SOME PROPERTIES OF LATTICES IN A LIE GROUP

BY

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1. Introduction

Let G be a connected Lie group, Γ a discrete subgroup and G/Γ be the space of left cosets. Given any right Haar measure μ over G, μ induces a measure $\bar{\mu}$ over G/Γ. If $\bar{\mu}(G/\Gamma)$ is finite, Γ is called a *lattice.* If G/Γ is compact, Γ certainly being a lattice is called a c-lattice. Let $\mathcal{S}(G)$ be the set of all lattices of G. We give $\mathcal{S}(G)$ a topology induced from the notion of *limit of lattices* introduced by Chabauty in [2]. We denote $A(G)$ to be the group of all open continuous automorphisms of G with the compact open topology. It is clear that $A(G)$ operates continuously on $\mathcal{S}(G)$. In [2], Chabauty conjectured that given any lattice Γ of G, $A(G)\Gamma$ with the induced topology from $\mathcal{S}(G)$ is homeomorphic to the homogeneous space $A(G)/N(\Gamma)$, where

$$N(\Gamma) = \{\alpha : \alpha \in A(G), \alpha(\Gamma) = \Gamma\}.$$

In [11], the author proved that if Γ is a finitely generated lattice of G such that the restriction map[2]

$$H^1(G, \hat{G}) \xrightarrow{\text{res}} H^1(\Gamma, \hat{G})$$

is surjective, then $A(G)\Gamma$ is homeomorphic to $A(G)/N(\Gamma)$. Here we shall study this conjecture in linear Lie groups. We shall establish the following.

THEOREM A. *Let Γ be a finitely generated lattice of a linear Lie group which is semi-simple without compact factor. If the set* $\operatorname{tr}(\Gamma) = \{\operatorname{trace}(\gamma) : \gamma \in \Gamma\}$ *is discrete, then $A(G)\Gamma$ is homeomorphic to $A(G)/N(\Gamma)$.*

Let μ be a fixed right Haar measure. There is a map $v : \mathcal{S}(G) \to \mathbf{R}$, defined by $v(\Gamma) = \bar{\mu}(G/\Gamma)$. In general v is not continuous. For an example, see [6]. However the following is true.

THEOREM B. *Let G be a connected semi-simple Lie group without compact factor and Γ_0 a lattice of G. If $\mathcal{W}$ is a subset of $\mathcal{S}(G)$ containing Γ_0 such that the restriction of v on $\mathcal{W}$ is continuous at Γ_0, then there exists a neighborhood $\mathcal{V}$ of Γ_0 in $\mathcal{W}$ and a positive integer n such that each $\Gamma \in \mathcal{V}$ is contained in at most n discrete subgroups of G.*

2. Some density properties

Suppose G is a topological group. A subgroup H of G will be said to have Selberg property (or property (S)) if for any neighborhood U of e in

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[2] $\hat{G}$ denotes the Lie algebra of G and the action of G on $\hat{G}$ is given by the adjoint representation.

G and any element x in G, there exists an integer $n > 0$ with $x^n \in UHU$. As shown by Selberg [9], every lattice has property (S). We show the converse for nilpotent groups.

PROPOSITION 2.1. *Let G be a connected nilpotent Lie group and H a subgroup of G with property* (S). *Then $G/\bar{H}$ is compact.*

Proof. Without loss of generality, we may assume that G is simply connected. First consider the case that $G = \mathbf{R}^r$. Let $\bar{H} = \mathbf{R}^s \oplus \mathbf{Z}^t$. Suppose that $\mathbf{R}^r/\bar{H}$ is not compact. Then $\mathbf{R}^r = A \oplus B$ where $\bar{H} \subset A \approx \mathbf{R}^{s+t}$, $B \approx \mathbf{R}^l$, $l > 0$. Let $\pi : \mathbf{R}^r \to \mathbf{R}^r/A$ be the natural projection map. From [1], we know that $\pi(H)$ has property (S) in $\mathbf{R}^r/A$. But $\pi(H) = \{0\}$ and $\mathbf{R}^r/A \approx \mathbf{R}^l$, $l > 0$ which is absurd. We now return to the general case. Let $G_2 = [G, G]$ and $\lambda : G \to G/G_2$ be the natural projection map. It is clear that $G/G_2 \approx \mathbf{R}^r$, for some r and $\lambda(H)$ has property (S). By what we have just proved above, $\lambda(G)/\overline{\lambda(H)}$ is compact. Hence $G/\overline{G_2 H}$ is compact. By a theorem of Malcev [7], we obtain that $G/\bar{H}$ is compact.

