Q-SIMPLICIAL SPACES[1, 2, 3]

BY

R. J. Knill

The quasicomplexes of Lefschetz have proven useful in investigating the fixed point properties of exotic spaces [1], [3], [11]. The purpose of this paper is to report a simplification of Leftschetz's quasicomplexes which is used to give a unified treatment of several forms of the Lefschetz fixed point theorem, including that part of the Schauder-Leray theory to which the Leftschetz theorem is pertinent. No attempt is made to consider ramifications of the related concept of the local degree theory of Leray, for my spaces will not always be locally connected.

In this study, I adopt the point of view apparently taken by Lefschetz [5] when he defined quasicomplexes: the key step in the classical proof of the Lefschetz theorem for simplicial complexes involves the subdivision operator, and if one is to extend the classical result he could do so by looking for a substitute for this step. The result of such an effort is the concept of "Q-simplicial spaces". Here the Q stands not for "quasi" but for the coefficient field of which the Lefschetz number is a member. Q-simplicial spaces enjoy some advantages over the quasicomplexes defined by Lefschetz [5], and the weak semi-complexes of Thompson [13]. Their definition is syntactically simpler. They are not necessarily compact: a locally convex topological vector space is Q-simplicial. Open subsets of, retracts of, and infinite products of Q-simplicial spaces are all again Q-simplicial.

The principal technique used to derive these properties uses the fact that the *category* of Q-simplicial spaces and continuous maps has a certain class of infinite limits. My formulation of this class of limits is not categorical, however, it being simpler and more direct to define these limits in topological terms. The formulation is more in keeping with the concept, introduced by Klee [8], of "approachable sets," and so I have adopted that terminology for describing my limits.

The proof of the Lefschetz theorem itself for Q-simplicial spaces is conceptually simplified in that no reference to chain homotopies is necessary; instead, use is made of the continuity axiom for Čech theory.

The last section is concerned with examples which show (1) Q-simplicial spaces do not have a local characterization, (2) the class of Q-simplicial spaces depends in an essential way upon the characteristic of Q.

Received January 8, 1968.

[1] Reported to the American Mathematical Society in cohomological form as *Cochain-simple spaces*, August, 1964.

[2] Reported to the Symposium on Infinite-Dimensional Topology of Louisiana State University, April, 1967.

[3] Research supported by a National Science Foundation Research grant.

1. Preliminaries

I will generally follow the conventions and notation of Spanier [12], other conventions to be made explicit. In particular I will mean by *chain complex* a chain complex C_* over a field Q which is non-negative and is equipped with an augmentation $\varepsilon : C_0 \to Q$. Any chain map in this paper, $\varphi : C_* \to C'_*$, will be assumed to preserve augmentation: $\varepsilon\varphi_0 = \varepsilon$. Let $\tilde{C}_*$ be the reduced chain complex of C_*, and let $\tilde{\varphi} : \tilde{C}_* \to \tilde{C}'_*$ be induced by φ. Let $\tilde{H}_q(C_*) = H_q(\tilde{C}_*)$, $q \geq 0$. If K is a simplicial complex, then $C_*(K)$ is the oriented chain complex of K, with coefficients in Q. An element of $C_*(K)$ is a *chain on* K. If s is a simplex of K, let $\bar{s}$ be the simplicial complex formed by the faces of s.

A *carrier* $\Gamma : K \to C_*$ of a simplicial complex K into a chain complex C_* is a function which assigns to each simplex s of K a subcomplex $\Gamma(s)$ of the chain complex C_* such that for a face t of $s \in K$, $\Gamma(t)$ is subcomplex of $\Gamma(s)$. Γ is *acyclic* if $\tilde{H}(\Gamma(s)) = 0$ for $s \in K$. A chain map $\varphi : C_*(K) \to C_*$ is *carried* by Γ if $\varphi(C_*(s)) \subset \Gamma(s)$ for $s \in K$. The classical theorem on acyclic carriers states that if Γ is an acyclic carrier, then there exists at least one chain map carried by Γ and any two chain maps carried by Γ are chain homotopic. The derived complex, K', of a simplicial complex, K, is the simplicial complex of all finite sets $\{s_0, s_1, \cdots, s_n\}$ such that $s_0 \subset s_1 \subset \cdots \subset s_n \in K$.

