THE FIRST OBSTRUCTION TO EMBEDDING A 1-COMPLEX IN A 2-MANIFOLD

BY

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1. Introduction

Shapiro [4] defines the first obstruction $m^n(X)$ to embedding a simplicial complex X into Euclidean n-space, taking values in $H^n(R^*X; G)$, where R^*X is the reduced deleted product of X and where $G = Z$ if n is even, and where G is the locally trivial integer sheaf, twisted by the fundamental class of the covering $RX \to R^*X$, where RX is the deleted product of X, if n is odd. This obstruction is called $\varphi^m(X)$ by Wu [6], [7], [8], and was first known to Van Kampen [5]. The vanishing of this obstruction is a necessary condition for the existence of a piecewise linear embedding of X into R^m [7]. Wu has shown that this condition is also sufficient if $m \geq 2 \dim X$ [8], while Shapiro has proved an even stronger result [4].

In the present paper we define the obstruction to embedding a finite one-dimensional complex into a two-dimensional connected oriented manifold without boundary. (The case of a non-oriented manifold is only slightly more complicated actually; the sheaf $G^*[K, f]$ has an additional twisting on it, and everything else goes through.) If $f : X \to M$ is any map, where (X, A) is a 1-dimensional simplicial pair, and if $f \mid A$ is a differentiable embedding, we define an obstruction to homotoping f, rel A, to an embedding; $\gamma(f) \in I^2(X, A; f)$. The graded functor I^* is defined in (3.5). Since $\gamma(f)$ depends only on the homotopy class, rel A, of f (cf. 3.5.1), and $\gamma(f) = 0$ if f is already an embedding (cf. 3.1.1), we have immediately that $\gamma(f) = 0$ is a necessary condition for f to be homotoped, rel A, to an embedding. This is unfortunately not sufficient, however, as we see in example (6.3).

In Section 5, we show how to compute cohomology with coefficients in sheaves over simplicial complexes by taking cochains and coboundaries; this technique is used when defining the obstruction cocycle (cf. 3.1.1).

In a later paper, we shall define the first obstruction $\gamma(f) \in I^n(X, A; f)$ to finding an embedding g, homotopic rel A to f, if f is a map from a simplicial complex X to an n-dimensional manifold M which is already an embedding on a subcomplex A. The vanishing of this obstruction will always be necessary for the existence of g, and will also be sufficient if $\dim X \leq n - 3$ and $\dim X + \dim (X - A) \leq n$. If $\dim X + \dim (X - A) > n$, there will presumably be higher obstructions; but these will certainly not help solve the special case of a 1-complex in a 2-manifold.

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2. The sheaf $G^*[K, f]$

2.1. Throughout this section, we let K be a 1-dimensional simplicial complex, $|K|$ the underlying space, M a connected oriented 2-dimensional manifold without boundary, and $f : |K| \to M$ a continuous function. We also assume that the intersection of the closed stars of any two distinct vertices of K is contractible.

DEFINITION (2.1.1). If S is any set, let $Z[S]$ be the Abelian group of all formal sums in S. If $y_0, y_1 \in M$, let $\pi_1(M, y_0, y_1)$ be all homotopy classes of paths from y_0 to y_1. Then $Z[\pi_1(M, y_0, y_1)]$ is a left $Z[\pi_1(M, y_0)]$-module and a right $Z[\pi_1(M, y_1)]$-module, free on one generator in both cases.

DEFINITION (2.1.2). Let $F[f]$ be the sheaf over $|K|^2$ whose stalk over every (x_0, x_1) is $Z[\Pi_1(M, fx_0, fx_1)]$. If v_0 and v_1 are any vertices of K, let St (v_0) and St (v_1) be the open stars. If σ is any path from fv_0 to fv_1, let

$$h^\sigma : \mathrm{St}\ (v_0) \times \mathrm{St}\ (v_1) \to F[f]$$

be the cross section which sends each (x_0, x_1) to $[a_0 \sigma a_1^{-1}]$, where a_i is a path through St (v_i) from x_i to v_i, for $i = 0$ or 1. Let $F[f]$ have that topology such that each h^σ is a continuous cross-section.

