ON A QUESTION OF AYOUB, CHOWLA AND WALUM CONCERNING CHARACTER SUMS

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Let p be a prime $\equiv 3 \pmod 4$, and let

$$S(k) = \sum_{n=1}^{p-1} \left(\frac{n}{p}\right) n^k,$$

where

$$\left(\frac{n}{p}\right)$$

is Legendre's symbol. The authors mentioned in the title have pointed out [1] that $S(0) = 0$, $S(1) < 0$, and $S(2) < 0$. They have also proved that for $k = 3$ and for some other small values of k, there are infinitely many $p \equiv 3 \pmod 4$ for which $S(k) > 0$ and infinitely many for which $S(k) < 0$. They raise the question whether a similar result holds for other values of k. In this note, using methods similar to theirs, we answer this question for all real $k > 2$.

THEOREM 1. *For each real $k > 2$, there are infinitely many primes $p \equiv 3 \pmod 4$ for which $S(k) > 0$ and infinitely many for which $S(k) < 0$.*

This is an immediate consequence of the following theorem.

THEOREM 2. *Let f be a real-valued function on $[0, 1)$ such that f'' exists and is non-decreasing, non-constant, and integrable on $[0, 1)$, and such that $\delta = f(1-) - f(0) > 0$. Then for infinitely many primes $p \equiv 3 \pmod 4$,*

$$S(f; p) = \sum_{n=1}^{p-1} \left(\frac{n}{p}\right) f(n/p) \tag{1}$$

is positive and for infinitely many it is negative.

Proof of Theorem 2. We can expand f in a Fourier series with period 1:

$$f(x) = a_0/2 + \textstyle\sum_{m=1}^{\infty} (a_m \cos 2\pi mx + b_m \sin 2\pi mx) \tag{2}$$

for $0 < x < 1$. Using the facts that

$$\sum_{n=1}^{p-1} \left(\frac{n}{p}\right) \cos \frac{2\pi mn}{p} = 0, \qquad \sum_{n=1}^{p-1} \left(\frac{n}{p}\right) \sin \frac{2\pi mn}{p} = \left(\frac{m}{p}\right) \sqrt{p},$$

we obtain (by substitution of (2) in (1))

$$\frac{1}{\sqrt{p}} S(f; p) = \sum_{m=1}^{\infty} \left(\frac{m}{p}\right) b_m .$$

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Now

$$
\begin{aligned}
b_m &= 2\int_0^1 f(x)\sin 2\pi mx\,dx \\
&= -\frac{1}{\pi m}f(x)\cos 2\pi mx\Big]_0^1 + \frac{1}{\pi m}\int_0^1 f'(x)\cos 2\pi mx\,dx \\
&= -\frac{\delta}{\pi m} + \frac{1}{2\pi^2 m^2}\left\{f'(x)\sin 2\pi mx\Big]_0^1 - \int_0^1 f''(x)\sin 2\pi mx\,dx\right\} \\
&= -\delta/\pi m + c_m,
\end{aligned}
$$

where

$$c_m = -\frac{1}{2\pi^2 m^2}\int_0^1 f''(x)\sin 2\pi mx\,dx.$$

Our hypotheses yield the conclusions that $c_m = o(m^{-2})$ and that $c_m > 0$. We have

$$\text{(3)} \qquad \frac{1}{\sqrt{p}}S(f;p) = -\frac{\delta}{\pi}L(1,\chi_p) + \sum_{m=1}^{\infty}\left(\frac{m}{p}\right)c_m,$$

where

$$L(s,\chi_p) = \sum_{n=1}^{\infty}\left(\frac{n}{p}\right)n^{-s}.$$

Now Bateman, Chowla and Erdös [2] have proved that for an infinite sequence of primes $p \equiv 3 \pmod 4$, $L(1, \chi_p) \to 0$ and that for another infinite sequence of such primes, $L(1,\chi_p) \to +\infty$. In [3], Mrs. P. T. Joshi has shown that this result is still valid if the p's are further required to lie in any consistent arithmetic progression $an + b$ (with $(a, b) = 1$). Since in (3) the series is dominated in absolute value by $\sum c_m < \infty$, the right side can be made negative for infinitely many $p \equiv 3 \pmod 4$. On the other hand, we can find an N so large that

$$D = \textstyle\sum_{m=1}^{N} c_m - \sum_{m=N+1}^{\infty} c_m > 0.$$

It is easy to see that if p belongs to a certain arithmetic progression P, we will have

$$p \equiv 3 \pmod 4 \quad \text{and} \quad \left(\frac{m}{p}\right) = 1 \text{ for } m = 1, 2, \cdots, N.$$

It will then follow that

$$\sum_{m=1}^{\infty}\left(\frac{m}{p}\right)c_m \geq D \qquad (p \in P).$$

Applying the extended result from [3], we see that there are infinitely many $p \in P$ for which $(\delta/\pi)L(1, \chi_p) < D$, and for these p, the right side of (3) is positive. This completes the proof.

REFERENCES

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