A *family* α of subsets of a set Y is any set of subsets of Y. Let $\bigcup \alpha$ (respectively, $\bigcap \alpha$) denote the union (intersection) of the sets of α. If $X \subset Y$, let $\mathrm{st}_\alpha(X)$ be the family

$$\mathrm{st}_\alpha(X) = \{A : A \cap X \neq \emptyset, A \in \alpha\}.$$

Let $\mathrm{St}_\alpha(X) = \bigcup \mathrm{st}_\alpha(X)$, and let $\alpha | X$ be the family $\{A \cap X : A \in \alpha\}$. If $f : Y' \to Y$ is a function, let $f^{-1}\alpha = \{f^{-1}A : A \in \alpha\}$. If β is a family of subsets of Y, β *refines* α ($\beta > \alpha$) if for each $B \in \beta$ there exists $A_B \in \alpha$ such that $B \subset A_B$; β *strongly refines* α ($\beta \gg \alpha$) if for $B \in \beta$, $A_B \in \alpha$ may be chosen so that $\mathrm{St}_\alpha(B) \subset A_B$.

For a family α of subsets of a set Y and subset X of Y, I will need to consider the following two simplicial complexes.

1.1. X_α is the simplicial complex formed by the finite nonempty subsets $s \subset X$ such that $s \subset A$ for some $A \in \alpha$.

1.2. $N(\alpha)$ is the nerve of α: the simplicial complex of all finite nonempty subsets s of α such that $\bigcap s \neq \emptyset$.

Note that if $W \subset X$, then $C_*(W_\alpha) \subset C_*(X_\alpha)$. If c is a chain in $C_*(X_\alpha)$, let $\sup(c)$ be the intersection of those sets $W \subset X$ such that $c \in C_*(W_\alpha)$. If c is a chain in $C_*(N(\alpha))$, let $\sup(c)$ be the union

$$\sup(c) = \bigcup \mathrm{car}(c)$$

where $\mathrm{car}(c)$ is the intersection of those subfamilies β of α such that $c \in C_*(N(\beta))$.

The important relationship of X_α to $N(\alpha)$ is that X_α and $N(\alpha \mid X)$ have the same homology, according to the following theorem of Dowker [2].

1.3 THEOREM (Dowker). *Let α be a family of subsets of X. Then there are chain maps*

$$k^\alpha : C_*(N(\alpha)) \to C_*(X_\alpha)$$

$$l^\alpha : C_*(X_\alpha) \to C_*(N(\alpha))$$

which are chain homotopy inverses to each other and which satisfy the following:

1.3.1. *For a chain c on $N(\alpha)$,*

$$\sup (k^\alpha(c)) \subset \sup (c)$$

1.3.2. *For a chain c on X_α,*

$$\sup (l^\alpha(c)) \subset \mathrm{St}_\alpha (\sup (c))$$

2. Q-simplicial spaces and the Lefschetz theorem

Throughout the remainder of this paper Y is a regular Hausdorff space, and X is a subset of Y. A family α of subsets of Y is a *covering* of X if $X \subset \bigcup \alpha$. Let $\mathrm{Cov}_Y(X)$ be the class of all coverings of X by families of open subsets of Y, and for $Y = X$, Cov (Y) means $\mathrm{Cov}_Y(Y)$. Suppose that W is another subset of Y and that β and γ are families of subsets of Y. A chain map

$$\omega : C_*(X_\beta) \to C_*(W_\gamma)$$

is *subordinate* to a family α of subsets of Y if for a chain c on X_α,

$$\sup (\omega(c)) \subset \mathrm{St}_\alpha (\sup (c)).$$

2.1 DEFINITION. A regular Hausdorff space Y is *Q-simplicial at a compact subset* X if for any $\alpha \in \mathrm{Cov}(Y)$, there exists an $\alpha^\# = \alpha^\#(X) \in \mathrm{Cov}_Y(X)$ such that $\alpha^\# > \alpha$, and such that for every $\gamma \in \mathrm{Cov}(Y)$ there is a chain map

$$\omega : C_*(X_{\alpha^\#}) \to C_*(Y_\gamma)$$

which is subordinate to α. Y is *Q-simplicial* if it is Q-simplicial at every compact subset.

Evidently a quasi-complex Y of Lefschetz [5] is Q-simplicial for every Q. As with Lefschetz's definition of quasi-complex, Q-simplicial spaces could have been defined in terms of nerves of open coverings. To see this one need only to apply to the theorem of Dowker. The reason for my preference of X_α over $N(\alpha \mid X)$ is that if $\beta > \alpha$, then X_β is a subcomplex of Y_α. Let

$$\pi_\alpha^\beta : C_*(X_\beta) \to C_*(Y_\alpha)$$

denote the inclusion.