2.2. If $a \in \pi_1(M, y_0, y_1)$ is represented by a path σ from y_0 to y_1, let $a^{-1} \in \pi_1(M, y_1, y_0)$ be represented by the path σ^{-1}. We can find an involution T of the sheaf $F[f]$ consistent with the interchange of coordinates $T : X^2 \to X^2$ as follows: $T(\sum n_i a_i) = \sum n_i a_i^{-1}$.

DEFINITION (2.2.1). Let $F^*[f]$ be that sheaf of Abelian groups over $R^*|K|$ obtained from $F[f]|RX$ by identifying each element with its image under T.

2.3. Let NK be that subcomplex of K^2 consisting of all $\sigma \times \tau$ such that σ and τ are non-disjoint cells of K. For any $y \in M$ let $1_y \in \pi_1(M, y, y)$ be represented by the trivial path: let S be that unique subsheaf of $F[f]|\,|NK|$ which is locally an integer sheaf and whose stalk over (x, x) is $Z[1_{fx}]$ for every $x \in |K|$. S exists by the contractability of intersections of closed stars property (cf. 2.1).

DEFINITION (2.3.1). Let $G[K, f]$ be that subsheaf of $F[f]$ such that

$$G[K, f]|\,|NK| = S \quad \text{and} \quad G[K, f]|(|K|^2 - |NK|) = F[f]|(|K|^2 - NK).$$

Note that $G[K, f]$ is not a locally trivial sheaf unless M is simply connected.

DEFINITION (2.3.2). Let $G^*[K, f]$ be that sheaf over $R^*|K|$ obtained from $G[K, f]|R|K|$ by identification via T; $G^*[K, f] \subset F^*[f]$.

3. The obstruction cocycle

3.1. Assume that M is a connected oriented 2-dimensional manifold without boundary, that (X, A) is a 1-dimensional simplicial pair, and that

$f : |X| \to M$ is a continuous function which is a differentiable embedding on A [3]. We want to know whether f can be homotoped to an embedding on all of $|X|$, holding $f|A$ fixed.

Modify f slightly, and subdivide (X, A) into (K, L), such that f is differentiable, and K has the property that the intersections of closed stars are contractible; and such that whenever x_1 and x_2 are distinct points of $|X|$ which both map to a point $z \in M$ under f, x_1 and x_2 lie in the interiors of disjoint 1-cells σ_1 and σ_2 of K, and $f(\sigma_1)$ meets $f(\sigma_2)$ transversely at z.

Definition (3.1.1). We define the obstruction class

$$c(f) \in C^2(J^*K, J^*L; G^*[K, f] | J^*K)$$

as follows: For any $x_1 \neq x_2$ in $|X|$ such that $f(x_1) = f(x_2) = z$, let σ_1 and σ_2 be the 1-cells which contain x_1 and x_2 respectively. The orientations of σ_1 and σ_2, in that order, determine, via f restricted to small neighborhoods of x_1 and x_2 respectively, a local orientation of M at z. Let $\varepsilon[x_1, x_2] = +1$ or -1 depending on whether this orientation agrees with the preassigned orientation of M or not. Let

$$a[x_1, x_2] \in C^2(JK, JL; G[K, f] \mid JK)$$

be that cochain which takes on the value

$$\varepsilon[x_1, x_2]1_z \in Z[\pi_1(M, z, z)] \quad \text{on } \sigma_1 \times \sigma_2$$

and

$$-\varepsilon[x_1, x_2]1_z = \varepsilon[x_2, x_1]1_z \quad \text{on } \sigma_2 \times \sigma_1 .$$

Since $T_* a = aT$, we have that $a[x_1, x_2] = \Pi^* a^*[x_1, x_2]$ for a unique choice of a^*, where

$$\Pi^* : C^2(J^*K, J^*L; G^*[K, f] \mid J^*K) \to C^2(JK, JL; G[K, f] \mid JK)$$

is the monomorphism induced by the identification $\Pi : R|X| \to R^*|X|$; and also that $a^*[x_2, x_1] = a^*[x_1, x_2]$. Let $c(f) = \sum a^*[x_1, x_2]$, sum over all unordered double points $[x_1, x_2]$.

Definition (3.1.2). Let $\gamma(f) \in H^2(J^*K, J^*L; G^*[K, f] \mid J^*K)$ be the cohomology class of $c(f)$. See (5.1).

3.2. We shall establish that $\gamma(f)$ is independent of the choice of subdivision of (X, A) and depends only on the homotopy class, rel A, of f.

