MODULAR SUBGROUPS OF FINITE GROUPS II

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The subgroup M of the group G is said to be *modular in* G ($M \mathfrak{m} G$) if

$$(U \cup M) \cap V = U \cup (M \cap V) \quad \text{for all } U, V \subseteq G \text{ such that } U \subseteq V,$$

and

$$(U \cup M) \cap V = (U \cap V) \cup M \quad \text{for all } U, V \subseteq G \text{ such that } M \subseteq V.$$

In [5] we proved among other results that M/M_G is nilpotent and M^G/M_G is supersolvable for a modular subgroup M of a finite group G (M_G being the core, M^G the normal closure of M in G). One of the problems that remained open in [5] was to discover the exact structure of G/M_G, M^G/M_G, and M/M_G. In the present paper we solve this problem modulo the quasinormal Sylow subgroups of M/M_G. We prove the following

THEOREM. *Let M be modular in the (finite) group G, and let Q/M_G be a q-Sylow subgroup of M/M_G which is not quasinormal in G/M_G, q a prime.*

Then $G/M_G = Q^G/M_G \times K$, where Q^G/M_G is a P-group of order $p^n \cdot q$, p a prime, $p > q$, and $(|Q^G/M_G|, |K|) = 1$.

(For the definition of a P-group see [6, p. 12] or [5].)

An immediate consequence of this theorem is the following

COROLLARY. *Let M be modular in G, and let $M_G = 1$* (to make notation simpler).

Then $G = P_1 \times \cdots \times P_r \times K$, where P_i is a P-group of order $p_i^{n_i} \cdot q_i$, p_i, q_i primes, $p_i > q_i$, $(|P_i|, |P_j|) = (|P_i|, |K|) = 1$ $(i, j = 1, \cdots, r; i \neq j)$, and where $M = Q_1 \times \cdots \times Q_r \times (M \cap K)$, with Q_i being a q_i-Sylow subgroup of P_i, and $M \cap K$ being quasinormal in G.

This corollary gives the solution of the problem mentioned above modulo the quasinormal part $M \cap K$ of M, about which we cannot say very much (except, of course, that it is quasinormal in G). Since it is obvious that a subgroup Q is modular in a group G whenever G/Q_G has the structure given in the Theorem, we also cannot say anything about the structure of the complement K in G/M_G.

Some other consequences of the theorem are perhaps worth mentioning.

(1) *Let M be modular in G, and let Q/M_G be a Sylow subgroup of M/M_G. Then Q is modular in G.*

(2) *A minimal modular (but not normal) subgroup of a group is of prime power order.*

(3) *Let M be modular in G, and let q be a prime dividing $|M : M_G|$. Then there is a normal subgroup N of index q in G.*

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These three corollaries are generalizations of well known theorems for quasinormal subgroups (see [4]).

It is obvious that the main theorem would follow from the same theorem for the special case of a modular subgroup of prime power order (which we prove in Section 2) together with (1). However, we are not able to prove (1) directly (i.e. only using the lattice properties of the Sylow subgroup Q/M_G of the nilpotent group M/M_G). One reason for this might be that there is no lattice theoretical reason for Q to be modular in G. In fact, we shall give an example of a nilpotent modular subgroup M (with $M_G \neq 1$, of course) having a Sylow subgroup Q which is not modular in G. This example also will show that the (implicit) assumption $M_G = 1$ in (1) cannot be replaced by the weaker assumption that M is nilpotent. We shall, however, show that Q is modular in G, if Q is the Sylow subgroup of M belonging to the largest prime dividing the order of the nilpotent modular subgroup M. The proof of this theorem occupies Section 3, whereas in Section 4 we give the proof of the main theorem. In Section 1 we show that a modular subgroup is permutable with any subgroup having relatively prime order, a result which again illustrates the close connection between modularity and permutability.

All groups in this paper are finite; the notation is the same as in [5], except that the order of G is now denoted by $|G|$, the index of U in V is $|V:U|$, and the subgroup lattice of G is $\mathfrak{L}(G)$.

1. Modularity and permutability

It is well known that a quasinormal subgroup M of a group G (i.e. a subgroup permutable with any subgroup of G) is subnormal and modular in G [6, p. 5 and p. 7]. It is a result due to Heineken ([2, Satz 7]; see also the forthcoming revised edition of [6]) that the converse statement is also true. We summarize this in the following

LEMMA 1. *The subgroup M of the group G is quasinormal in G if and only if M is a subnormal and modular subgroup of G.*

Since there are modular subgroups which are not subnormal (for example the 2-Sylow subgroups of the nonabelian group of order 6), a modular subgroup is in general not quasinormal. But we can show that it is still permutable with many subgroups, namely with all subgroups of relatively prime order.

THEOREM 1. *Let M be modular in the group G, and let U be a subgroup of G such that $(|M|, |U|) = 1$.*

Then M is permutable with U, i.e. $UM = MU$.

Proof. If M is permutable with two subgroups U_1 and U_2 of G, then M is also permutable with $U_1 \cup U_2$. So we can assume that U is cyclic of order p^n, p a prime. By (2.1) of [5], we have that

$$[U \cup M/M] \simeq [U/U \cap M] \simeq \mathfrak{L}(U).$$

Hence $[U \cup M/M]$ is a chain of length n. By [5, Lemma 2],

$$| U \cup M : M | = q^n,$$

q a prime. Since $(| M |, p) = 1$, p^n divides $| U \cup M : M |$. Hence $p = q$, and so we have

$$| U \cup M : M | = p^n = | U : U \cap M |,$$

i.e. $UM = MU$. This proves the theorem.

In the following technical lemma we treat some of the situations which will later on occur as "minimal counterexamples" in the proofs of our theorems.

