INVARIANT AND EXTENDIBLE GROUP CHARACTERS

BY

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1. Let G be a finite group and let $N \triangle G$. Suppose $\chi \in \mathrm{Irr}(N)$, the set of irreducible complex characters of N and that χ is invariant under the action of G. We seek conditions sufficient to guarantee that χ can be extended to G, in other words that there exists $\theta \in \mathrm{Irr}(G)$ with $\theta \mid N = \chi$. A related question which is considered in §2 is the following. Suppose $N \subseteq H \triangle G$ and that χ can be extended to H. What conditions will guarantee that some extension of χ to H is invariant in G. In this paper we will provide sufficient conditions for both problems provided that N has a normal solvable subgroup N_0 such that $\chi \mid N_0$ is irreducible. In particular, our results apply if N is solvable. Although both of the theorems proved here may be true without this assumption, these proofs depend strongly on solvability.

We begin with some general remarks which are probably well known. (For instance see [1].) Let $\mathfrak{X}$ be a representation of N which affords χ. Since χ is invariant in G, for each $g \in G$ there exists a matrix Y_g such that for all $h \in N$,

$$(*) \qquad Y_g^{-1}\mathfrak{X}(h)Y_g = \mathfrak{X}(g^{-1}hg).$$

Since χ is irreducible, it follows from Schur's lemma that

$$(**) \qquad Y_{g_1} Y_{g_2} = f(g_1, g_2)Y_{g_1 g_2}$$

where f is a function from $G \times G$ into the complex numbers. We may choose the Y_g in the following manner. Pick a transversal T for the cosets of N in G with $1 \in T$ and define Y_t arbitrarily such that $(*)$ is satisfied for $1 \neq t \in T$. Set $Y_1 = I$, the unit matrix. For arbitrary $g \in G$, write $g = tn$ for $t \in T$ and $n \in N$ and put $Y_g = Y_t \mathfrak{X}(n)$.

The function f associated with this choice of the Y_g satisfies

$$f(t_1 n_1, t_2 n_2) = f(t_1, t_2)$$

for all $t_i \in T$ and $n_i \in N$ and because of this f defines a function $\bar{f}$ on $G/N \times G/N$ and $\bar{f}$ is a factor set of G/N. If $\bar{f}$ is a coboundary, then for some function $\bar{\alpha}$ on G/N we have

$$\bar{f}(x, y) = \bar{\alpha}(x)\bar{\alpha}(y)/\bar{\alpha}(xy)$$

for $x, y \in G/N$. This defines a function α on G which is constant on the cosets of N such that

$$f(g_1, g_2) = \alpha(g_1)\alpha(g_2)/\alpha(g_1 g_2).$$

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It follows from this and (**) that the map $g \rightarrow \alpha(g)^{-1}Y_g$ is a representation of G and since $\bar{f}(1, 1) = 1$, we have $\alpha(g) = 1$ for $g \in N$ and thus this representation is an extension of $\mathfrak{X}$.

Now by a theorem of Gaschütz (Thm. 15.8.3 of [4]), it follows that a condition sufficient to guarantee that $\bar{f}$ is a coboundary is that for each prime $p \mid [G:N]$, the restriction of $\bar{f}$ to some Sylow p-subgroup P/N of G/N is trivial. If for each such prime, a P can be found such that χ can be extended to P, then an extension $\mathfrak{X}_P$ of $\mathfrak{X}$ to P exists and for $t \in T \cap P$, we may let $Y_t = \mathfrak{X}_P(t)$. The resulting factor set $\bar{f}$ satisfies $\bar{f}(x, y) = 1$ for $x, y \in P/N$ and this proves

PROPOSITION 1. *If for each prime $p \mid [G:N]$, χ can be extended to the inverse image in G of some Sylow p-subgroup of G/N, then χ can be extended to G.*

A general result about extendible characters which we use repeatedly appears as Theorem 2 of [2]. We state it here as

PROPOSITION 2. *Suppose χ has an extension $\theta \in \mathrm{Irr}(G)$. Let $\beta \in \mathrm{Irr}(G/N)$ be viewed as a character of G. Then $\beta\theta \in \mathrm{Irr}(G)$, all $\beta\theta$ are distinct for distinct $\beta \in \mathrm{Irr}(G/N)$ and any $\theta_0 \in \mathrm{Irr}(G)$ with χ a constituent of $\theta_0 \mid N$ is of the form $\beta\theta$ for some $\beta \in \mathrm{Irr}(G/N)$.*

In particular, it follows from Proposition 2 that the set of extensions of χ to G is $\{\lambda\theta \mid \lambda$ is a linear character of $G/N\}$ where θ is any given extension.