Definition (3.2.1). If $F : |K| \times I \to M$ is a homotopy connecting f with a map f_1, we describe a function

$$F_{\#} : \pi_1(M, fx_0, fx_1) \to \pi_1(M, f_1 x_0, f_1 x_1) \qquad \text{for all } x_0, x_1 \in |K|;$$

if σ is a path from x_0 to x_1, $F_{\#}[\sigma] = [a_0^{-1}\sigma a_1]$, where a_i is the path from fx_i to $f_1 x_i$ given by $a_i(t) = F(x_i, t)$. We extend $F_{\#}$ to a sheaf homomorphism $F_{\#} : G^*[K, f] \to G^*[K, f_1]$ which is obviously an isomorphism.

3.3. Let $\{f_t\} = F : |K| \times I \to M$ be a homotopy of f. We say that F is a general homotopy if the following hold.

(3.3.1) F is a differentiable map from $K \times I$ to M.

(3.3.2) If $x_1 \neq x_2 \in |K|$ and $t \in I$ and if $f_t(x_1) = f_t(x_2) = z$, then one of the following two conditions holds:

(i) K has disjoint 1-cells σ_1 and σ_2 whose interiors contain x_1 and x_2 respectively, and $f_t(U_1)$ meets $f_t(U_2)$ transversely at z, for small neighborhoods U_i of x_i.

(ii) K has as a 1-cell σ and a vertex v outside σ such that $x_1 \in \operatorname{Int} \sigma$ and $x_2 = v$ (or the other way around), and a small neighborhood of (x_1, t) in $\sigma \times I$ meets a small neighborhood of (x_2, t) in $v \times I$ transversely at (z, t) under the map $F' : |K| \times I \to M \times R$, where $F'(x, u) = (F(x, u), u)$ for all $x \in |K|$ and $u \in I$; furthermore, $0 < t < 1$.

PROPOSITION (3.3.3). *If $F = \{f_t\}$ is a general homotopy of f which holds L fixed, then $F^* \gamma(f) = \gamma(f_1)$, where F^* is the isomorphism of cohomology induced by the sheaf isomorphism $F_{\#}$.*

Proof. It is sufficient to consider the case that there is only one $t \in (0, 1)$ such that f_t has a double point of type (3.3.2, ii), and that f_t has only one such double point, namely, $x_1 \neq x_2$, $x_1 \in \operatorname{Int} \sigma$, $x_2 = v$, $f_t(x_1) = f_t(x_2) = z$. This situation can be illustrated by the following picture:

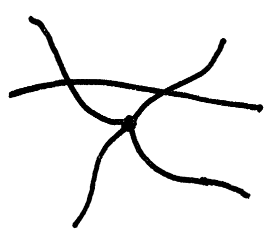

FIGURE 1. Before t

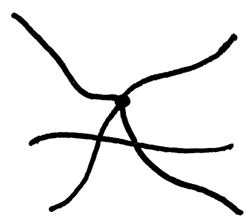

FIGURE 2. After t

Now, since F holds L fixed, either σ is not a cell of L, or v and all cells of which v is a face are not in L. Consider the map $F' : |K| \times I \to M \times R$ given in (3.3.2, ii). Let e_1 and e_2 be independent tangent vectors of M at z such that (e_1, e_2) determines the given orientation of M; let $e_3 = \partial/\partial t$ be a tangent vector of $M \times R$ at (z, t) which is perpendicular to $M \times \{t\}$ and which points in the direction of increasing t. Let s_2 be a tangent vector of σ at x_1 which orients σ, and let $s_3 = \partial/\partial t$ be the vector on $\sigma \times I$ at (x_1, t) which points in the direction of increasing t. Finally, let $s_1 = \partial/\partial t$ be the tangent of $v \times I$ at (x_2, t). We let $\varepsilon = +1$ or -1 depending on whether the orientation of $M \times R$ determined by $(F'_* s_1, (if_t)_* s_2, F'_* s_3)$ is the same as or different from that given by $(i_* e_1, i_* e_2, e_3)$, where $i : M \to M \times R$ sends each $y \in M$ to (y, t). Let $d \in C^1(JK, JL; G[K, f_t])$ be that cochain which takes on the value $\varepsilon 1_z$ on $\sigma \times v$ and $\varepsilon 1_z$ on $v \times \sigma$. Since $T_* d = dT$, $d = \Pi^* d^*$ for a unique $d^* \epsilon C^1(J^*K, J^*L; G^*[K, f_t])$; routine computation shows that

$$\delta d^* = (F^+_*)^{-1} c(f_1) - F^-_* c(f),$$

where F^- and F^+ are the portions of the homotopy F from f to f_t and from f_t to f_1, respectively. Thus $F_* \gamma(f) = \gamma(f_1)$.