The proof of the next lemma is a straightforward argument involving acyclic carriers.

2.2 LEMMA. *Suppose X is a compact subset of a regular Hausdorff space Y; then for any $\alpha \in \operatorname{Cov}(Y)$, there exists a $\gamma \in \operatorname{Cov}_Y(X)$ such that $\gamma \gg \alpha$. If β refines such a covering γ of X, then any augmentation preserving chain map*

$$C_*(X_\beta) \to C_*(Y_\alpha)$$

which is subordinate to γ, is chain homotopic to π_α^β.

Now I want to show that the Lefschetz fixed point theorem generalizes to Q-simplicial spaces if, roughly speaking, one takes care to consider only maps which have relatively compact image, and for which the Lefschetz number is defined as an element of a coefficient ring. It turns out that the choice of this coefficient ring can be crucial, and that the integers are not always the best choice.

Throughout, let P be a principal ideal domain, and let Q be the field of rationals of P. If $h : M \to M$ is a P-module endomorphism, let $h^1 = h$, let $h^{n+1} = h \circ h^n$, for $n \geq 1$, and let $N(h)$ be the union of the kernels of h^n, $n \geq 1$. h is of *finite type* if $M/N(h)$, modulo its torsion subgroup, is a free P-module of finite rank. In such a case let $\operatorname{tr}(h)$ be the trace of the automorphism of $(M/N(h)) \otimes_P Q$ which is naturally induced by h. This definition is due to Leray [7], and its usefulness is a consequence of the next proposition of Leray.

2.3 PROPOSITION. *Let $h : M \to M$ be a P-module endomorphism of finite type. If $h = h_1 \circ h_2$, then $h_2 \circ h_1$ is of finite type and*

$$\operatorname{tr}(h_2 \circ h_1) = \operatorname{tr}(h).$$

Furthermore if $P = Q$, and M is a finite-dimensional vector space over Q, then

$$\operatorname{tr}(h) = \operatorname{trace}(h).$$

2.4 DEFINITION. Suppose $f : Y \to Y$ is a map. Then *f has finite type over P* if the Čech cohomology homomorphism induced by f,

$$(f^*)^n : H^n(Y; P) \to H^n(Y; P),$$

has finite type for $n \geq 0$, and is non-zero for at most finitely many values of n. The *Lefschetz* index in P of such a map f is defined as

$$\Lambda_P(f) = \textstyle\sum_{n \geq 0} (-1)^n \operatorname{tr}(f^*)^n.$$

This is an element of P and by the universal coefficient theorem for Čech cohomology theory there is the proposition:

2.5 PROPOSITION. *If $f : Y \to Y$ is a map which has finite type over P, then f has a finite type over Q, and*

$$\Lambda_Q(f) = \Lambda_P(f).$$

2.6 DEFINITION. If $f : Y \to Y$ is a map, a *Q-admissible image* of f is a compact subset X of Y such that $X \supset f(Y)$ and such that the corestriction of f,

$f_X : Y \to X$, defined by $f_X(y) = f(y)$, induces a Čech cohomology homomorphism

$$(f_X^*)^n : H^n(X; Q) \to H^n(Y; Q)$$

which has a finite-dimensional image for all values of n, and which is zero for all but finitely many values of n.

2.7 THEOREM. *Suppose that Y is a regular Hausdorff space and $f : Y \to Y$ has finite type over P. If f has a Q-admissible image $X \subset Y$, such that Y is Q-simplicial at X, then $\Lambda_P(f) \in P$ and if $\Lambda_P(f) \neq 0$, then $f(x) = x$ for some $x \in X$.*

Proof. Since $(f^*)^n : H^n(Y; P) \to H^n(Y; P)$ is of finite type over P, then $\operatorname{tr}(f^*)^n \in P$, for $n \geq 0$, and so $\Lambda_P(f) \in P$.

Since X is compact, the universal coefficient theorem for Čech theory with field coefficients applies and so the induced Čech homology homomorphism

$$((f_X)_*)_n : H_n(Y; Q) \to H_n(X; Q)$$

has a finite-dimensional image for $n \geq 0$, it is zero for all but finitely many values of n, and

$$\operatorname{tr}(f_*)_n = \operatorname{tr}(f^*)^n, \qquad n \geq 0,$$

where

$$(f_*)_n : H_n(Y; Q) \to H_n(Y; Q)$$

is the Čech homology homomorphism induced by f. Thus

$$\Lambda_P(f) = \textstyle\sum_{n \geq 0} (-1)^n \operatorname{tr}(f_*)_n$$

Suppose that $fx \neq x$ for all $x \in X$. The remainder of the proof is concerned with showing that $\Lambda_P(f) = 0$.