LEMMA 2. *Let p, q, r be primes, $r > p > q$.*

(a) *Let M be a subgroup of the group G such that $| G | = pqr$, $| M | = q$, and $M^G = G$.*

(b) *Let G be as in* (a), *but let this time $| M | = pq$ and $M_G = 1$.*

(c) *Let $G = PQ \times RS$, where $| P | = p$, $| Q | = | S | = q$, $| R | = r$, and where PQ and RS are nonabelian; let $| M | = q^2$.*

In any of the above situations, M is not modular in G.

Proof. (a) Since $r > p > q$, by [3, Satz 2.11, p. 420], G' and G/G' are cyclic. As $M \ntriangleleft G$, G' is contained in the q-complement K of G. If G' were properly contained in K, then $G'M$ would be a proper normal subgroup of G containing M; this would contradict $M^G = G$. Hence $G' = K$, and therefore, again by [3, Satz 2.11, p. 420], $G = \{a, b\}$, where $a^{pr} = b^q = 1$, $b^{-1}ab = a^s$, $s^q \equiv 1 \ (m)$, $(pq, s - 1) = 1$. By [5, (2.7)], we can take $M = \{b\}$. Then $M \cup M^a = G$, since $(b^{-1})^a \cdot b = a^{-1}b^{-1}ab = a^{s-1}$, and therefore, as $(pq, s - 1) = 1$, $a \in M \cup M^a$. But $[G/M]$ obviously is not isomorphic to $[M^a/M \cap M^a]$; by [5, (2.1)], M is not modular in G.

(b) Again G' and G/G' are cyclic. Since $M_G = 1$, $G' \cap M = 1$, i.e. $M \simeq G/G'$ is cyclic. Let N be a conjugate of M, $N \neq M$. Since

$$M \cap N \triangleleft M \cup N = G,$$

we have $M \cap N = 1$. But then

$$[N \cup M/M] = [G/M] \not\simeq \mathfrak{L}(N) = [N/N \cap M],$$

and hence by [5, (2.1)], M is not modular in G.

(c) Assume M is modular in G. By [5, (2.7)], we can take M to be SQ. Clearly $M \cup PR = G$, and hence

$$[G/M] \simeq [PR/PR \cap M] \simeq \mathfrak{L}(PR).$$

Since PQ and RS are nonabelian, there exist q-Sylow subgroups Q_1 and S_1 of PQ and RS, respectively, such that $Q_1 \neq Q$ and $S_1 \neq S$. $Q_1 S_1$ is a q-Sylow subgroup of G, and clearly $Q_1 S_1 \cup QS = G$. Hence

$$[G/M] \simeq [Q_1 S_1/Q_1 S_1 \cap QS] \simeq \mathfrak{L}(Q_1 S_1).$$

Hence $\mathfrak{L}(PR) \simeq L(Q_1 S_1)$, which is impossible, since PR is cyclic of order pr and $Q_1 S_1$ is an elementary abelian q-group. This completes the proof of Lemma 2.

In the second theorem of this section we treat the groups of prime power order not permutable with a modular subgroup of G.

THEOREM 2. *Let M be modular in the group G, and let S be a q-subgroup of G, q a prime.*

If M is not permutable with S, then M is maximal in $M \cup S$, and $|M \cup S : M| = p$, p a prime, $p > q$.

Proof. Let G be a minimal counterexample to Theorem 2, and let M be maximal among the modular subgroups of G for which the theorem is false. By the minimality of G,

(1) $M \cup S = G.$

Let M_1 be a maximal subgroup of G containing M, and let $S_1 = S \cap M_1$. By [5, (2.1)], S_1 is a maximal subgroup of S and therefore modular in $[S/M \cap S]$. Again by [5, (2.1)], $M_1 \mathfrak{m} [G/M]$, and hence by [5, (2.3)],

(2) $M_1 \mathfrak{m} G.$

By [5, Lemma 1],

(3) $|G : M_1| = p_1$, p_1 a prime.

Furthermore, since $M_1 = (M \cup S) \cap M_1 = M \cup (S \cap M_1) = M \cup S_1$, the minimality of G implies that either $MS_1 = S_1M$ or $|M_1 : M| = q_1$, where q_1 is a prime, $q_1 > q$. In both cases,

(4) $|M_1 : M| = q_1^n$, q_1 a prime, $n \geq 0$.

Assume $p_1 = q_1$ (or especially $n = 0$); call $p_1 = q_1 = p$.

Then $|G : M| = p^{n+1}$, and therefore by [5, Satz 1], G/M_G is either a p-group or a P-group of order $p^{n+1}r$, $p > r$, r a prime. In the first case M would be subnormal in G, and hence by Lemma 1, $M \lhd_q G$. This is impossible, since $MS \neq SM$. In the second case q divides $|G : M_G|$, since $S \nsubseteq M_G$. If $q = p$, then $|S : S \cap M| = p^{n+1}$, since $n + 1$ is the dimension of the lattice $[S/S \cap M] \simeq [G/M]$. But then $|G : M| = |S : S \cap M|$, which is impossible. So we have $q \neq p$, hence $q = r$. SM_G/M_G is contained in some q-Sylow subgroup of G/M_G. These q-Sylow subgroups have order q; hence $M_G = M \cap SM_G$ is maximal in SM_G. By the modularity of M, M is maximal in $M \cup SM_G = M \cup S = G$. So $|M \cup S : M| = p$, $p > q$, which is a contradiction.—Hence

(5) $p_1 \neq q_1$, and $n \geq 1$.

We now distinguish two cases.