COROLLARY 2.2. *Let G be a connected nilpotent Lie group and Γ a discrete subgroup. Then Γ is a lattice iff Γ has property* (S).

In [11], we see that Chabauty's conjecture is always true for nilpotent Lie groups. For solvable groups, we have the following.

THEOREM 2.3. *Let G be a connected solvable Lie group and Γ a lattice such that $\mathfrak{A}(\mathrm{Ad}\ \Gamma)$, the Zariski closure in the ambient real linear group, contains* $\mathrm{Ad}\,(G)$. *Then $A(G)\Gamma$ is homeomorphic to $A(G)/N(\Gamma)$.*

Proof. G/Γ is compact; in particular Γ is finitely generated. Furthermore by [8, Theorem 8.1],

$$H^1(G, \hat{G}) \xrightarrow{\text{res}} H^1(\Gamma, G)$$

is an isomorphism. Hence from [11], $A(G)\Gamma$ is homeomorphic to $A(G)/N(\Gamma)$

3.

Groups considered in this section are linear Lie groups. Given any subset T of $GL(m, \mathbf{R})$, we denote $\mathrm{tr}\,(T) = \{\mathrm{trace}\,(t) : t \in T\}$ and $l(T)$ be the linear span in $M_m(\mathbf{R})$.

LEMMA 3.1. *Let Γ be a connected semi-simple Lie group and H a finitely generated subgroup of G such that $\mathfrak{A}(H) \supset G$. If $r_n : H \to G$ is a sequence of trace-preserving homomorphisms with $r_n \to 1_H$, then there exists $\alpha_n \in A(G)$ such that $\alpha_n \to 1_G$ and $r_n = \alpha_r|_H$ for large n.*[3]

Proof. Since $l(H) \supset \mathfrak{A}(H) \supset G$, $l(H) = l(G)$. From $\lim_n r_n = 1_H$, it follows that there exists an integer $n_0 > 0$ such that $l(r_n(H)) = l(G)$ for $n > n_0$. In the sequel, n is assumed to be $> n_0$. Define $\beta_n : l(G) \to l(G)$

[3] The argument used in the proof essentially follows [9, Lemma 4].

by

$$\beta_n(\sum_{i=1}^q s_i h_i) = \sum_{i=1}^q s_i r_n(h_i), \quad \text{for } s_i \in \mathbf{R},\ h_i \in H,\ 1 \leq i \leq q.$$

We have to verify that β_n is well defined. Let $B : l(G) \times l(G) \to \mathbf{R}$ be defined by $B(x, y) = \text{trace } xy$, $x, y \in L(G)$. Since G is semi-simple, $l(G)$ is a semi-simple associative algebra and B is a nonsingular bilinear form. Let $\sum_{i=1}^q s_i h_i = 0$. For any $\sum_{j=1}^v t_j k_j \in l(G)$, $t_j \in \mathbf{R}$, $k_j \in H$, $1 \leq j \leq v$, $(\sum_{i=1}^q s_i h_i)(\sum_{j=1}^v t_j k_j) = 0$. As r_n preserves trace,

$$B(\sum_{i=1}^q s_i r_n(h_i), \sum_{j=1}^v t_j r_n(k_j)) = 0.$$

However $l(r_n(H)) = l(G)$, and B is nonsingular, we have that $\sum_{i=1}^q s_i r_n(h_i) = 0$. Thus β_n is well defined. It is obvious that β_n is an algebra homomorphism. Since β_n is surjective and $l(G)$ is of finite dimension, β_n is an isomorphism.

$$\mathfrak{A}(r_n(H)) = \mathfrak{A}(\beta_n(H)) = \beta_n(\mathfrak{A}(H)) \supset \beta_n(G).$$

Hence $\mathfrak{A}(G) \supset \beta_n(G)$. But $\dim G = \dim \beta_n(G)$ and $\mathfrak{A}(G)^0 = G$ where $\mathfrak{A}(G)^0$ is the topological connected component of e in $\mathfrak{A}(G)$, so this yields $\beta_n(G) = G$. Clearly $\beta_n \to 1_{l(G)}$. Set $\alpha_n = \beta_n|_G$; the proof of the lemma is thus completed.