3.4. Let (K', L') be any subdivision of (K, L). Then

$$(J^*K, J^*L) \subset (J^*K', J^*L'),$$

and $G^*[K, f]$ is a subsheaf of $G^*[K', f]$. We thus have two chain maps

$$i_* : C^*(J^*K, J^*L; G^*[K, f]) \to C^*(J^*K, J^*L; G^*[K', f])$$

and

$$i_* : C^*(J^*K', J^*L'; G^*[K', f]) \to C^*(J^*K, J^*L; G^*[K', f]).$$

If f satisfies the generality conditions described in (3.1) on the cell-structures of K', that is, all double points lie in interiors of products of 1-cells of K', then it is obvious that $i_* c(f) = i^* c'(f)$, where c' is computed using the cell structure of K'.

LEMMA (3.4.1). *The homology maps induced by the above chain maps i_* and i^* are both isomorphisms.*

Proof. It is sufficient to consider the case that K' is obtained from K by adding just one more vertex; namely, in the interior of a specific cell of K. Call this vertex $\sigma_\circ$; call the portions of σ above and below $\sigma_\circ$, σ^+ and σ^-, respectively. Call the upper and lower end-points of σ v^+ and v^-, respectively; then $\partial \sigma^+ = v^+ - \sigma_\circ$ and $\partial \sigma^- = \sigma_\circ - v^-$.

According to Shapiro [4], we can find a strong deformation retraction $\{g_t\}$ of $R^*|K|$ onto J^*K which is also a s.d.r. of $R^*|L|$ onto J^*L and which does not allow any point of $R^*|K| - J^*K$ to stray out of $N^*K =$ the image of $NK - \Delta|K|$ under the identification $\Pi : R|K| \to R^*|K|$. Restricting $\{g_t\}$ to J^*K', we get a homotopy of maps from J^*K' to J^*K' such that g_0 is the identity. Let

$$\{h_t : g_t^{-1} G^*[K, f] \to G^*[K, f]\}$$

be the unique sheaf homotopy over $\{g_t\}$ such that h_0 is the identity; then h_t is a sheaf isomorphism for all t. Let $i : J^*K \to J^*K'$ be the inclusion, and let i be the identity sheaf map over i. Then we have the diagram

$$H^*(J^*K', J^*L'; G[K, f]) \underset{(h_1)_* g_1^*}{\overset{i^*}{\rightleftarrows}} H^*(J^*K, J^*L; G[K, f]).$$

Now $i^*(h_1)_* g_1^*$ is the identity, since $g_1 | J^*K$ and $h_1 | J^*K$ are identities. But $(h_1)_* g_1^* i^*$ is the identity, since $\{h_t\}$ is a sheaf homotopy over $\{g_t\}$ and h_0 and g_0 are identities (cf. 4.1.3). Thus i^* is an isomorphism.

Now consider the exact sequence of sheaves over J^*K':

$$0 \to G^*[K, f] \to G^*[K', f] \to Q \to 0;$$

the quotient sheaf Q takes on non-zero values only on $N^*K - N^*K'$, and is locally trivial on that set. Now (N^*K, N^*K') is relatively homeomorphic to a disjoint union of ordered pairs of the form $(\tau \times \sigma^+, \tau \times \sigma^\circ \cup v^+ \times \sigma^+)$, where τ is some cell which has v^+ as an end-point (similarly with σ^- and v^-); it is at this point that we need the fact that intersections of closed stars in K are contractible. Now $\tau \times \sigma^\circ \cup v^+ \times \sigma^+$ is a strong deformation retract of $\tau \times \sigma^+$, so

$$H^*(N^*K, (N^*K \cap J^*L') \cup N^*K'; Q) = 0;$$

by the long exact sequence of cohomology induced by the exact sequence of sheaves, we have that

$$i_* : H^*(J^*K', J^*L'; G^*[K', f]) \to H^*(J^*K', J^*L'; G[K, f])$$

is an isomorphism and we are done.