For any $\beta \in \operatorname{Cov}_Y(X)$, and for any $\delta > f^{-1}\beta$, f defines the simplicial map $Y_\delta \to X_\beta$ which takes a simplex $s \in Y_\delta$ to the simplex $f(s) \in X_\beta$. Let

$$f_\beta^\delta : C_*(Y_\delta) \to C_*(X_\beta)$$

be the chain map induced by this simplicial map.

Now choose $\alpha \in \operatorname{Cov}_Y(X)$, $\beta \in \operatorname{Cov}_Y(X)$, $\gamma \in \operatorname{Cov}_Y(X)$ and $\delta \in \operatorname{Cov}(Y)$ so that

(i) $\operatorname{St}_\alpha(f(A)) \cap A = \emptyset$ for $A \in \alpha$.
(ii) The natural projection

$$(\pi_\alpha)_* : H_*(X; Q) \to H_*(X_\alpha; Q)$$

is monic on the image of $(f_X)_* : H_*(Y; Q) \to H_*(X; Q)$.

(iii) Let $\gamma \in \operatorname{Cov}_Y(X)$ be such that $\gamma \gg \alpha$, and choose $\beta \in \operatorname{Cov}_Y(X)$ so that it refines both γ and $(f^{-1}\gamma)^\#$.

(iv) Let $\delta \in \operatorname{Cov}(Y)$ be such that $\delta > f^{-1}\beta$ and image $(f_\beta^\delta)_* =$ image $(\pi_\beta)_*(f_X)_*$.

Then there is the following diagram of transformations of vector spaces over Q.

$$\begin{array}{ccccc}
H_*(X;Q) & \xrightarrow{i_*} & H_*(Y;Q) & \xrightarrow{(f_X)_*} & H_*(X;Q) \\
{\scriptstyle (\pi_\beta)_*}\downarrow & \text{(II)} & \downarrow & \text{(III)} & \downarrow{\scriptstyle (\pi_\beta)_*} \\
H_*(X_\beta\,;Q) & \xrightarrow{\omega_*} & H_*(Y_\delta\,;Q) & \xrightarrow{(f_\beta^\delta)_*} & H_*(X_\beta\,;Q) \\
 & {\scriptstyle (\pi^\beta_{f^{-1}\alpha})_*}\searrow \quad \text{(I)} & \downarrow & \text{(IV)} & \downarrow \\
 & & H_*(Y_{f^{-1}\alpha}\,;Q) & \xrightarrow{(f_\alpha^{f^{-1}\alpha})_*} & H_*(X_\alpha\,;Q)
\end{array}$$

The vertical arrows are the natural projections, i_* is induced by the inclusion $X \subset Y$, and ω_* is induced by a chain map

$$\omega : C_*(X_\beta) \to C_*(Y_\delta)$$

which is subordinate to $f^{-1}\gamma$. All of the numbered subdiagrams are commutative except for (II), diagram (I) being commutative as a consequence of Lemma 2.2 with α in that proposition replaced by $f^{-1}\alpha$ and γ in that proposition replaced by $f^{-1}\gamma$ in the diagram. My first step in showing that $\Lambda_p(f) = 0$ is to show that

(v) $\quad \operatorname{tr}\,(f_*)_n = \operatorname{tr}\,((f_\beta^\delta \circ \omega)_*)_n\,.$

Let

$$r_* : H_*(X_\alpha\,;Q) \to H_*(X;Q)$$

be any linear transformation such that

$$((\pi_\alpha)_*\, r_*\, (\pi_\alpha)_*)_n = ((\pi_\alpha)_*)_n$$

Since $(\pi_\alpha)_*$ is monic on the image of $(f_X)_*$ then

$$(r_*\,(\pi_\alpha)_*\,(f_X)_*)_n = ((f_X)_*)_n$$

From the commutativity of diagrams (I), (III), and (IV), and from

$$(\pi^\beta_{f^{-1}\alpha}\,\pi_\beta)_* = (\pi_{f^{-1}\alpha})_*$$

it follows that

$$(r_*\,(\pi_\alpha)_*\,(f_X)_*)_n = (r_*\,(\pi_\alpha^\beta)_*\,(f_\beta^\delta)_*\,\omega_*\,(\pi_\beta)_*)_n\,.$$

