# HAAR SERIES AND ADJUSTMENT OF FUNCTIONS ON SMALL SETS

BY

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## 1. Introduction

D. E. Menshov proved that a measurable function finite almost everywhere on $[0, 2\pi]$ can be changed on a set of measure less than $\varepsilon$ to a function whose Fourier series converges uniformly [3]; see also [1, Chapter VI].

One may ask whether an analogous result holds for orthonormal systems other than the trigonometric system. For the Walsh functions an affirmative answer was given by B. D. Kotlyar [2] and, with different techniques, but later, by the author [4]. For Haar functions the question is trivial; the Haar-Fourier series of every continuous function converges uniformly and a finite measurable function agrees with a continuous function except on a set of measure less than $\varepsilon$.

Nevertheless, one aspect of our results on Walsh functions suggests a non-trivial question about Haar functions. In the cited paper, we constructed subsets $W$ of the Walsh functions with the following property: Every continuous (or finite, measurable) function can be adjusted on a small set so that the modified function has a uniformly convergent Walsh-Fourier series involving only those Walsh functions in $W$.

In this paper, we characterize families of Haar functions which have an analogous property.

DEFINITION 1. Let $\Phi$ be an orthonormal set of functions in $L^2[0, 1]$ not necessarily complete. Let $M(\Phi)$ be the closed linear manifold of $L^2[0, 1]$ spanned by $\Phi$. Then $\Phi$ has property U if, given a continuous function $f$ on $[0, 1]$ and an $\varepsilon > 0$, there exists a function $g$ such that

(a) $g \in M(\Phi)$,
(b) $g(x) = f(x)$ except on a set of measure less than $\varepsilon$,
(c) the expansion of $g$ in the system $\Phi$ converges uniformly.

Our objective is to determine which subsystems of the Haar functions have property U. We shall investigate also a similar question involving absolute convergence of Haar series.

DEFINITION 2. An orthonormal set has property A if it satisfies the conditions of Definition 1 relative to absolute convergence instead of uniform convergence.

Recently A. A. Talayan [7] constructed certain orthonormal sets having

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property A. We shall show that the Haar functions and certain subsets of the Haar functions have property A.[2]

Our results are contained in the following theorem.

THEOREM 1. *Let $H = \{h_n\}$ be a family of Haar functions, total in measure on* [0, 1]. *Let f be a continuous function on* [0, 1] *and let $\varepsilon > 0$ be given. Then there exists a function g such that*

(a) $g(x) = f(x)$ *except on a set of measure less than* $\varepsilon$,
(b) $g(x) = \sum_{n=1}^{\infty} c_n h_n(x)$, *the series converging uniformly and absolutely.*

COROLLARY. *For families of Haar functions the following are equivalent:*

(a) *totality in measure* (TM),
(b) *property* U,
(c) *property* A.

*Proof.* According to Theorem 1, TM $\Rightarrow$ U and TM $\Rightarrow$ A. Now U $\Rightarrow$ TM. This is immediate from the definition of TM. Also A $\Rightarrow$ TM as can be seen by an easy application of Egoroff's Theorem.

## 2. Adjustment of step functions

We begin by quoting two results that will be needed.

THEOREM A. *Let $H = \{h_n\}$ be a family of Haar functions and let $E_n$ denote the support of $h_n$. Then H is total in measure on a set $G \subset [0, 1]$ if and only if $G \subset \limsup E_n$ except perhaps for a set of measure zero.*

THEOREM B. *If a sequence of functions is total in measure on a set G, it remains so when a finite number of its elements are removed.*

Theorem A was proved by Robert E. Zink and the author [5]. Theorem B is due to A. A. Talayan [6].

From now on, $H = \{h_n\}_1^{\infty}$ will denote a family of Haar functions that is total in measure of [0, 1]. $\mu(S)$ will denote the Lebesgue measure of $S$. $I(n, j)$ will denote the dyadic interval $[j \cdot 2^{-n}, (j + 1) \cdot 2^{-n})$. If

$$\sum_1^{\infty} c_n h_n(x)$$

is the Haar-Fourier series of $f$ we shall set

$$s_k(x; f) = \sum_1^k c_n h_n(x), \qquad a_k(x; f) = \sum_1^k |c_n h_n(x)|.$$

LEMMA 1. *Let I be a subinterval of* [0, 1]. *Let N be a positive integer and let $\varepsilon > 0$. Then, there exist $h_{n_1}, h_{n_2}, \cdots, h_{n_k}$ in $\{h_n\}_N^{\infty}$ such that their supports $E_{n_i}$ are disjoint, contained in I, and*

$$\mu(I - \cup_1^k E_{n_i}) < \varepsilon.$$

[2] In a conversation with the author, Y. Katznelson sketched a proof that the trigonometric functions do not have property A.