We now prove Theorem A. Let $\{\alpha_n(\Gamma)\}$ be a sequence of lattices of G converging to Γ in $\mathcal{S}(G)$. From [11], we know that there exists $r_n : \Gamma \to \alpha_n(\Gamma)$ a homomorphism, such that $r_n \to 1_\Gamma$. Suppose for the moment that r_n preserves trace for large n. Then by the preceding lemma, there is an $\alpha_n' \in A(G)$ such that $r_n = \alpha_n'|_\Gamma$.

$$\alpha_n'(\Gamma) \subset \alpha_n(\Gamma) \quad \text{and} \quad \bar{\mu}(G/\alpha_n'(\Gamma)) = \bar{\mu}(G/\Gamma) = \bar{\mu}(G/\alpha_n(\Gamma)).$$

It follows that $\alpha_n'(\Gamma) = \alpha_n(\Gamma)$. By the same lemma, $\alpha_n' \to 1_G$. This shows that the map

$$\phi : A(G)\Gamma \to A(G)/N(\Gamma), \qquad \phi(\alpha(\Gamma)) = \alpha N(\Gamma)$$

is continuous at Γ. By action of $A(G)$, ϕ is continuous. It is clear that ϕ^{-1} is always continuous. Hence $A(G)\Gamma \approx A(G)/N(\Gamma)$. Thus in order to complete the proof, it remains to show that r_n preserves trace for large n. Let $\gamma_1, \cdots, \gamma_{2q}$ be a set of generators of Γ with $\gamma_{2i} = \gamma_{2i-1}^{-1}$, $1 \leq i \leq q$. Let ω be a word on $2q$ elements $w_1, \cdots, w_{2q}$. Given any $\omega = w_{i_1} \cdots w_{i_a}$, we define $\omega(\gamma) = \gamma_{i_1} \cdots \gamma_{i_a}$ and $W : M_m(\mathbf{R})^{2q} \to M_m(\mathbf{R})$ by

$$W(X) = X_{i_1} \cdots X_{i_a}, \quad \text{where } X = (X_1, \cdots, X_{2q}) \in M_m(\mathbf{R})^{2q}.$$

Since the polynomial rings with coefficients in a field are Noetherian, there are finitely many words $\omega_1, \cdots, \omega_b$ such that $\operatorname{tr} W(X) - \operatorname{tr} \omega_i(\gamma) = 0$, $1 \leq i \leq b$ implies $\operatorname{tr} W(X) - \operatorname{tr} \omega(\gamma) = 0$ for all word ω. But $\operatorname{tr}(\Gamma) = \operatorname{tr}(\alpha_n(\Gamma))$ is discrete and $r_n \to 1_\Gamma$. It follows that $\operatorname{tr} W_i(r_n(\gamma)) - \operatorname{tr} \omega_i(\gamma) = 0$, $1 \leq i \leq b$ holds for large n where $r_n(\gamma) = (r_n(\gamma_1), \cdots, r_n(\gamma_{2q}))$. Therefore r_n preserves trace for large n.

Corollary 3.2. *Let* Γ *be as in Theorem* A *and* $\Gamma_1 \in \mathcal{S}(G)$ *with* $\Gamma_1 \supset \Gamma$. *Then* $A(G)\Gamma_1 \approx A(G)/N(\Gamma_1)$.