Case 1. M_1 is the only maximal subgroup of G containing M. Let $x \in G \backslash N_G(M)$. Then $M \cup M^x \mathfrak{m} G$, by [5, (2.7)], $M \cup M^x > M$, and

$(M \cup M^x) \cup S = G$. By the maximality of M, $(M \cup M^x)S = S(M \cup M^x)$ or $|G : M \cup M^x| = p, p > q$, p a prime. If $M \cup M^x$ were properly contained in M_1, we would therefore have

$$p_1 q_1^{n_1} = |G : M \cup M^x| = |S : (M \cup M^x) \cap S| = q^{n_1+1},$$

where $n_1 > 0$. But this is impossible, since $p_1 \neq q_1$.—Hence $M_1 \subseteq M \cup M^x$, since M_1 is the only maximal subgroup of G containing M. This is true for any $x \in G \backslash N_G(M)$; hence $D(M) = \bigcap_{x \in G \backslash N_G(M)} (M \cup M^x) \geq M_1 > M$. By [5, Lemma 4] (and since $M \ntriangleleft_q G$), $|M^G : M_G| = rs$, r, s primes. If M^G were properly contained in M_1, we would again get $p_1 = q_1$, which contradicts (5).—So $M^G \geq M_1$. If $M^G = G$, then $M = M_1$, which contradicts the second part of (5). But if $M^G = M_1$, $[G/M]$ would be a chain, and therefore by [5, Lemma 2] $p_1 = q_1$. So Case 1 leads to a contradiction.—We are therefore left with

Case 2. There is a maximal subgroup $M_2 \neq M_1$ of G containing M; let

$$S_2 = M_2 \cap S.$$

By (2), (3), (4), and (5), $M_2 \mathfrak{m} G$, $|G : M_2| = p_2$, $|M_2 : M| = q_2^m$, $p_2 \neq q_2$ primes, and $m \geq 1$. We consider $M_1 \cap M_2$. Since

$$(M_1 \cap M_2) \cap S = S_1 \cap S_2 \triangleleft S,$$

we have $M_1 \cap M_2 \mathfrak{m} G$. If M were properly contained in $M_1 \cap M_2$, then the maximality of M would yield that $(M_1 \cap M_2)S = S(M_1 \cap M_2)$ and hence $|G : M_1 \cap M_2| = q^2$. This again contradicts (5).—So

(6) $M = M_1 \cap M_2$, hence $n = m = 1$.

Now $[G/M]$ is isomorphic to the subgroup lattice of the elementary abelian q-group $S/S_1 \cap S_2$. Hence no maximal subgroup of G containing M can be normal in G, since $M \ntriangleleft G$. By [5, Lemma 1], G/M_{i_G} is nonabelian of order $r_i s_i$, $r_i > s_i$, r_i, s_i primes $(i = 1, 2)$. Let N_i/M_{i_G} be the normal subgroup of index s_i of G/M_{i_G}.

Assume, $S \subseteq N_i$ for $i = 1$ or 2. By the modularity of M we get $N_i = (M \cap N_i) \cup S$ and $M \cap N_i \mathfrak{m} N_i$. Since G was a minimal counterexample,

$$|N_i : M \cap N_i| = |S : M \cap S| = q^2 \quad \text{or} \quad |N_i : M \cap N_i| = p, \quad p > q;$$

but this is impossible, since $|N_i : M \cap N_i| = |G : M| = p_i q_i$.—So S is not contained in N_i, and hence

(7) $s_i = |G : N_i| = q$ $(i = 1, 2)$.

Now $M_{1_G} \cap M_{2_G} = M_G$, by (6). Furthermore

$$|M : M \cap M_{i_G}| = |M_i : M_{i_G}| = |G : N_i| = q,$$

and hence $|M : M_G| = q$ or q^2. If $|M : M_G| = q$, then $N_1 \cap M = M_G$, and hence $[N_1/M_G] \simeq [G/M]$. So N_1/M_G has exactly $q + 1$ minimal subgroups, which is impossible, since $|N_1/M_G| = r_1 q_1$ with $r_1 > s_1 = q$.—Hence $|M : M_G| = q^2$, and therefore

$$G/M_G = M_{1_G}/M_G \times M_{2_G}/M_G,$$

where $|M_{1_G}/M_G| = r_2 q, r_2 > q, |M_{2_G}/M_G| = r_1 q, r_1 > q$, and $|M/M_G| = q^2$. By (5), $r_1 \neq r_2$, and therefore by Lemma 2, (c), M/M_G cannot be modular in G/M_G.

This contradiction shows that also Case 2 cannot occur. This is a final contradiction and so we have proved the theorem.

2. Modular subgroups of prime power order

In this section we prove the main theorem for modular subgroups of prime power order.

THEOREM 3. *Let M be modular in the group G, and let $|M| = q^n$, q a prime. If M is not quasinormal in G, then*

$$G/M_G = M^G/M_G \times K,$$

where M^G/M_G is a P-group of order $p^n q$, p a prime, $p > q$, and $(|M^G/M_G|, |K|) = 1$.

Conversely, if M is a subgroup of G for which G/M_G has the above structure, then M is modular in G.

Proof. To prove the first part of Theorem 3, let G be a minimal counterexample, and let M be a modular subgroup of G for which the theorem is false.

Since then also G/M_G, M/M_G is a counterexample to the theorem, we have

(1) $M_G = 1$,

by the minimality of G.

We have to distinguish two cases.

Case 1. $M^G = G$. Then G is supersolvable [5, Satz 4]. Let p be the largest prime dividing $|G|$, and let P be the p-Sylow subgroup of G. If $q = p$, then $M \subseteq P$, and hence $G = M^G = P$ would be a p-group. By Lemma 1, M would be quasinormal in G, a contradiction.—Hence

(2) $p > q$.

Let N be a minimal normal subgroup of G contained in P. Then $|N| = p$. If MN/N would be quasinormal in G/N, then $G/N = (MN/N)^{G/N}$ would be a q-group. So any maximal subgroup of G containing MN would be normal in G, and therefore $MN = G$, by the assumption in Case 1. Hence M would be a maximal subgroup of G. By [5, Lemma 1], $G/M_G = G$ would be a P-group of order pq, which is impossible.—Hence the induction hypothesis yields that

$$G/M_1 = (MN)^G/M_1 \times K \quad (\text{with } M_1/N = (MN/N)_{G/N}),$$

where $(MN)^G/M_1$ is a P-group of order $r^n q$, r a prime, $r > q$, and $(|K|, rq) = 1$.