So

$$\operatorname{tr}\,(f_*)_n = \operatorname{tr}\,(r_*\,(\pi_\alpha^\beta)_*\,(f_\beta^\delta)_*\,\omega_*\,(\pi_\beta)_*)_n$$

and from Proposition 2.5, this becomes

$$\operatorname{tr}\,(f_*)_n = \operatorname{tr}\,((\pi_\beta)_*\,r_*\,(\pi_\alpha^\beta)_*)_n((f_\beta^\delta\,\omega)_*)_n\,.$$

However, image $((f_\beta^\delta)_*)_n$ = image $((\pi_\beta)_*(f_X)_*)_n$, and $((\pi_\beta)_*\,r_*\,(\pi_\alpha^\beta)_*)_n$ is the identity on the image of $(\pi_\beta)_* (f_X)_*$, so (v) holds:

$$\operatorname{tr}(f_*)_n = \operatorname{tr}((f_\beta^\delta \omega)_*)_n .$$

From Dowker's theorem, it follows that for $\beta' = \beta \mid X$,

$$\begin{aligned}\operatorname{tr}((f_\beta^\delta \omega)_*)_n &= \operatorname{tr}((k^{\beta'} l^{\beta'} f_\beta^\delta \omega)_*)_n \\ &= \operatorname{tr}((l^{\beta'} f_\beta^\delta \omega k^{\beta'})_*)_n .\end{aligned}$$

So from the Hopf trace formula one has

$$\Lambda_P(f) = \textstyle\sum_{n\geq 0} (-1)^n \operatorname{tr}(l^{\beta'} f_\beta^\delta \omega k^{\beta'})_n$$

where

$$(l^{\beta'} f_\beta^\delta \omega k^{\beta'})_n : C_n(N(\beta')) \to C_n(N(\beta')).$$

However from (i) one may show that for any oriented simplex σ on $N(\beta')$, $(l^{\beta'} f_\beta^\delta \omega k^{\beta'})_n(\sigma)$ is zero, so $\Lambda_P(f) = 0$.

3. Q-simplicial spaces

The purpose of this section is to develop properties of Q-simplicial spaces which do not depend on the choice of Q.

In [8], Klee defined the notion of an approachable subset of a topological vector space, and used it to develop a local degree theory. The concept itself has been around a long time, one of the earliest users of it having been Leray and Schauder to extend Brouwer degree and its applications to Banach spaces. The next definition is an extension of the concept of Klee [8].

3.1 Definition. Let $\mathfrak{C}$ be a class of pairs of spaces (W, V) such that $V \subset W$. A compact subset X of a space Y is *approachable* by $\mathfrak{C}$ if for every $\alpha \in \operatorname{Cov}_Y(X)$, there exists a pair $(W, V) \in \mathfrak{C}$ and maps

$$f : X \to V, \qquad g : W \to Y$$

such that for $x \in X$, $g \circ f(x)$ and x are in a common set $A_x \in \alpha$. If $\mathfrak{D}$ is a class of spaces we say that X is approachable in Y by $\mathfrak{D}$ if it is approachable by the class of pairs (W, W), such that $W \in \mathfrak{D}$.

3.2 Definition. For a field Q, let $\mathfrak{C}_Q$ be the class of pairs (W, V) such that W is a regular Hausdorff space, V is compact, and W is Q-simplicial at V.

3.3 Theorem. *If Y is a regular Hausdorff space and X is a compact subset of Y, then Y is Q-simplicial at X if and only if X is approachable by $\mathfrak{C}_Q$. Y is Q-simplicial if and only if every compact subset is approachable by a class of Q-simplicial spaces.*

Proof. If Y is Q-simplicial at X, then $(Y, X) \in \mathfrak{C}_Q$ and, a fortiori, X is approachable by $\mathfrak{C}_Q$. Conversely, suppose that X is approachable in Y by $\mathfrak{C}_Q$. If $\alpha \in \operatorname{Cov}(Y)$, then let $\beta \gg \beta' \gg \alpha$ be a sequence of coverings in