Proof. Let $\{\alpha_n(\Gamma_1)\}$ be a sequence of lattices converging to Γ_1. Let $r_n : \Gamma_1 \to \alpha_n(\Gamma_1)$ be the sequence of homomorphisms constructed in [11]. Clearly $[\Gamma_1 : r_n^{-1}(\alpha_n(\Gamma))]$ is bounded. Since Γ_1 is finitely generated, Γ_1 has only finitely many subgroups with bounded index. Hence there is a subgroup Γ_0 of Γ with finite index and $r_n(\Gamma_0) \subset \alpha_n(\Gamma)$ for all n. Since tr (Γ) is discrete, as shown above that $r_n|_{\Gamma_0}$ preserves trace for large n. By Lemma 3.1, there exists $\beta_n \in A(G)$ with $\beta_n|_{\Gamma_0} = t_n|_{\Gamma_0}$ for large n and $\beta_n \to 1_G$. Then

$$\beta_n^{-1} \circ r_n \to 1_{\Gamma_1} \quad \text{and} \quad \beta_n^{-1} \circ r_n(\Gamma_1) \supset \Gamma_0 .$$

By [10], there are only finitely many lattices containing Γ_0. Therefore $\beta_n^{-1} \circ r_n = 1_{\Gamma_1}$ for large n. It follows that $\beta_n(\Gamma_1) = \alpha_n(\Gamma_1)$ for large n. Same argument as used in the above proof leads to the conclusion

$$A(G)\Gamma_1 \approx A(G)/N(\Gamma_1).$$

4.

In this section, we shall give a proof of Theorem B which essentially follows that given in [10] with some modification. Suppose the theorem is false. Then there is a sequence $\{\Gamma_n\}$ of lattices in $\mathcal{W}$ such that $\Gamma_n \to \Gamma_0$ and the cardinal number of the set $a(\Gamma_n) = \{\Gamma : \Gamma \in \mathcal{S}(G),\ \Gamma \supset \Gamma_n\}$ increases with n unboundedly. Since $\lim_n \Gamma_n = \Gamma_0$, there is a compact subset K of G such that $\Gamma_n \cap K$ generates a subgroup of type (P) [10]. By the main lemma in [10], $\bigcup_{n=1}^{\infty} a(\Gamma_n)$ is uniformly discrete.[4] Hence $[\Gamma:\Gamma_n]$ is bounded for all n and all $\Gamma \in a(\Gamma_n)$. Thus we may assume that the cardinal number of the set

$$a(\Gamma_n:l) = \{\Gamma:\Gamma \in \mathcal{S}(G),\ [\Gamma:\Gamma_n] = l\}$$

is not bounded for certain fixed positive integer l. Since $\mathcal{S}(G)$ is separable metric [11] and Γ_0 is contained in only finitely many lattices of G [10], there exists a subsequence $\{i_n\}$ of $\{n\}$ with $b(\Gamma_{i_n}:l) \subset a(\Gamma_{i_n}:l)$ and there is $\Gamma' \in \mathcal{S}(G)$ such that

(1) the cardinal number of $b(\Gamma_{i_n};l) = 2^n$,
(2) $\Gamma' \supset \Gamma_0$,
(3) $d(\Gamma, \Gamma') < 1/n$ for $\Gamma \in b(\Gamma_{i_n};l)$ where d is a fixed metric which induces the topology of $\mathcal{S}(G)$.

Let $\{H_n\}$ be the sequence of lattices of G defined by

$$\{H_{2^n+1}, \cdots, H_{2^{n+1}}\} = b(\Gamma_{i_n};l).$$

It is clear that $\lim H_n = \Gamma'$. Since $\bar{\mu}(G/\Gamma_{i_n}) \to \bar{\mu}(G/\Gamma_0)$, by the assumption,

[4] A set $\mathcal{S}$ of discrete subgroups is uniformly discrete if there is a neighborhood V of e such that $V \cap \Gamma = \{e\}$ for all $\Gamma \in \mathcal{S}$.

we have

$$\bar{\mu}(G/H_n) \to (1/l)\bar{\mu}(G/\Gamma_0).$$

From [2], we know that $\liminf \bar{\mu}(G/H_n) \geq \bar{\mu}(G/\Gamma')$. Therefore $\infty > [\Gamma':\Gamma_0] \geq l$. Let $\alpha_1, \cdots, \alpha_k$ $(k \geq l)$ be a set of representatives of Γ'/Γ_0 and $\beta_1, \cdots, \beta_m$ in Γ_0 such that $\alpha_i^{-1}\beta_j\alpha_i \in \Gamma_0$, $1 \leq i \leq k$, $1 \leq j \leq m$ and $\{\beta_1, \cdots, \beta_m\}$ generates a subgroup of type (P). Since $H_n \to \Gamma'$, there exists $\alpha_i(n), \beta_j(n) \in H_n$ such that $\alpha_i(n) \to \alpha_i$, $\beta_j(n) \to \beta_j$. As $\bigcup_{n=1}^{\infty} a(\Gamma_n)$ is uniformly discrete, $\alpha_i(n)$, $\beta_j(n)$ are uniquely determined for large n. Further we have