Since $M^G = G$, $K = 1$, and hence we have

(3) G/M_1 is a P-group of order $r^n q$, r a prime, $r > q$.

Since $MN/N \simeq M$, $(|M|, |N|) = 1$, and MN/N is a q-Sylow subgroup of G/N,

(4) M is a q-Sylow subgroup of G.

If $r = p$, $|G : M| = p^{n+1}$, by (3) and (4). But then [5, Satz 1] implies that G is a P-group, a contradiction.—Hence

(5) $r \neq p$.

Then by (3), $N = P$. Let R be an r-Sylow subgroup of G. Then R is permutable with any conjugate M^x of M (Theorem 1). Since G is supersolvable and $r > q$, $R \lhd RM^x$ for all $x \in G$. Since $M^G = G, R \lhd G$. Let R_1 be a minimal normal subgroup of G contained in R; $|R_1| = r$. If MR_1/R_1 were quasinormal in G/R_1, MR_1/R_1 would be normal in G/R_1 (by (4)), hence $MR_1 = G$, by the assumption in Case 1. But N is clearly not contained in MR_1.—So, as before, the induction hypothesis implies that

(6) G/M_2 is a P-group of order $s^m q$, s a prime, $s > q$,

where $M_2/R_1 = (MR_1/R_1)_{G/R_1}$. This is only possible if $R = R_1$, since otherwise p, q, and r would divide the order of the P-group G/M_2. Hence

(7) $G = PRM$, where $|P| = p$, $|R| = r$ (and therefore $s = p$).

Especially, $n = 1$ in (3), $m = 1$ in (6). Now $M_1 \cap M_2$ has order a power of q and is therefore contained in M. By (1), $M_1 \cap M_2 = 1$, and therefore $|G|$ is either prq or prq^2. The first case is impossible by Lemma 2, (a); in the second case $G = M_1 \times M_2$, and so Lemma 2, (c) gives a contradiction.—Hence Case 1 cannot occur. We are left with

Case 2. $M^G \neq G$. Let U be a proper subgroup of G containing M^G. If M were quasinormal in U, then $M \lhd\lhd M^G \lhd G$, and hence by Lemma 1, $M \lhd_q G$, which is impossible. So the induction hypothesis yields that $U/M_U = M^U/M_U \times K$, where M^U/M_U is a P-group of order $p^n q$, $p > q$, and $(|K|, pq) = 1$. Hence M is a q-Sylow subgroup of U, especially of M^G. Therefore $M^{M^G} = M^U = M^G$ and $M_{M^G} = M_G = 1$. So finally we have shown

(8) M^G is a P-group of order $p^n q$, p a prime, $p > q$,

and

(9) if $M^G \subseteq U \subset G$, then $U = M^G \times K$, where $(|K|, pq) = 1$.

Let P be a p-Sylow subgroup of M^G. We want to show that M^G is a Hall subgroup of G. So assume that P is not a p-Sylow subgroup of G, and let S be a p-Sylow subgroup of G containing P. Then $|M^G S| = p^m \cdot q$, and so by

(9) $M^G S = G$. Hence $|G : M| = p^m$, and therefore by [5, Satz 1] G is a P-group. But this implies that $M^G = G$, contrary to the assumption in Case 2.—So

(10) P is a p-Sylow subgroup of G.

Assume M is not a q-Sylow subgroup of G. Let T_1 be a q-subgroup of G containing M as a maximal subgroup. By (9) $T_1 M^G = G$, and hence by (8), $|G| = p^n \cdot q^2$. Since M is not subnormal in G (by Lemma 1), there is a q-Sylow subgroup T of G not containing M. Let $H = T \cup M$. Then

$$[H/M] \simeq [T/T \cap M] \simeq \mathfrak{L}(T).$$

So if T is cyclic, $[H/M]$ is a chain of length 2, and so by [5, Lemma 3] $|H : M|$ is a prime power. But $|H : M| = pq$, a contradiction.—Hence T is elementary abelian, and therefore M is the intersection of exactly $q + 1$ maximal subgroups of H. One of these (namely $M^G \cap H$) is normal in H. Hence $M \lhd H$, which is impossible, since M is not contained in the q-Sylow subgroup T of H.—So

(11) M is a q-Sylow subgroup of G.

By (10) and (11), M^G is a normal Hall subgroup of G. By a theorem of Zassenhaus and Schur [3, 18.1, p. 126] there is a complement K to M^G in G. Since $(|K|, q) = 1$, $KM = MK$ (by Theorem 1), and therefore $|KM| = q\,|K|$. Hence p does not divide $|KM|$, and therefore $M^G \cap KM = M$. So $M \lhd KM$, i.e. $K \subseteq N_G(M)$. This is true for any complement K to M^G, especially for all the conjugates of K. Hence $K^G \subseteq N_G(M)$. So $K^G \cap M^G \subseteq N_{M^G}(M) = M$ (since M^G is a P-group). Since $|M| = q$ and $M \ntriangleleft G$, $K^G \cap M^G = 1$. Hence $K^G = K$ (i.e. $K \lhd G$), and therefore $G = M^G \times K$, $(|M^G|, |K|) = 1$. This contradicts the choice of G and M.

So we have shown that also Case 2 cannot occur, and this completes the proof of the first part of Theorem 3.