$\mathrm{Cov}_Y(X)$, and let $(W, V) \in \mathfrak{C}_Q$ be such that there exist maps

$$f : X \to V, \qquad g : W \to Y$$

such that for $x \in X$, $g \circ f(x)$ and x are in a common set $B_x \in \beta$. Then $g^{-1}(\beta) \in \mathrm{Cov}_W(f(X))$. Let

$$\alpha' = g^{-1}(\beta) \cup \{W \backslash f(X)\}$$

and let $\alpha^{\#} = \alpha^{\#}(X)$ be defined as

$$\alpha^{\#} = f^{-1}((\alpha')^{\#}(V)).$$

If $\gamma \in \mathrm{Cov}_Y(X)$, then let $\gamma' = g^{-1}(\gamma)$ and define a chain map

$$\omega : C_*(X_{\alpha^{\#}}) \to C_*(Y_\beta)$$

as the composition

$$C_*(X_{\alpha^{\#}}) \xrightarrow{f^{\alpha^{\#}}_{(\alpha')^{\#}}} C_*(V_{(\alpha')^{\#}}) \xrightarrow{\omega'} C_*(W_{\gamma'}) \xrightarrow{g^{\gamma'}_{\gamma}} C_*(Y_\gamma)$$

where ω' is subordinate to α'. Then ω is subordinate to α, for one may compute that

$$\sup \omega(c) \subset \mathrm{St}_\beta(\mathrm{St}_\beta(\sup(c))) \subset \mathrm{St}_\alpha(\sup(c)).$$

The second part of the theorem follows from the first part.

3.4 Corollary. *Suppose that X is a compact subset of a regular Hausdorff space Y. Then Y is Q-simplicial at X if either*

3.4.1. *Y is Q-simplicial at a compact set X' which contains X,*

3.4.2. *Y' is Q-simplicial at X, for some subspace Y' of Y,*

3.4.3. *there is a finer topology on Y, which induces the same topology on X as the original, and in which Y is Q-simplicial at X.*

3.5 Corollary. *Any neighborhood retract of a Q-simplicial space is a Q-simplicial space.*

Proof. From 3.3, it follows that any retract of a Q-simplicial space is Q-simplicial. To complete the proof, it suffices to observe that if Y is Q-simplicial at X then every neighborhood U of X in Y is Q-simplicial at X, for if $\alpha \in \mathrm{Cov}(U)$, one may let $\alpha' = \alpha \cup \{Y \backslash X\}$ and choose $\alpha^{\#}$ as $(\alpha')^{\#}(X) | U$.

3.6 Corollary. *Every convex subset Y of a locally convex topological vector space is Q-simplicial.*

Proof. Every compact subset X of Y is approachable by the class of Euclidean spaces.

3.7 Corollary. *Every neighborhood extensor, Y, for the class of compact spaces is Q-simplicial. In particular every absolute neighborhood retract for normal spaces is Q-simplicial.*

Proof. Every compact subset of Y is approachable by the class of compact convex subsets of locally convex topological vector spaces.

Comment. The nature of these results seems to indicate that the property of being Q-simplicial is a local property. Not so. If Y is a space such that every point has an open Q-simplicial neighborhood, then Y is not necessarily Q-simplicial as Example 4.1 will show.

3.8 Corollary. *Every totally disconnected regular Hausdorff space Y is Q-simplicial.*

Proof. Every compact subset of Y is approachable by the class of finite Hausdorff spaces.

Suppose that X_1, X_2 are compact subsets of regular Hausdorff spaces Y_1 and Y_2, and α_1, α_2 are families of subsets of Y_1 and Y_2. Let $\alpha = \alpha_1 \times \alpha_2$ be defined by

$$\alpha_1 \times \alpha_2 = \{A_1 \times A_2 : A_1 \in \alpha_1, A_2 \in \alpha_2\}$$

and let $X = X_1 \times X_2$. Then the Eilenberg-Zilber theorem [4] yields as a special case:

3.9 Theorem. *There are natural chain equivalences*

$$u : C_*(X_\alpha) \to C_*((X_1)_{\alpha_1}) \otimes C_*((X_2)_{\alpha_2}),$$

$$v : C_*((X_1(_{\alpha_1}) \otimes C_*((X_2)_{\alpha_2}) \to C_*(X_\alpha)$$

where by "natural" is meant that for $W = W_1 \times W_2$ a subset of X,

$$u(C_*(W_\alpha)) \subset C_*((W_1)_{\alpha_1}) \otimes C_*((W_2)_{\alpha_2}),$$

$$v(C_*((W_1)_{\alpha_1}) \otimes C_*((W_2)_{\alpha_2})) \subset C_*(W_\alpha).$$

3.10 Corollary. *If X_i is Q-simplicially imbedded in Y_i, for $i = 1, 2$, then $X = X_1 \times X_2$ is Q-simplicially imbedded in $Y = Y_1 \times Y_2$.*