THEOREM 3. *Let G/N be a p-group and suppose that a Sylow p-subgroup of G is abelian. Suppose there exists $N_0 \triangle N$ such that N_0 is solvable and $\chi \mid N_0$ is irreducible. Then χ is extendible to G.*

Proof. By double induction on $|N|$ and for each value of $|N|$ on $[G:N]$. Note that the theorem is trivial if $|N| = 1$ or if $[G:N] = 1$. We may assume that $N_0 = \mathrm{S}(N)$, the maximal solvable normal subgroup of N, so $N_0 \triangle G$. If $N_0 < N$ then by induction and Proposition 1, $\psi = \chi \mid N_0$ can be extended to φ on G so χ and $\varphi \mid N$ are both extensions of ψ to N. Thus $\chi = \lambda(\varphi \mid N)$ where λ is a linear character of N/N_0. Since χ and $\varphi \mid N$ are both invariant in G, it follows from Proposition 2 that λ is invariant in G and we have $K \triangle G$, $N_0 < K \subseteq N$, where $K = \ker \lambda$. Since N/K is abelian, the theorem applies in G/K to λ and by induction, λ can be extended to μ on G and it follows that $\mu\varphi$ is an extension of χ to G. We may assume then that $N_0 = N$ is solvable.

Since we may assume $N > 1$, let $M < N$, $M \triangle G$ be maximal so that N/M is an elementary abelian q-group for some prime q. We have by Clifford's Theorem that $\chi \mid M = e \sum_{i=1}^{t} \theta_i$ where the θ_i are distinct irreducible characters of M which are conjugate in N and form a full orbit under the action of G. If $t > 1$, let T be the inertia group of θ_1 in G so $T \cap N$ is the inertia group of θ_1 in N and $t = [G:T] = [N:N \cap T]$ so $NT = G$ and since $N \cap T \triangle T$ and N/M is abelian, we have $N \cap T \triangle G$. By the maximality of M we have $N \cap T = M$ and thus the induced character $\theta_1^N = \chi$. Also T/M is a p-group and $|M| < |N|$

so by the inductive hypothesis, θ_1 can be extended to ψ on T. Now $\psi^G(1) = t\psi(1) = t\theta(1) = \chi(1)$ and since

$$0 \neq [\psi^G \mid M, \theta_1] = [\psi^G \mid N, \theta_1^N] = [\psi^G \mid N, \chi]$$

we must have $\psi^G \mid N = \chi$ and we have produced an extension of χ.

We may suppose then that $t = 1$ so $\chi \mid M = e\theta$ and θ is invariant in G. If $q = p$, then since $M < N$, we can apply the inductive hypothesis and extend θ to ψ on G. Since G/M is a p-group, it is abelian and thus every irreducible constituent of θ^G is an extension of θ by Proposition 2. It follows that any irreducible constituent of χ^G has degree equal to $\theta(1) \leq \chi(1)$. We must therefore have equality and thus χ is extendible.

We now consider the remaining case which is $q \neq p$. Suppose that χ is not extendible to G. We claim that if φ is any irreducible constituent of θ^G, then $p\chi(1) \mid \varphi(1)$. Let $G_0 \supseteq N$ have index p in G so $[G_0:N] < [G:N]$ and by the second inductive hypothesis, χ can be extended to G_0. Since G_0/N is abelian, every irreducible character of G_0 whose restriction to N contains χ as a constituent is an extension of χ. Thus if $\varphi \mid N$ has constituent χ, $\varphi \mid G_0$ reduces to a sum of p irreducible conjugate constituents, each of which is an extension of χ and so $\varphi(1) = p\chi(1)$ in this case. Now $[\varphi \mid N, \theta^N] \neq 0$ so some irreducible constituent χ_0 of $\varphi \mid N$ is a constituent of θ^N and by the above, we may assume that $\chi_0 \neq \chi$. Assume $\chi \mid M$ is reducible so $e > 1$. Since N/M is elementary abelian it follows (for instance by Proposition 2.5 of [5]), that for some subgroup L, $M \subseteq L < N$, of index q in N, $\chi \mid L$ is reducible and hence χ vanishes on $N - L$. Since χ is invariant in G, it vanishes on $N - L^g$ for every $g \in G$. Thus χ vanishes on $N - \bigcap_{g \in G} L^g = N - M$. Thus