3.5. We now can let $I^*(X, A; f)$ be defined to be the limit, over all sufficiently fine subdivisions (K, L) of (X, A), of $H^*(J^*K, J^*L; G^*[K, f])$. By (3.1) and (3.4), we can define the obstruction class $\gamma(f) \in I^2(X, A; f)$.

Remark (3.5.1). If $F : |X| \times I \to M$ is a homotopy of f with f_1 which holds $|A|$ fixed, F induces an isomorphism

$$F_* : I^*(X, A; f) \to I^*(X, A; f_1),$$

and, by (3.3.3), $F_* \gamma(f) = \gamma(f_1)$.

Remark (3.5.2). If (Y, B) is a simplicial pair, and $i : (Y, B) \to (X, A)$ is a one-to-one map, then i induces a homomorphism

$$i^* : I^*(X, A; f) \to I^*(Y, B; fi),$$

and $i^*\gamma(f) = \gamma(fi)$.

Remark (3.5.3). If M is a submanifold of another connected oriented 2-manifold M', then the inclusion $i : M \to M'$ induces a function

$$i_\# : \pi_1(M, y_0, y_1) \to \pi_1(M', y_0, y_1)$$

for all $y_0, y_1 \in M$, which induces a homomorphism

$$i_* : I^*(X, A; f) \to I^*(X, A; if);$$

and $i_* \gamma(f) = \gamma(if)$.

APPENDIX

4. Sheaf homotopies

4.1. Let X and Y be spaces, let A and B be sheaves of Abelian groups over X and Y respectively, and let $f : X \to Y$ be a map.

DEFINITION (4.1.1). If $h : f^{-1}B \to A$ is a homomorphism of sheaves over X, we say h is a sheaf map over f, from B to A. Then h induces a homomorphism of cohomology $h_\#$, defined to be the composition

$$H^*(Y;B) \xrightarrow{f^*} H^*(X; f^{-1}B) \xrightarrow{h_*} H^*(X;A).$$

DEFINITION (4.1.2). If $F : X \times I \to Y$ is a homotopy of f, we say a sheaf map

$$H : F^{-1}B \to A \times I$$

is a homotopy of h if $H((x, 0), b) = (h(x, b), 0)$ for all $(x, b) \in f^{-1}B$. If $F(x, 1) = f_1(x)$ for all $x \in X$, let $(h_1(x, b), 1) = H((x, 1), b)$ for all $(x, b) \in f_1^{-1}B$ Then F is a homotopy of f with f_1, and we say that H is a homotopy, over F, of h with h_1.

PROPOSITION (4.1.3). *Homotopic sheaf maps induce the same homomorphism in cohomology.*

Proof. Suppose that F is a homotopy of f with f_1, H is a homotopy, over F, of a sheaf map h with a sheaf map h_1. For any $t \in I$, let $i_t(x) = (x, t)$ for all $x \in X$. Now $i_t^{-1}(A \times I)$ is a sheaf over X, and there is a canonical isomorphism $i^t_\# : A \to i_t^{-1}(A \times I)$. Now we have the commutative diagram:

$$\begin{tikzcd}
H^*(X;A) & \\
H^*(X \times I; A \times I) \arrow[u, "i^0_\#"] \arrow[d, "i^1_\#"'] & H^*(Y;B). \arrow[l, "H_\#"'] \arrow[ul, "h_\#"'] \arrow[dl, "(h_1)_\#"] \\
H^*(X;A) &
\end{tikzcd}$$

According to Carl Bredon [1, page 203], $i^0_\# = i^1_\#$, and we are done.

5. Cohomology with coefficients in a sheaf as homology of a graded complex

Let (X, A) be a locally finite regular pair, and let G be a sheaf over X which is trivial over the interior of every cell. For any integer $k \geq 0$, let $C^k = C^k(X, A; G)$ be the set of all cochains c, such that c is a function which assigns to every k-cell σ of X a cross-section of G over $\operatorname{St}\sigma$, the open star of σ, which is zero if $\sigma \subset A$. We shall define a coboundary $\delta : C^k \to C^{k+1}$. Let each cell of X have a once-and-for-all orientation. If τ is any $(k+1)$-cell of X, and if $c \in C^k$ is a cochain which is nonzero only on σ, and if σ is a face of τ, let $\delta c(\tau) = \varepsilon(c(\sigma) \mid \mathrm{St}\,\tau)$ where $\varepsilon = \pm 1$ is the incidence number.