The second part of the theorem is a trivial consequence of the following

LEMMA 3. *Let $G = H \times K$, $(|H|, |K|) = 1$, and let M be a modular subgroup of H. Then M is modular in G.*

Proof. Since $(|H|, |K|) = 1$, by [6, Theorem 4, p. 5]

$$\mathfrak{L}(G) = \mathfrak{L}(H) \times \mathfrak{L}(K).$$

Let $U, V \in \mathfrak{L}(G)$ such that $U \subseteq V$; then $U = U_1 \times U_2$, $V = V_1 \times V_2$, where $U_1 \subseteq V_1 \subseteq H$ and $U_2 \subseteq V_2 \subseteq K$. Since M is modular in H, we get

$$\begin{aligned}(U \cup M) \cap V &= ((U_1 \cup M) \cap V_1) \times (U_2 \cap V_2)\\ &= (U_1 \cup (M \cap V_1)) \times U_2\\ &= U \cup (M \cap V),\end{aligned}$$

the first condition to be satisfied by a modular subgroup. The other one is proved in the same way.

The second part of Theorem 3 now follows immediately. Since M^G/M_G is a P-group (and therefore has a modular subgroup lattice), $G/M_G = M^G/M_G \times K$, and $(|M^G/M_G|, |K|) = 1$, by Lemma 3 M/M_G is modular in G/M_G. Hence by [5, (2.3)] M is modular in G. This completes the proof of the theorem.

Obviously, Theorem 3 gives criteria for quasinormality of modular subgroups of prime power order. We formulate two of them as corollaries to the theorem.

COROLLARY 1. *A modular subgroup of prime power order which is not a Sylow subgroup of the group G is quasinormal in G.*

COROLLARY 2. *Let M be a modular q-subgroup of G, q a prime, and let $|M : M_G| \neq q$. Then M is quasinormal in G.*

As another consequence of Theorem 3 we generalize a result about the normalizer of a quasinormal q-subgroup to the modular case.

COROLLARY 3. *Let M be a modular subgroup of order q^n of G, q a prime. Then $|G : N_G(M)| = p^m$, p a prime, $p \geq q$, and $N_G(M)$ contains every element of order relatively prime to p and q.*

Proof. For quasinormal subgroups, this is well known (with $p = q$; see for example [4, p. 169]). If M is not quasinormal in G, then by Theorem 3 $G/M_G = M^G/M_G \times C/M_G$, where M^G/M_G is a P-group of order $p^m q$, p a prime, $p > q$, and $(|C/M_G|, |M^G/M_G|) = 1$. Clearly C normalizes M; hence every element of order prime to p and q is contained in $N_G(M)$. Furthermore $N_G(M) = MC$ (since M^G/M_G is a P-group), and therefore

$$|G : N_G(M)| = p^m.$$

We conclude this section with an example showing that M^G in Theorem 3 is in general not a direct factor of G. Since M^G is a normal Hall subgroup of G, it certainly has a complement C in G; but C need not be normal in G.

Example 1. Let G be the direct product of a nonabelian group PQ (where $|P| = p > q = |Q|$, p, q primes) and a nonabelian group RS (where $|S| = q > r = |R|$, r a prime), and let $M = QS$ be a q-Sylow subgroup of G. Then $M_G = S$, $M^G = PQS$, $G/M_G = M^G/M_G \times RM_G/M_G$, and hence M is modular in G, by the second part of Theorem 3. But since $R \ntriangleleft G$, G is not the direct product of M^G and some complement of it.

3. Nilpotent modular subgroups

Let M be a nilpotent modular subgroup of the group G, and let Q be a q-Sylow subgroup of M. We would like to prove that Q is again modular in G. By the second part of Theorem 3, we only have to show that G/Q_G has the structure given for G/M_G in the first part of this theorem. Therefore we try to carry out the program given in the proof of Theorem 3, for the q-Sylow subgroup Q of M. It is clear that this will be somewhat more difficult, and, in fact, the following example shows that it is not quite possible.

Example 2. Let $G = PQ \times R$, where PQ is a nonabelian group of order pq, p and q primes, $p > q$, and $|R| = p$. Let $M = QR$. M is nilpotent, and by [5, (2.3)], M is modular in G. If Q were modular in G, [5, (2.1)] would imply that

$$\mathfrak{L}(PR) \simeq [PR \cup Q/Q] = [G/Q] = [\bar{Q}R \cup Q/Q] \simeq \mathfrak{L}(\bar{Q}R),$$

for any q-Sylow subgroup $\bar{Q} \neq Q$ of G. But a group of order p^2 and a cyclic group of order pq, $p > q$, do not have isomorphic subgroup lattices.—So Q is not modular in G.

This example shows that a q-Sylow subgroup Q of a nilpotent modular subgroup M of a group G need not be modular in G again. We shall, however, show that Q is modular in G if q is the largest prime dividing the order of M.

We begin with the following

LEMMA 4. *Let M be a nilpotent modular subgroup of the group G, and let Q be a q-Sylow subgroup of M, q a prime.*

If Q is not quasinormal in G, then Q is a q-Sylow subgroup of G and $|Q : Q_G| = q$.

Proof. Let G be a minimal counterexample to Lemma 4, and let M, Q be such that the lemma is false. Assume,

$$Q \lhd \lhd G.$$

We want to show that this implies that $Q \lhd_q G$, which will give a contradiction. So let X be a p-subgroup of G, p a prime. If $MX = XM$, then also $QX = XQ$, by [3, Satz 4.8, p. 676] (if $p = q$), or since Q is a subnormal, hence normal, q-Sylow subgroup of MX (if $p \neq q$). If $MX \neq XM$, then by Theorem 2 we have that M is maximal in $M \cup X$. If Q is a q-Sylow subgroup of $M \cup X$, then again Q would be normal in $M \cup X$, and hence $QX = XQ$. If not, then let R be a q-Sylow subgroup of $M \cup X$ containing Q. Then $Q = M \cap R$ is maximal in R, by [5, (2.1)]. Hence $Q \lhd R$, $Q \lhd M$, and so again $Q \lhd M \cup X$, i.e. $QX = XQ$. So in any case Q is permutable with X; hence $Q \lhd_q G$, which is not the case.—Therefore

(1) Q is not subnormal in G.