$$[e\theta, e\theta] = [\chi \mid M, \chi \mid M] = [N:M]$$

so $[N:M] = e^2$. Now $e\chi$ is a constituent of θ^N and has degree $e^2\theta(1) = \theta^N(1)$. Therefore $\theta^N = e\chi$ and χ is the unique irreducible constituent of θ^N. This contradicts $\chi \neq \chi_0$ and thus we must have $\chi \mid M$ is irreducible and since N/M is abelian we must have $\chi_0 = \lambda\chi$ for some linear character λ of N/M by Proposition 2. If χ_0 is not invariant in G, then it is in an orbit of size divisible by p and hence $p\chi_0(1) \mid \varphi(1)$. Since $\chi(1) = \chi_0(1)$, the claim follows in this case. If χ_0 is invariant in G then by Proposition 2 we must have λ invariant in G also. Now let $K = \ker \lambda$. For $x \in N$, $\lambda(x)$ determines the coset Kx and it follows that $N/K \subseteq \mathbf{Z}(G/K)$ so $M[G, N] < N$. By the maximality of M we have $[G, N] \subseteq M$ and thus $N/M \subseteq \mathbf{Z}(G/M)$ and therefore N/M is a direct factor of G/N so λ can be extended to a linear character μ of G. We have then, $\chi = \lambda^{-1}\chi_0$ is a constituent of $\mu^{-1}\varphi \mid N$. Thus $(\mu^{-1}\varphi)(1) = \varphi(1) = p\chi(1)$ and the claim is fully established.

Now let P/M be a Sylow p-subgroup of G/M. Since θ is invariant in P, it can be extended to ψ on P by the inductive hypothesis. If φ is an irreducible constituent of ψ^G, then θ is a constituent of $\varphi \mid M$ and therefore $p\chi(1) \mid \varphi(1)$.

Hence $p\chi(1)$ divides $\psi^G(1) = [G:P]\theta(1)$ so

$$pe\theta(1) \mid [G:P]\theta(1) \quad \text{and} \quad p \mid [G:P],$$

a contradiction by the choice of P. The proof is complete.

To apply the theorem to the case where G/N is not a p-group, we define a certain set of primes $\pi(\chi)$, associated with χ. If $\mathfrak{X}$ is any representation of G which affords χ, we may define $\lambda(n) = \det \mathfrak{X}(n)$ for $n \in N$, and we call the resulting linear character of N the determinant $\det \chi$ of χ. It is clear that $\det \chi$ is independent of the choice of $\mathfrak{X}$. Now λ is an element of the group $\hat{N}$ of linear characters of N and by $o(\lambda)$ we mean the order of λ as an element of this group. We now set

$$\pi(\chi) = \{p \mid p \text{ divides } \chi(1) \text{ or } o(\det \chi)\}.$$

Clearly $\pi(\chi) \subseteq \pi(|N|)$.

LEMMA 4 (Gallagher). *Let $\lambda = \det \chi$ and suppose G/N is a p-group. If $p \nmid \chi(1)$ and λ can be extended to G, then χ can be extended to G.*

Actually, Gallagher proves this as Theorem 5 of [2] without any assumption on G/N except that $([G:N], \chi(1)) = 1$. His proof consists of showing it first when G/N is supersolvable and then using the Brauer-Tate Theorem to obtain the general result. The general theorem follows easily from that part of it stated here as Lemma 4 together with Proposition 1.

LEMMA 5. *Suppose G/N is a p-group and χ is a linear character with $p \nmid o(\chi)$. Then χ can be extended to G.*

Proof. Let $K = \ker \chi$. Then as was seen in part of the proof of Theorem 3, $N/K \subseteq \mathbf{Z}(G/K)$. Now $|N/K| = o(\chi)$ which is prime to p so it follows that N/K is a direct factor of G/K and χ can be extended.