*Proof.* $\{h_n\}_N^\infty$ is total in measure by Theorem B. Therefore, by Theorem A, if

$$J = \text{interior of } I \cap \limsup_{n \geqq N} E_n ,$$

then $I - J$ is a null set.

Each point of $J$ is contained in infinitely many sets $E_n$. Since $\mu(E_n) \to 0$ as $n \to \infty$, the family of supports

$$\mathcal{E} = \{E_n : n \geqq N, E_n \subset \text{interior of } I\}$$

is a covering of $J$ in the sense of Vitali. Therefore, by the Vitali Covering Theorem, there exist $E_{n_1}, E_{n_2}, \cdots, E_{n_k}$ in $\mathcal{E}$ which satisfy the assertion of the lemma.

LEMMA 2. *Let $\chi$ be the characteristic function of $I(n, j)$. Let $m$ and $N$ be given positive integers. Then there exists a function $g$ with the following properties.*

(a) *$g$ is a linear combination of the functions $\{h_n\}_N^\infty$.*
(b) *$g(x) \equiv 0$ outside of $I(n, j)$.*
(c) *$g(x) = \chi(x)$ except on a set of measure less than $2^{-n-m}$.*
(d) *$|g(x)| < 2^{m+1}$ for all $x$.*
(e) *$|s_k(x; g)| \leqq a_k(x; g) < 2^{m+1}$ for all $x$ and $k$.*

*Proof.* Choose a number $\delta$ such that $0 < \delta < \frac{1}{2}$ and

$$\delta + \delta^2 + \cdots + \delta^{m+1} > 1 - 2^{-m}.$$

This is possible since

$$\lim_{\delta \to 1/2} (\delta + \delta^2 + \cdots + \delta^{m+1}) = 1 - 2^{-m-1} > 1 - 2^{-m}.$$

Let $\delta = \frac{1}{2} - \varepsilon$. By Lemma 1, there exist $h_{n_1}, h_{n_2}, \cdots, h_{n_k}$ in $\{h_n\}_N^\infty$ with disjoint supports $E_{n_i}$ contained in $I(n, j)$ such that

$$\mu(I(n, j) - \textstyle\bigcup_1^k E_{n_i}) < 2\varepsilon \cdot 2^{-n} \tag{1}$$

Define

$$l_1(x) = \textstyle\sum_1^k h_{n_i}(x) |h_{n_i}(x)|^{-1}.$$

Then

$$\begin{aligned} l_1(x) &= 1, & x &\in P_1, \\ &= -1, & x &\in Q_1, \\ &= 0, & &\text{otherwise,} \end{aligned}$$

where $P_1$ and $Q_1$ are finite unions of dyadic intervals and $\mu(P_1) = \mu(Q_1)$. Because of the latter fact, it follows from (1) that

$$\mu(P_1) = \mu(Q_1) > (\tfrac{1}{2} - \varepsilon)2^{-n} = \delta \cdot 2^{-n}.$$

We now apply the same technique to each component of $Q_1$. We obtain

$l_2(x)$, a linear combination of the functions $\{h_n\}_{N_2}^{\infty}$, $N_2 > \max_{1 \leq i \leq k} n_i$, such that

$$l_2(x) = \begin{cases} 2, & x \in P_2, \\ -2, & x \in Q_2, \\ 0, & \text{otherwise,} \end{cases}$$

where $P_2$ and $Q_2$ are finite unions of dyadic intervals, $P_2 \cup Q_2 \subset Q_1$ and

$$\mu(P_2) = \mu(Q_2) > \delta \cdot \mu(Q_1)$$

Then $l_1(x) + l_2(x) = 1$ on $P_1 \cup P_2$ and $|l_1(x) + l_2(x)| \leq 1 + 2$ for all $x$.

We continue in this way. After $m + 1$ steps we obtain

$$g(x) = l_1(x) + l_2(x) + \cdots + l_{m+1}(x),$$

a function which obviously satisfies (a) and (b). $g(x) = 1$ on the set $\bigcup_1^{m+1} P_i$ whose measure is

$$\textstyle\sum_1^{m+1} \mu(P_i) > 2^{-n} \sum_1^{m+1} \delta^i > 2^{-n}(1 - 2^{-m}).$$

Thus $g(x) = \chi(x)$ except perhaps on a set of measure less than $2^{-n-m}$. Furthermore, it is clear from the construction that

$$|g(x)| \leq 1 + 2 + 4 + \cdots + 2^m < 2^{m+1}$$

and that

$$|s_k(x; g)| \leq a_k(x; g) < 2^{m+1} \qquad \text{for all } x \text{ and } k.$$

By applying Lemma 2 in an obvious way, we can approximate step functions in the sense of Lemma 2. We shall omit the routine proof and just state the result.