Proof. If $\alpha \in \mathrm{Cov}_Y(X)$, there are coverings

$$\alpha_1 \in \mathrm{Cov}_Y(X_1) \quad \text{and} \quad \alpha_2 \in \mathrm{Cov}_Y(X_2)$$

such that $\alpha_1 \times \alpha_2$ refines α. Then we may choose $\alpha^\# = \alpha^\#(X)$, as

$$\alpha^\# = \alpha_1^\# \times \alpha_2^\#.$$

3.11 Corollary. *If $[Y_i]_{i \in I}$ is an indexed set of Q-simplicial spaces then their product space, $Y = \prod_I Y_i$, is a Q-simplicial space.*

Proof. It suffices to observe that every compact subset of Y is approachable in Y by the class of product spaces of finite subfamilies of $[Y_i]_{i \in I}$, and by 3.10, these products are Q-simplicial spaces.

Comment. In general though the Lefschetz fixed point theorem may hold for two compact spaces, it need not hold for their product [9].

4. Examples

I give two examples in this section. The first amounts to the observation about an example of Wilder [14]. The second shows that the class of Q-simplicial spaces depends on the characteristic of Q.

4.1 *Example.* Suppose that $X = S_1 \cup S_2 \cup A$, is the compact subspace of the complex plane defined by letting S_1 and S_2 be the circles of radii 1 and 2, respectively, and letting A be the spiral whose points z are of the form,

$$z = \left(\frac{1}{\pi} \text{arc tan } \theta + 1 \frac{1}{2}\right) e^{2\pi i \theta}, \qquad -\infty < \theta < \infty.$$

Then the cohomology of X is given by

$$H^0(X; Q) = Q, \quad H^1(X; Q) = Q \oplus Q$$

and the Lefschetz number of the map $f: X \to X$ defined by rotating X in itself one-half revolution is homotopic to the identity, so,

$$\Lambda_Q(f) = -1.$$

Thus X is not Q-simplicial for any Q, although it is locally Q-simplicial for every Q since every point has a neighborhood homeomorphic to a product of an interval with a totally disconnected space.

Now, let T be the additive group of reals modulo the integers, $T = R/Z$, with the quotient topology. For $x \in R$, let $[x]$ denote the equivalence class of x, and for any integer n, let $n[x] = [nx]$. Let N be the set of natural numbers, directed by the relation n divides m (denoted $n \mid m$). In this relation, m is regarded as larger than n. Let $\mathfrak{D}$ be the inverse system of compact groups and group homomorphisms defined by letting

$$\mathfrak{D} = [X_n \,;\, p_n^m]_{n \in N}$$

where $X_n = T$ for $n \in N$, where $m = nq$, q, $n \in N$, and where

$$h_q = p_n^m : X_m \to X_n$$

is defined by $p_n^m(t) = qt$, $t \in T$. The solenoid S is the inverse limit $S = \lim_{\leftarrow} \mathfrak{D}$. For each $n \in N$, let

$$p_n : S \to X_n = T$$

be the limit homomorphism.

4.2 Proposition. *S is Q-simplicial if Q is any field of characteristic zero.*

Proof. It will suffice to find a cofinal sequence $\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots$ in Cov (S) which satisfies

1. For $m = nq$, $\alpha_m > \alpha_n$ and the family $\{\alpha_m : m = nq, q = 1, 2, \cdots\}$ is cofinal in Cov (S).

2. If $m = nq$, there is a chain map.

$$\omega : C_*(S_{\alpha_n}) \to C_*(S_{\alpha_m})$$

such that ω is subordinate to α_n.

For $n = 1, 2, \cdots$, we let

$$\alpha_n = \{A(k, n) : k = 1, 2, \cdots, n\}$$

where

$$A(k, n) = p_n^{-1}(\{[t] : (k-1)/n < t < (k+1)/n\})$$

for $k = 1, 2, \cdots, n$. For $m = nq$, then $p_n = p_n^m p_m$ and for $[t] \in T$,

$$(p_n^m)^{-1}[t] = \{[t + in/m] : i = 1, \cdots, q\}$$

Thus, for each $x \in A(k, n)$, and for $i = 1, \cdots, q$, there is a unique $x^{(i)} \in A(k + in, m)$ such that $p_m(x^{(i)}) = p_n(x)$.