COROLLARY 6. *Suppose $N_0 \triangle N$ is solvable and $\chi \mid N_0$ is irreducible. For every prime $p \in \pi(\chi)$ which divides $[G:N]$, suppose that a Sylow p-subgroup of G is abelian. Then χ is extendible to G.*

Proof. By Proposition 1, we may assume that G/N is a p-group. If $p \notin \pi(\chi)$, then by Lemma 5, $\det \chi$ can be extended to G so by Lemma 4, χ is extendible. If $p \in \pi(\chi)$ then a Sylow p-subgroup of G is abelian and Theorem 3 applies.

2. Throughout this section we suppose that $N \subseteq H \triangle G$, $N \triangle G$ and that $\chi \in \mathrm{Irr}(N)$ is invariant in G and is extendible to H. In order to be able to apply Theorem 3, we shall also assume that a solvable $N_0 \triangle N$ exists with $\chi \mid N_0$ irreducible. If for every prime $p \mid [G:N]$ with $p \in \pi(\chi)$ we have a Sylow p-subgroup of G abelian, then by the previous results it is clear that χ can be extended to G and thus some extension of χ to H is invariant in G. It is our purpose here to obtain the latter conclusion under the weaker hypothesis that for those $p \in \pi(\chi)$ which divide both $[G:H]$ and $[H:N]$, a Sylow p-subgroup of G is abelian. We begin with two lemmas.

LEMMA 7. *Let $H \subseteq D \subseteq G$, $D \triangle G$ where $([G:D], [H:N]) = 1$. Suppose that some extension θ of χ to H is invariant in D. Then some extension of χ to H is invariant in G.*

Proof. If $g \in G$ then θ^g is an extension of χ to H which is also invariant in D. By Proposition 2, $\theta^g = \lambda\theta$ where $\lambda \in \mathrm{Irr}(H/N)$ is linear and invariant in D. It follows that $L = N[D, H] \subseteq \ker \lambda$ so $\theta \mid L = \theta^g \mid L$ and $\theta \mid L$ is invariant in G. Let $\mathcal{S}$ be the set of characters of H which extend $\theta \mid L$. Since D centralizes H/L and fixes θ, D fixes every element of $\mathcal{S}$ so G/D acts on $\mathcal{S}$. Since H/L is abelian, $\mathrm{Irr}(H/L) = C$ is a group. Now C acts transitively on $\mathcal{S}$ by multiplication and $(|G/D|, |C|) = 1$. The actions of these two groups on $\mathcal{S}$ are compatible with the action of G/D on C in the sense of Glauberman's Lemma (Theorem 4 of [3]) and that lemma applies to yield the result.

LEMMA 8. *There exists an extension θ of χ to H such that $\pi(\theta) \subseteq \pi(\chi)$. If no prime of $\pi(\chi)$ divides $[H:N]$ then θ is unique. In any case, if some extension of χ to H is invariant in G, we may take θ invariant in G also.*

Proof. Choose any extension θ_0 of χ to H. If θ_1 is any other extension then $\theta_1 = \mu\theta_0$ where μ is a linear character of H/N. Let $f = \chi(1)$ so $\det \theta_1 = \mu^f \det \theta_0$. If $o(\mu)$ is divisible by some prime not in $\pi(\chi)$ then so is $o(\mu^f)$. In this case if $\pi(\theta_0) \subseteq \pi(\chi)$ then $\pi(\theta_1) \not\subseteq \pi(\chi)$ and this establishes the uniqueness if $[H:N]$ is divisible by no prime of $\pi(\chi)$.

Now, in general, let $\lambda = \det \theta_0$ and let m be the full $\pi(\chi)$ part of $o(\lambda)$ so λ^m has $\pi(\chi)'$ order. Let $\mu = \lambda^m$ and note that $\mu \mid N = (\det \chi)^m$ has $\pi(\chi)$ order so we must have $\mu \mid N = 1$. Let $n = o(\mu)$, a $\pi(\chi)'$ number and choose an integer a with $afm \equiv -1 \bmod n$. Let $\theta = \mu^a\theta_0$. Since $\mu \mid N = 1$, θ is an extension of χ and if θ_0 is invariant in G, so is θ. Now $\det \theta = \mu^{af}\lambda = \lambda^{amf+1} = \lambda^{kn}$ for some integer k. Since $\lambda^{mn} = 1$, $o(\det \theta) \mid m$ and thus $\pi(\theta) \subseteq \pi(\chi)$, proving the lemma.