LEMMA 3. *Let $f$ be constant on each of the dyadic intervals $I(n, j)$, $0 \leq j < 2^n$. Let $m$ and $N$ be given positive integers. Then there exists a function $g$ with the following properties.*

(a) *$g$ is a linear combination of the functions $\{h_i\}_N^{\infty}$.*
(b) *$g(x) = f(x)$ except on a set of measure less than $2^{-m}$.*
(c) $\max_x |g(x)| < 2^{m+1} \max_x |f(x)|$.
(d) $\max_x |s_k(x; g)| \leq \max_x a_k(x; g) < 2^{m+1} \max_x |f(x)|$ *for all $k$.*

## 3. Proof of Theorem 1.

Let a continuous function $f$ and a positive number $\varepsilon$ be given. $f$ may be represented as the sum of a series

$$f(x) = \textstyle\sum_{r=0}^{\infty} f_r(x)$$

where $f_r$ is a step function constant on the intervals $I(n_r, j)$ and $\{n_r\}$ is increasing so fast that

$$|f_r(x)| < 2^{-2r}, \qquad r > 0.$$

Choose $p$ such that $2^{-p} < \varepsilon$. For each $r$, apply Lemma 3 to $f_r$ with $m = r + p + 1$. In this way we may obtain a sequence of functions $\{g_r\}$ such that

(i) $g_r(x) = f_r(x)$ except on a set of measure less than $2^{-r-p-1}$,
(ii) $g_r(x)$ is a linear combination of the Haar function in $H$,
(iii) the Haar functions involved in $g_r$ have greater subscripts than those involved in $g_1, g_2, \cdots, g_{r-1}$,
(iv) $\max_x |g_r(x)| < 2^{-2r} \cdot 2^{r+p+2} = 2^{-r+p+2}$, $r > 0$,
(v) $\max_x | s_k(x;\, g_r) | \leqq \max_x a_k(x;\, g_r) < 2^{-2r}2^{r+p+2} = 2^{-r+p+2}$, for all $k$, $r > 0$.

Set

$$g(x) = \sum_{r=0}^{\infty} g_r(x). \tag{2}$$

The series converges uniformly because of (iv). $g(x) = f(x)$ except for a set of measure less than

$$\sum_0^{\infty} 2^{-r-p-1} = 2^{-p} < \varepsilon.$$

If we replace each $g_r$ by its expression in terms of Haar functions in $H$, we obtain from (2) a Haar series for $g$. Since a subsequence of its partial sums converges uniformly to $g$, this Haar series is the Haar-Fourier series of $g$. Write the series as

$$\sum_0^{\infty} c_n h_n(x). \tag{3}$$

There is an increasing sequence of positive integers $\{\nu_r\}$ such that

$$\sum_{n=1}^{\nu_r} c_n h_n(x) = \sum_{i=1}^{r} g_i(x).$$

If $\nu_r \leqq k < \nu_{r+1}$ then

$$s_k(x;\, g) = \sum_{i=1}^{r} g_i(x) + s_k(x;\, g_{r+1}). \tag{4}$$

The sum on the right side of (4) converges uniformly to $g(x)$ as $k \to \infty$. According to (v), $\max_x | s_k(x;\, g_{r+1}) | < 2^{-r+p+2}$ and so

$$\lim_{k\to\infty} s_k(x;\, g_{r+1}) = 0 \quad \text{uniformly}.$$

Therefore

$$\lim_{k\to\infty} s_k(x;\, g) = g(x) \quad \text{uniformly}.$$

It remains to show that the series (3) converges absolutely. It suffices to prove that the increasing sequence $\{a_{\nu_r}(x;\, g)\}$ is bounded. Using (v), we have for all $x$,

$$a_{\nu_r}(x;\, g) = a_{\nu_0}(x;\, g_0) + \sum_{j=1}^{r} a_{\nu_j}(x;\, g_j) < \max_x a_{\nu_0}(x;\, g_0) + \sum_{j=1}^{\infty} 2^{-r+p+2}$$
$$= \max_x a_{\nu_0}(x;\, g_0) + 2^{p+2} = \text{constant}.$$

This concludes the proof of Theorem 1.

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