$, we may argue as for p odd. Thus $| C(x) \cap \langle z \rangle | = 2$ and in the permutation representation of G on the cosets of B, x fixes $\gamma = | A |/2^{n-1}$ points, and $\gamma = 2\delta$, where δ is odd. Let $| B | = 2^m \rho$, where ρ is odd.

Then $2^m \rho_1 + \gamma = | A |$, for some ρ_1.

Thus either $| A |$ is exactly divisible by 2 or $m = 1$. This is not the case by (5).

Hence we have $AT_1 = A$. Now $| T_2 |$ is odd because AT_2 is supersoluble and if $| T_2 |$ is even there exists an element of order 2 in T_2 normalized by A. Since $A \cap T_2 = 1$ we have a contradiction to (11).

Thus $AT_2/T_2 \trianglelefteq C(t)/T_2$ and an S_2-subgroup R of $C(t)$ contains a normal cyclic subgroup $Y \leqq A$ such that R/Y is also cyclic.

(13) If R is non-abelian or if R is abelian and $|Y| > |R/Y|$, an S_2-subgroup of $C(t)$ is an S_2-subgroup of G.

For if R is non-abelian or if R is abelian with $|Y| > |R/Y|$, $\langle t \rangle$ is a characteristic subgroup of R. Clearly, then R is an S_2-subgroup of G.

(14) An S_2-subgroup S of G is not an extension of a cyclic group by a cyclic group.

We apply the result of [2] to show that if S has a normal cyclic subgroup S_1 such that S/S_1 is cyclic of order $\geqq 4$, G is soluble. We thus have that, applying (12) and (13), an S_2-subgroup of $C(t)$ is abelian and $|Y| \leqq |R/Y|$. Now let $X \leqq C$ be an S_2-subgroup of C, $t \in N(C)$, $X = \langle x \rangle$. Then $|C(t) \cap X| \geqq |Y| \geqq 4$. It is clear that if $x^{2^m} = 1$, $t^{-1}xt = x^{1+2^{m-1}}$. Thus $x^2 \in C(t)$. Let $Y = \langle y \rangle$, $y^{2^n} = 1$, $y^{2^{n-1}} = t$. Then by (13) $n \leqq m - 1$. Now $[x^2, y] = 1$ because an S_2-subgroup of $C(t)$ is abelian.

It follows that $y^x = y^{-1}x^{2j}$. Note that $t \notin Z(\langle x, y \rangle)$.

Let $x^{2^{m-1}} \in U$ where U is an elementary abelian group, $U \leqq \langle x, y \rangle$. Then $|U| \leqq 4$ since if $|U| \geqq 8$, $U \cap \langle x^2, y \rangle = \langle t, x^{2^{m-1}} \rangle$. But

$$C(t) \cap \langle x, y \rangle = \langle x^2, y \rangle.$$

Calculate $C(U) \cap \langle x, y \rangle$. Let $t_1 \in U \backslash \langle x^{2^{m-1}} \rangle$ be an involution. Then

$$C(U) \cap \langle x, y \rangle = C(t_1) \cap \langle x, y \rangle = \langle t_1 \rangle \times \langle x^2 \rangle.$$

But then $C(U) \cap \langle x, y \rangle$ is of type $(2^n, 2)$ where $n > 1$. For if $n = 1$, $|\langle x, y \rangle| = 8$ and an S_2-subgroup of G is dihedral of order 8, a contradiction.

Now apply Theorem 4 of [4]. Since $x^{2^{m-1}}$ is not weakly closed in $\langle x, y \rangle$, there exists $g \in G$ of odd order such that

$$g \in N(C(U) \cap \langle x, y \rangle) \cap N(U)$$

such that $x^{2^{m-1}g} \neq x^{2^{m-1}}$. But $\langle x^{2^{m-1}} \rangle$ is a characteristic subgroup of

$$C(U) \cap \langle x, y \rangle = \langle t_1 \rangle \times \langle x^2 \rangle.$$

This is a contradiction and completes the proof.

Note. Professor N. Itô has communicated a proof of the following unpublished result to the authors.

THEOREM. *Let G be a simple doubly transitive group such that the stabilizer B of a single point is metacyclic. Then $G \cong PSL(2, p)$, for some prime $p > 3$.*

Using this result the above proof can be considerably shortened. In particular, steps (10)–(13) can be eliminated. Also the use of the main result in [5] can then be avoided.

Some examples. The following groups were introduced by Schur [8]. They contain a normal subgroup $H \cong SL(2, p)$, $p > 3$ a prime. There are two cases:

1. If $p \equiv -1 \pmod 4$, let $U(p)$ denote the group of all 2×2 matrices over $GF(p)$ of determinant ± 1.

2. If $p \equiv +1 \pmod 4$, let $U(p) \leqq GL(2, p^2)$ be the group generated by the following matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma^{-1} \end{pmatrix}$$

where $\sigma \in GF(p^2)$ is a primitive $2(p-1)$ root of unity.

It can be shown, see [8], that $U(p)/Z(U(p) \cong PGL(2, p))$. We show that the groups $U(p)$ can be factored into a product AB where A is cyclic and B is metacyclic. $|U(p)| = 2p(p^2 - 1)$. Again there are two cases:

1. $p \equiv -1 \pmod 4$. Let $\alpha \in GF(p)$ be a primitive $(p-1)/2$ root of unity. Let

$$B = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha^{-1} \end{pmatrix} \right\rangle.$$

Then $|B| = p(p-1)$ and B is metacyclic, $|B'| = p$.

We construct a matrix of order $2(p+1)$ such that the cyclic subgroup A it generates intersects B trivially. Let $\rho \in GF(p^2)$ be a primitive $2(p+1)$ root of unity. Then

$$x = \begin{pmatrix} \rho & 0 \\ 0 & -\rho^{-1} \end{pmatrix} \in GL(2, p^2)$$

has order $2(p+1)$. Let

$$y = \frac{1}{\rho^2 + 1} \begin{pmatrix} \rho & 1 \\ -1 & \rho \end{pmatrix}.$$

Then $y^{-1}xy \in U(p)$ as may be verified. Now $A = \langle y^{-1}xy \rangle$. Since the unique element of order 2 in A is central in $U(p)$ and $|A \cap B| \leqq 2$, it is clear that $A \cap B = 1$.

2. $p \equiv 1 \pmod 4$. Let B be the group generated by the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma^{-1} \end{pmatrix}.$$

Then $|B| = 2p(p-1)$ and B is metacyclic.

We construct a matrix of order $(p+1)$ as follows. If $\rho^4 = \tau$,

$$x_1 = \begin{pmatrix} \tau & 0 \\ 0 & -\tau^{-1} \end{pmatrix} \in GL(2, p^2)$$

has order $(p+1)$ since $p \equiv 1 \pmod 4$ and again conjugation gives $y^{-1}x_1 y \in U(p)$. Similar argument shows that $A \cap B = 1$ and $U(p) = AB$.

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University of Illinois
Urbana, Illinois