THEOREM (5.1). *The homology of the complex $C^*(X, A; G)$ is canonically isomorphic to $H^*(X, A; G)$.*

Proof. We first establish the theorem for the case that (X, A) is a simplicial pair. It is sufficient to show that the cover of open stars of vertices is an acyclic cover. [2, page 175]. Let σ be any cell of X. We must show that $H^p(\mathrm{St}\,\sigma; G) = 0$ for all $p > 0$. Let

$$\mathrm{Int}\,\sigma = A_0 \subset A_1 \subset \cdots \subset A_n = \mathrm{St}\,\sigma$$

be a resolution of $\mathrm{St}\,\sigma$; that is, for all $1 \leq i \leq n$, $A_i - A_{i-1}$ is precisely the interior of some cell σ_i, and $\dim \sigma_i \geq \dim \sigma_{i-1}$. Now $H^*(A_i, A_{i-1}; G) = 0$, since $G \mid \mathrm{Int}\,\sigma_i$ is trivial and A_{i-1} is a s.d.r. of A_i. (s.d.r. = strong deformation retract.) Thus $H^*(\mathrm{St}\,\sigma; G) = H^*(\mathrm{Int}\,\sigma; G)$, which is 0 in dimensions higher than 0.

If X is not a simplicial complex, let $\{(X_n, A_n)\}_{n \geq 0}$ be a sequence of subdivisions of (X, A), where $X_0 = X$, and such that for each $n \geq 1$, the n-skeleton of X_n is a simplicial subdivision of the n-skeleton of X, and every cell of dimension not n of X_{n-1} is also a cell of X_n. For any integer $p \geq 0$, let X_n^p denote the p-skeleton of X_n, and let $(X_\infty, A_\infty) = \bigcup_{n=0}^{\infty} (X_n^n, A_n^n)$, a simplicial subdivision of (X, A). For any $n \geq 0$, let H_n^* denote the homology of the graded complex $C^*(X_n, A_n; G)$, and let $(\sigma)_n$ and $(\partial\sigma)_n$ denote the subdivisions of σ and $\partial\sigma$, respectively, induced by the subdivision X_n; where σ is any cell of X. If σ is any $(n+1)$-cell of X, let $K^*[\sigma]$ denote the cokernel of the epimorphism

$$e[\sigma] : C^*((\sigma)_{n+1}, (\partial\sigma)_{n+1}; G \mid \sigma) \to C^*((\sigma)_n, (\partial\sigma)_n; G \mid \sigma)$$

induced by subdivision. Since $G \mid \mathrm{Int}\,\sigma$ is trivial, $e[\sigma]$ induces an isomorphism in homology. Routine computation shows that the cokernel of the epimorphism

$$e : C^*(X_{n+1}, A_{n+1}; G) \to C^*(X_n, A_n; G)$$

is the direct product of $K^*[\sigma]$, over all $(n+1)$-cells σ of X. Thus $H_{n+1}^* = H_n^*$. It follows by induction that $H_n^* = H_0^*$ for all $n \geq 0$. Let p be any integer. $X_{p+1}^p = X^p$, and so $H_0^p = H_{p+1}^p = H_\infty^p$ which equals $H^p(X, A; G)$ since X_∞ is a simplicial complex. Thus $H_0^* = H^*(X, A; G)$.

6. Some examples

We compute the obstruction for a few examples.

Example 6.1. Let X be the barycentric subdivision of a simplicial complex homeomorphic to the disjoint union of $S^1 \subset R^2$ and the closed interval I. Let f and g be maps from X to R^2, both of which send the S^1 part of X to S^1 via the identity map, and where $f(t) = (t - \frac{1}{2}, 0)$ and $g(t) = (t + \frac{1}{3}, 0)$

for all $t \in I$. See Figures 3 and 4. Obviously g is not homotopic to an embedding rel $A = S^1 \cup \partial I$, while f is already an embedding. The pair (J^*X, J^*A) is equivalent, by excision and homotopy, to the pair $(S^1 \times I, S^1 \times \partial I)$ and

$$I^*(X, A; f) \simeq I^*(X, A; g) \simeq H^*(S^1 \times I, S^1 \times \partial I; Z)$$

(cf. 3.5). The obstruction $\gamma(f)$ is zero, while $\gamma(g)$ is a generator of the infinite cyclic group $I^2(X, A; g)$. This example shows that the relative case is worthy of consideration; in the absolute theory, the obstruction of g is zero.