THEOREM 9. *Suppose $N_0 \triangle N$ is solvable and $\chi \mid N_0$ is irreducible. Also suppose that for every prime $p \in \pi(\chi)$ which divides both $[H:N]$ and $[G:H]$ that a Sylow p-subgroup of G is abelian. Then some extension θ of χ to H is invariant in G.*

Proof. By double induction on $|G|$, and for each value of $|G|$, on $[H:N]$. Note that if $H = N$ the theorem is trivial since we can take $\theta = \chi$. Let θ_0 be an extension of χ to H with $\pi(\theta_0) \subseteq \pi(\chi)$. Let $L = NH'$ so that the kernel of every linear character of H/N contains L. It follows that $\theta_0 \mid L$ is invariant in G. If $p \in \pi(\theta_0 \mid L)$ divides both $[G:H]$ and $[H:L]$, then since $\pi(\theta_0 \mid L) \subseteq \pi(\chi)$ and $[H:L] \mid [H:N]$, a Sylow p-subgroup of G is abelian and if $L > N$ then $[H:L] < [H:N]$ and induction applies to yield the result, since we may assume that $N_0 = \mathbf{S}(N)$ so $N_0 \triangle L$. We therefore may assume that $L = N$ so H/N is abelian. Also suppose $H > N$.

If H/N is not a q-group for some prime q, let Q/N be a Sylow q-subgroup of H/N for some $q \mid [H:N]$. Then $Q \triangle G$, $Q < H$ and if $p \in \pi(\chi)$ divides both $[Q:N]$ and $[G:Q]$, then we must have $q = p$ and $p \mid [G:H]$ so a Sylow p-subgroup of G is abelian and by induction, χ can be extended to ψ on Q with ψ invariant in G, since $[Q:N] < [H:N]$. By Lemma 8 we may assume that $\pi(\psi) \subseteq \pi(\chi)$. Now $\psi = \lambda(\theta_0 \mid Q)$ where λ is a linear character of Q/N. Since H/N is abelian, λ can be extended to μ on H and it follows that $\mu\theta_0$ is an extension of ψ. Induction applies again because $[H:Q] < [H:N]$ and the theorem follows in this case. We therefore assume that H/N is a q-group.

If $q \notin \pi(\chi)$, then by Lemma 8 there exists a unique extension θ of χ to H such that $\pi(\theta) \subseteq \pi(\chi)$ and thus θ is necessarily invariant in G and we are done. Suppose then that $q \in \pi(\chi)$. Assume now that $H/N \not\subseteq \mathbf{Z}(G/N)$. Let $D/N = \mathbf{C}_{G/N}(H/N)$ so $N < H \subseteq D \triangle G$. Since $D < G$, we may assume by induction that θ_0 is invariant in D. We claim $q \nmid [G:D]$, or else by hypothesis, a Sylow q-subgroup of G is abelian and $\mathbf{C}(H/N)$ contains a full Sylow q-subgroup of G, contradicting $q \mid [G:D]$. We therefore have $([G:D], [H:N]) = 1$ and Lemma 7 applies to yield the theorem.

We may assume then that $H/N \subseteq \mathbf{Z}(G/N)$ and every linear character of H/N is invariant in G. It follows that the inertia groups of all the extensions of χ to H are equal. Let T be this common inertia group. Now let M be any subgroup of G which satisfies $H \subseteq M < G$. By induction, some extension of χ to H is invariant in M so $M \subseteq T$ and it follows that T/H is the unique maximal subgroup of G/H. Thus G/H is a p-group and if $p \neq q$ then Lemma 7 applies with $H = D$ and we are done. Finally, if $p = q$, then G/N is a q-group and a Sylow q-subgroup of G is abelian so Theorem 3 applies and χ is extendible to G. The theorem follows immediately from this.

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