By (1), there is a q-Sylow subgroup S of G which does not contain Q. Since q divides $|M : M \cap S|$, but does not divide $|M \cup S : S|$, M is not permutable with S. By Theorem 2, $|M \cup S : M| = r$, r a prime, $r > q$. Hence Q is a q-Sylow subgroup of $M \cup S$, and therefore

(2) Q is a q-Sylow subgroup of G.

Assume there is a proper normal subgroup N of G containing Q. Then Q is a q-Sylow subgroup of the nilpotent modular subgroup $M \cap N$ of N. If Q were quasinormal in N, then by Lemma 1, $Q \lhd \lhd G$, contradicting (1). By the minimality of G we have that $|Q : Q_N| = q$. Since Q is a Sylow subgroup of N, we have $Q_N = Q_G$, and now $|Q : Q_G| = q$ (together with (2)) gives a contradiction to the choice of Q.—Hence

(3) $Q^G = G$

Certainly $M \neq Q$, since otherwise Theorem 3 would give a contradiction. So let $p \neq q$ be a prime dividing $|M|$, and let P be a p-Sylow subgroup of M. We want to show that $P \lhd G$, and for this we show that P is normalized by any q-Sylow subgroup T of G. If $T = Q$, then clearly T normalizes P. If $T \neq Q$, then (as before S) T is not permutable with M, and hence

$$|T \cup M : M| = s,$$

s a prime, $s > q$. Let $H = M \cup T$. Then by [5, Lemma 1], $|M : M_H| = t$, t a prime. Since $T \nsubseteq M_H$, $t = q$. But then $P \subseteq M_H$, which implies that $P \lhd H$. So in any event $T \subseteq N_G(P)$, and by (3) we get

(4) $P \lhd G$.

Now $QP/P \lhd_q G/P$ would by (2) imply that $QP \lhd G$. By (3) we would get that $G = QP = M$, which is not the case. So the induction hypothesis yields $|QP/P : (QP/P)_{G/P}| = q$. Let $(QP/P)_{G/P} = K/P$. Then $K \cap Q$ is a q-Sylow subgroup of the nilpotent (since $K \subseteq M$) normal subgroup K of G, and hence $K \cap Q \lhd G$. Since $|Q : K \cap Q| = |KQ : K| = q$, we have that $K \cap Q = Q_G$. But this (together with (2)) contradicts the choice of Q. This completes the proof of the lemma.

We are now able to prove

LEMMA 5. *Let M be a nilpotent modular subgroup of the group G, and let Q be a q-Sylow subgroup of M which is not quasinormal in G, q a prime.*

(a) *Then $|Q^G : Q_G| = p^n q$, p a prime, $p > q$.*

(b) *If $q > 2$, then Q^G/Q_G is a P-group of order $p^n q$, $p > q$.*

Proof. We give the proofs for (a) and (b) simultaneously, as far as possible.

So let G be a minimal counterexample to (a) or (b), and let M, Q be such that (a) or (b) is false.

Then also G/Q_G, M/Q_G, Q/Q_G is a counterexample to (a) or (b), and hence

(1) $Q_G = 1$,

by the minimality of G. Now Lemma 4 implies that

(2) Q is a q-Sylow subgroup of G and $|Q| = q$.

Assume there is a proper normal subgroup R of G containing Q. By the minimality of G (and since Q is not quasinormal in R) we have that $|Q^R : Q_R| = p^n q$, p a prime, $p > q$ (or that Q^R/Q_R is a P-group, in case (b)). But since Q is a q-Sylow subgroup of R, $Q^R = Q^G$, $Q_R = Q_G$, and this gives a contradiction.—Hence

(3) $Q^G = G$.

Especially,

(4) $M^G = G$.

By Theorem 3,

(5) $M \neq Q$.

Now let N be a minimal normal subgroup of G. Assume

$$QN \lhd_q G.$$

Then $QN \lhd G$, hence by (3), $QN = G$. Since $M \subset G$, $N \nsubseteq M$. So $N \cap M_G = 1$, and therefore $|M_G| = |M_G N : N|$ divides $|G : N| = q$. As $Q \ntriangleleft G$, $M_G = 1$. But then by [5, Satz 4], G is supersolvable. Hence $|N| = t$, t a prime, and therefore $|G| = tq$, which is impossible by (5).—So we have shown

(6) QN is not quasinormal in G.

We now separate the proofs of (a) and (b), and we first prove (a).

By (6) and the minimality of G we have (using (2) and (3))

$$|(QN/N)^{G/N} : (QN/N)_{G/N}| = |G : N| = p^n q,$$

p a prime, $p > q$. If $M_G \neq 1$, we could choose $N \subseteq M_G$. Since M is nilpotent, this would imply that $Q \subseteq C_G(N) \lhd G$. By (3) $C_G(N) = G$, and hence $|N| = r$, where r is a prime, and $G = N \times R$, where R is an r-complement in G (obviously, $p \neq r \neq q$). But then $Q \subseteq R \subset G$, a contradiction.—Hence

(7) $M_G = 1$.

By [5, Satz 4], G is supersolvable, and hence

(8) $|N| = r$, r a prime, $p \neq r \neq q$.

By (5), (7), and (8) the p-Sylow subgroup P of M is different from 1. If $P \lhd_q G$, then P would be normalized by any q-Sylow subgroup of G [4, p. 169], and therefore by $Q^G = G$. This contradicts (7).—Hence P is not quasinormal in G, and therefore by Lemma 4 P is a p-Sylow subgroup of G and $|P : P_G| = |P| = p$. So finally

(9) $|G| = pqr$, $r > p > q$.

But now Lemma 2, (b) gives a contradiction which proves part (a) of the lemma.