Example 6.2. Let X be a suitable complex homeomorphic to the circle, say, with six 1-cells. Then the pair $(J^*X, J^*X \cap N^*X)$ is equivalent, in

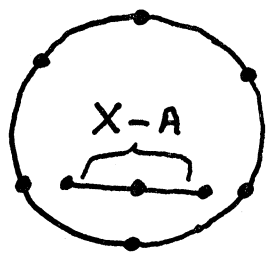

FIGURE 3

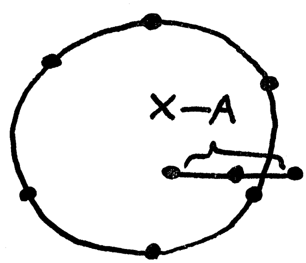

FIGURE 4

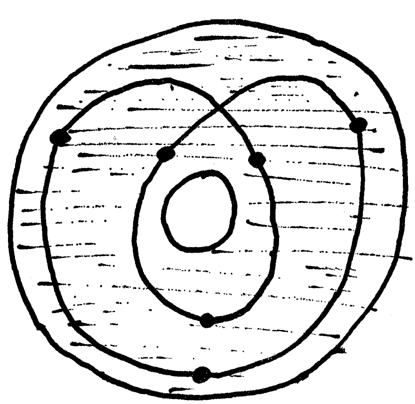

FIGURE 5

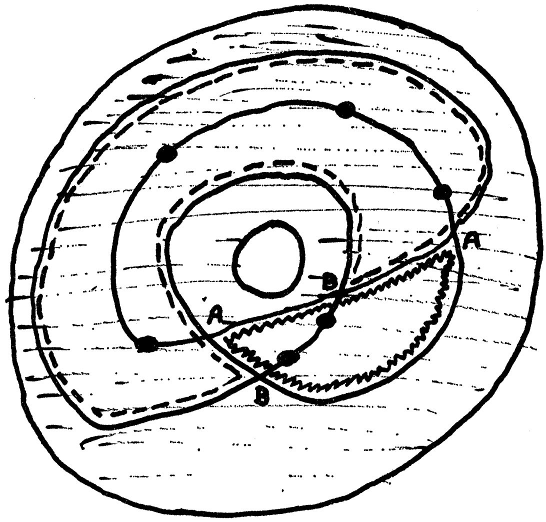

FIGURE 6. The two double points labeled "A" cancel, as illustrated by the jagged loop; the dotted loop illustrates the cancellation of the "B" points.

fact, homeomorphic, to $(E, \partial E)$, where E is the Moebius band. We can find a general map $f : X \to S^1 \times R$ which is homotopic to the two-fold covering of X to S^1 (see Figure 5) which has only one double point which is general. Then $I^*(X; f) \simeq H^*(E; G)$, where G is a sheaf which is locally isomorphic to $Z\pi_1(S^1)$ (the group ring of the fundamental group) everywhere except on the boundary, where it is isomorphic to the trivial integer sheaf. Routine computation shows that $I^0(X;f) = 0$, $I^1(X;f) = 0$, and $I^2(X;f) \simeq Z\pi_1(S^1)/L$ where L is the subgroup generated by a, a^{-1}, and all elements of the form $x + x^{-1}$. Here a is the generator of $\pi_1(S^1)$. Thus $I^2(X;f) \simeq Z_2 + F$ where F is a free Abelian group with infinitely many generators. The obstruction $\gamma(f)$ is the 2-torsion element of $I^2(X; f)$, represented by the coset $1 + L$.

Example 6.3. Let X be the same complex as in Example 6.2; let

$$g : X \to S^1 \times R$$

be a map homotopic to the three-fold covering of X to S^1. Clearly g is not homotopic to an embedding; yet $\gamma(g) = 0$. This we can see by finding a map homotopic to g which has two pairs of cancelling double points and no other double points (cf. 5.1). See Figure 6. The problem may be viewed as follows: g is homotopic to the three-fold covering $S^1 \to P_1$; but that map is "unstable" in the sense that there is no three-fold covering from S^n to P_n for $n > 1$.

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