(a) $\alpha_i(n)^{-1}\beta_j(n)\alpha_i(n) \to \alpha_i^{-1}\beta_j\alpha_i$,

(b) $\beta_j(2^n + 1) = \cdots = \beta_j(2^{n+1})$,

(c) $\alpha_i^{-1}(m)\beta_j(m)\alpha_i(m) \in \Gamma_{i_n}$, $2^n + 1 \leq m \leq 2^{n+1}$,

(d) the subgroup $B(n)$ generated by $\{\beta_1(n), \cdots, \beta_m(n)\}$ is of type (P), where n is sufficiently large.

From (a) through (c), we see that $\alpha_i(s)\alpha_i(t)^{-1}$, $2^n + 1 \leq s, t \leq 2^{n+1}$, belongs to the normalizer $N(B(n))$ of $B(n)$ in G for large n. Again by the main lemma in [10], and the fact that normalizer of subgroup of type (P) is discrete [10], $\bigcup_{n=1}^{\infty} N(B(n))$ is uniformly discrete. It follows that $\alpha_i(s) = \alpha_i(t)$, $2^n + 1 \leq s, t \leq 2^{n+1}$ for large n. Since $\Gamma_{i_n} \to \Gamma_0$, $\alpha_i(n) \to \alpha_i$, $1 \leq i \leq k$ and $\alpha_i^{-1}\alpha_j \notin \Gamma_0$, we must have that $\alpha_i(m)^{-1}\alpha_j(m) \notin \Gamma_{i_n}$, $2^n + 1 \leq m \leq 2^{n+1}$ for large n. Since $k \geq l$, and $[H_m:\Gamma_{i_n}] = l$, $H_{2^n+1} = \cdots = H_{2^{n+1}}$ holds for large n which contradicts our choice of $\{H_n\}$. Thus the proof is completed.

REFERENCES

1. A. BOREL, *Density properties for certain subgroups of semi-simple groups without compact factor*, Ann. of Math. (2), vol. 72, 1960, pp. 179–188.
2. C. CHABAUTY, *Limite d'ensemble et geometrie des nombres*, Bull. Soc. Math. France, vol. 78 (1950), pp. 143–151.
3. C. CHEVALLEY, *Theory of Lie groups I*, Princeton University Press, Princeton, 1946.
4. H. GARLAND AND M. GOTO, *Lattices and the adjoint group of a Lie group*, vol. 124 (1966), pp. 450–460.
5. S. HELGASON, *Differential geometry and symmetric space*, Academic Press, N. Y., 1962.
6. A. M. MACBEATH AND S. SWIERCZKOWSKI, *Lattices in a compactly generated group*, Canad. J. Math., vol. 7 (1960), pp. 427–438.
7. A. I. MALCEV, *On a class of homogeneous spaces*, Amer. Math. Soc. Transl. no. 9 (1962), pp. 276–307.
8. G. D. MOSTOW, *Cohomology of topological groups and solvmanifolds*, Ann. of Math., vol. 73 (1961), pp. 20–48.
9. A. SELBERG, *Discontinuous groups in symmetric spaces, contributions to function theory*, Tata Institute of Fundamental Research, pp. 147–164.
10. H. C. WANG, *On a maximality property of discrete subgroups with fundamental domain of finite measure*, Amer. J. Math., vol. 89 (1967), pp. 124–132.
11. S. P. WANG, *Limit of lattices in a Lie group*, Trans. Amer. Math. Soc., vol. 133 (1968), pp. 519–526
12. A. WEIL, *Remark on the cohomology of groups*, Ann. of Math., vol. 80 (1964), pp. 149–158.

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