To prove (b), we again consider G/N. By (6) and the minimality of G this time we have that

(10) G/N is a P-group of order $p^n q$, $p > q$.

MN/N is a nilpotent subgroup of G/N containing the q-Sylow subgroup QN/N. By the structure of a P-group,

(11) $MN = QN$.

Assume $N \nsubseteq M$. Then $M_G \cap N = 1$, and hence $|M_G| = |M_G N : N|$ divides $|MN : N| = q$, i.e. $M_G = 1$. This would imply that G is supersolvable, hence that $|N| = r$, r a prime, and therefore $|MN| = qr$. Since $N \nsubseteq M$, $M = Q$, which contradicts (5).—Hence

(12) $M = QN$.

Since M is nilpotent, $Q \subseteq \underline{C}_G(N) \lhd G$, and therefore by (3),

(13) $N \subseteq \underline{Z}(G)$, $|N| = r$, r a prime.

By (2) and by part (a) of the lemma, $r = p$, and so

(14) $|G| = p^{n+1}q$, $p > q$.

By (12), N is the q-complement of M. Since N was an arbitrary minimal normal subgroup of G, we have shown that

(15) N is the only minimal normal subgroup of G.

Let P be the p-Sylow subgroup of G. Suppose x is an element of order p^2 in P; let $X = \{x\}$. Since P/N is elementary abelian (by (10)), we have that $X \supseteq N$. Again by (10), $X \lhd G$. Hence Q operates on X and centralizes x^p. By [3, Satz 5.12 (a), p. 437] Q centralizes X. But this is impossible, since XQ/N is a P-group.—Hence

(16) $\exp P = p$.

We consider $Z(P)$. By (16) $Z(P)$ is an elementary abelian p-group of order p^m, $m \geq 1$. Clearly $N \subseteq Z(P)$, and Q leaves N invariant. By Maschke's Theorem [3, p. 122] $Z(P) = N \times S$, where S is invariant under Q. This implies $S \lhd G$, which contradicts (15), unless $S = 1$. Hence

(17) $Z(P) = N$.

By (10), (16), and (17) P is an extraspecial p-group of exponent p. Hence [3, Satz 13.7, p. 353] there is a nonabelian subgroup P_1 of order p^3 of G containing N. By (10), $P_1 \lhd G$. Consider $H = P_1 Q$. Let $P_1 = \{a, b, c\}$, $a^p = b^p = c^p = 1$, $a^{-1}b^{-1}ab = c$, $ac = ca$, $bc = cb$ (cf. [1, p. 145]), and let $Q = \{x\}$. Then $N = P_1' = \{c\}$. Since H/N is a P-group, we have $a^x \equiv a^s (N)$,

$b^x \equiv b^s(N)$, where $s \not\equiv 1\ (p)$, $s^q \equiv 1\ (p)$. Hence $a^x = a^s \cdot c^u$, $b^x = b^s \cdot c^v$, where $0 \leq u, v < p$. By (13) $c^x = c$. Hence

$$c = c^x = (a^{-1}b^{-1}ab)^x = a^{-s}b^{-s}a^sb^s = c^{s^2},$$

i.e. $s^2 \equiv 1\ (p)$. But since $s^q \equiv 1\ (p)$ and $q > 2$, we get $s \equiv 1\ (p)$, which is a contradiction.—This completes the proof of Lemma 5.

Remark. The assumption $q > 2$ in part (b) of Lemma 5 cannot be omitted (although we used it only in the last line of the proof). This is shown by the following

Example 3. Let $p > 2$ be a prime, and let $P = \{a, b, c\}$ with $a^p = b^p = c^p = 1$, $a^{-1}b^{-1}ab = c$, $ac = ca$, and $bc = cb$. Let σ be the automorphism of P defined by $a^\sigma = a^{-1}$, $b^\sigma = b^{-1}(c^\sigma = c)$, and let G be the semidirect product of P and $\{\sigma\}$. Then $G/\{c\}$ is a P-group of order $2p^2$, and hence $M = \{c, \sigma\}$ is modular in G. Clearly M is nilpotent, and $Q = \{\sigma\}$ is a 2-Sylow subgroup of M. Since $Q \ntriangleleft G$, $Q_G = 1$. Furthermore $\{a, \sigma\}$ is nonabelian of order $2p$, and hence $a \in Q^G$. For the same reason $b \in Q^G$, and therefore $Q^G = G$. Hence Q^G/Q_G is not a P-group.

We are now able to prove the result announced at the beginning of this section. In order to avoid giving the same argument twice we use Theorem 5 (which is proved in the next section) in the proof of Theorem 4 (of course, we shall not use Theorem 4 in the other proof).

THEOREM 4. *Let M be a nilpotent modular subgroup of the group G, let q be the largest prime dividing $|M|$, and let Q be the q-Sylow subgroup of M. Then Q is modular in G.*

Proof. Let G be a minimal counterexample to Theorem 4, and let M, Q be such that the theorem is false.

Since then also G/Q_G, M/Q_G, Q/Q_G is a counterexample to the theorem, we have

(1) $Q_G = 1$.

Furthermore by Lemma 4,

(2) Q is a q-Sylow subgroup of G, and $|Q| = q$.

Now if $M_G = 1$, then $Q \mathfrak{m} G$ by Theorem 5. Hence

(3) $M_G \neq 1$.

If QM_G/M_G were quasinormal in G/M_G, then $QM_G \triangleleft G$ (by (2)). As $QM_G \subseteq M$ is nilpotent, Q would be normal in G. This is obviously not the case. Hence QM_G/M_G is not quasinormal in G/M_G, but since (by the minimality of G) it is modular in G, we have by Theorem 3 that

(4) $G/M_G = (QM_G)^G/M_G \times K/M_G$, where $(QM_G)^G/M_G$ is a P-group of order p^nq, p a prime, $p > q$, and $(|K/M_G|, pq) = 1$.