We define

$$\omega : C_*(S_{\alpha_n}) \to C_*(S_{\alpha_m})$$

by letting, for $\sigma = (x_0, \cdots, x_n) \in C_*(S_{\alpha_n})$,

$$\omega(\sigma) = (1/q) \textstyle\sum_{i=1}^q (x_0^{(i)}, x_1^{(i)}, \cdots x_n^{(i)})$$

If $\sup(\sigma) \subset A_n^k$, then $\sup \omega(\sigma) \subset A_n^k$, and so σ is subordinate to α_n.

4.3 THEOREM. *The solenoid S has the following properties*:

4.3.1. *For any rational number r, there is a function $f : S \to S$ such that $\Lambda_Q(f) = r$, when Q is the field of rationals.*

4.3.2. *If the characteristic of Q is non-zero, then there is a function $f : S \to S$ such that $\Lambda_Q(f) = 1$, but such that $f(x) \neq x$ for all $x \in S$.*

4.3.3. *S is Q-simplicial if and only if the characteristic of Q is zero.*

Proof of 4.3.1 (See [15, exercise f, p. 296]). Let Q be the field of rationals. Then $H_0(S; Q) = Q = H_1(S; Q)$ and $H_n(S; Q) = 0$ for $n \neq 0, 1$. For any rational number p/q, p and q integers, and any $a \in S$, the equation

$$px = qa$$

has a unique solution $x_a = x$, and the function $f(a) = x_a$ is a continuous function such that the induced homomorphism

$$(f_*)_1 : H_1(S; Q) \to H_1(S; Q)$$

satisfies

$$(f_*)_1(Z) = (p/q)Z, \quad Z \in H_1(S; Q).$$

Thus $\Lambda_Q(f) = 1 - p/q$, so if r is a preassigned rational number choose p and q so that $r = 1 - p/q$.

Proof of 4.3.2. If Q is a field of characteristic $q \neq 0$, note that if any $n \in N$, and $m = qn$, then

$$0 = ((p_n^m)_*)_1 : H_1(T; Q) \to H_1(T; Q),$$

so that $((p_n)_*)_1 = 0$ for all $n \in N$. Thus

$$H_1(S; Q) = 0,$$

and for any map $f : S \to S$, $\Lambda_Q(f) = 1$. Choose f to be a translation to obtain that $f(x) \neq x$ for $x \in S$.

Proof of 4.3.3. This is immediate from 4.3.1 and 4.3.2.

Bibliography

1. T. R. Brahna, *Products of quasi-complexes*, Proc. Amer. Math. Soc., vol. 7 (1956), pp. 954–958.
2. C. H. Dowker, *Homology groups of relations*, Ann. of Math. (2), vol. 56 (1952), pp. 84–95.
3. E. Dyer, *A fixed point theorem*, Proc. Amer. Math. Soc., vol. 7 (1956), pp. 662–672.
4. S. Eilenberg and J. A. Zilber, *On products of complexes*, Amer. J. Math., vol. 75 (1953), pp. 200–204.
5. S. Lefshetz, *Algebraic topology*, New York, 1942.
6. J. Leray, *Sur la forme des espaces topologiques et sur les points fixes des representations*, J. Math. Pures Appl. (9), vol. 24 (1945), pp. 95–167.
7. ———, *Theorie des points fixes: indice total et nombre de Lefschetz*, Bull. Soc. Math. France, vol. 87 (1959), pp. 221–233.
8. Victor Klee, *Leray-Schauder theory without local convexity*, Math. Ann., vol. 141 (1960), pp. 286–296.
9. R. Knill, *Cones, products and fixed points*, Fund. Math., vol. 60 (1967), pp. 35–46.
10. Ernest Michael, *Some extension theorems for continuous functions*, Pacific J. Math., vol. 3 (1953), pp. 789–806.
11. J. Segal, *A fixed point theorem for the hyperspace of a snakelike continuum*, Fund. Math., vol. 50 (1961-62), pp. 237–248.
12. E. Spanier, *Algebraic topology*, New York, 1966.
13. R. Thompson, *On the semicomplexes of F. Browder*, Bull. Amer. Math. Soc., vol. 73 (1967), p. 532.
14. R. Wilder, *Some mapping theorems with applications to non-locally connected spaces*, Algebraic geometry and topology, A symposium in honor of S. Lefschetz, Princeton University Press, Princeton, 1957.
15. S. Eilenberg and N. Steenrod, *Foundations of algebraic topology*, Princeton University Press, Princeton, 1952.

Tulane University
New Orleans, Louisiana