Let $N = (QM_G)^G$. Since q is the maximal prime divisor of $|M|$, $|N : M_G| = p^n q$, and $p > q$, M_G is a normal Hall subgroup of N. Hence there is a complement C to M_G in N. Now let Q^x, $x \in G$, be any conjugate of Q in G. Then $M_G Q^x \subseteq M^x$, and since M^x is nilpotent, $Q^x \subseteq C_G(M_G)$. Since C is isomorphic to N/M_G, C is a P-group of order $p^n q$, and therefore $C = Q^C \subseteq C_G(M^G)$. So $N = M_G \times C$, whence $C \lhd G$, i.e. $C = Q^G$. Finally by (4) we have therefore $G = K \times Q^G$, where $(|K|, |Q^G|) = 1$. But now by the second part of Theorem 3 Q is modular in G, a contradiction.—This proves Theorem 4.

4. Proof of the main theorem

For the proof of the main theorem we need a technical result. In order to formulate it we make the following

DEFINITION. Let X, Y be subgroups of the group G such that $Y \subseteq X$. Then we denote by $\sigma(X : Y)$ the number of prime factors of $|X : Y|$, i.e.

$$\sigma(X : Y) = \textstyle\sum_{i=1}^{n} r_i, \quad \text{if} \quad |X : Y| = \prod_{i=1}^{n} p_i^{r_i}, \qquad p_i \text{ primes.}$$

Now we can state

LEMMA 6. *Let M be modular in the group G, and let U be a subgroup of G. Then*

(a) $\sigma(U \cup M : M) = \sigma(U : U \cap M)$,

(b) $\sigma(U \cup M : U) = \sigma(M : U \cap M)$.

Proof. Obviously (b) is a consequence of (a). So we need only to prove (a). We do this by induction on the order of G.

Let $M \mathrel{\mathfrak{m}} G$, $U \subseteq G$; consider $U \cup M$. Clearly $M \mathrel{\mathfrak{m}} (U \cup M)$; let N be a maximal modular subgroup of $U \cup M$ containing M. By [5, Lemma 1] either $N \lhd U \cup M$, or N is maximal in $U \cup M$ and $|U \cup M : N| = p$, p a prime. In the first case

$$|M \cup U : N| = |N \cup U : N| = |U : U \cap N|,$$

and hence especially

(1) $\sigma(M \cup U : N) = \sigma(U : U \cap N)$.

In the second case $N \cap U$ is maximal in U, and hence

$$\sigma(U : U \cap N) = 1 = \sigma(U \cup M : N),$$

i.e. (1) also holds in this case. Since $|N| < |G|$, the induction hypothesis implies that

(2) $\sigma((U \cap N) \cup M : M) = \sigma((U \cap N) : (U \cap N) \cap M)$.

But since M is modular in G (and $M \subseteq N \subseteq U \cup M$), we have that

$$(U \cap N) \cup M = (U \cup M) \cap N = N,$$

and therefore (2) becomes

(3) $\sigma(N : M) = \sigma(U \cap N : U \cap M)$.

Now (1) and (3) yield the desired result.

THEOREM 5. *Let M be modular in the group G, and let Q/M_G be a q-Sylow subgroup of M/M_G, q a prime. Then Q is modular in G.*

Proof. Let G be a minimal counterexample to Theorem 5, let M be modular in G, and let q be the maximal prime divisor of $|M|$ for which the theorem is false.

Since then also G/M_G, M/M_G is a counterexample, we have

(1) $M_G = 1$.

By [5, Satz 2],

(2) M is nilpotent.

Let Q be the q-Sylow subgroup of M. By Lemma 4,

(3) Q is a q-Sylow subgroup of G and $|Q| = q$.

We want to show that Q^G is a P-group. By Lemma 5, (b), this is clearly true if $q > 2$. So assume (for the moment) that

(4) $q = 2$.

Since $M \neq Q$, there is a prime $r \neq q$ dividing $|M|$ and an r-Sylow subgroup R of M. Since $q = 2$, $r > q$, and therefore by the maximality of q, R is modular in G. If R would not be quasinormal in G, then by Theorem 3, $G = R^G \times K$, where R^G is a P-group of order $s^n r$, s a prime, $s > r$, and $(|K|, rs) = 1$. Now $Q \subseteq K$, since $s > r > q$. Clearly $M \cap K \mathrel{\mathrm{m}} K$ and $(M \cap K)_K = 1$ (by (1)). Hence the induction hypothesis yields that Q is modular in K. By Lemma 3, $Q \mathrel{\mathrm{m}} G$.—This is impossible, and hence we have shown that R is quasinormal in G. This holds for any prime r dividing $|M|$ which is different from q; hence

(4a) $T \lhd_q G$,

where T is the 2-complement of M.

Let $H = MQ^G$. Since $T \lhd_q G$ and $(|T|, |Q^x|) = 1$ for every conjugate Q^x of Q, $Q^x \subseteq N_G(T)$ for every $x \in G$, and hence

(4b) $T \lhd H$.

We consider H/T. M/T is modular in H/T, and has order q. If M/T were quasinormal in H/T, then $M \lhd H$ (by (3)), hence $Q \lhd H$ (by (2)), especially $Q \lhd Q^G \lhd G$. So Q would be a subnormal q-Sylow subgroup of G, hence $Q \lhd G$, which is impossible.—Hence M/T is not quasinormal in H/T, and therefore by Theorem 3,

$$H/T = M^H/T \times K,$$

where M^H/T is a P-group of order $p^n q$, p a prime, $p > q$, and $(|K|, pq) = 1$. Now $M^H \supseteq Q^H = Q^G$ (by (3)), and hence $M^H = H$. So H/T is a P-group, and therefore finally

(4c) $Q^G/T \cap Q^G$ is a P-group of order $p^n q$, $p > q$.

Since M is nilpotent, $Q \subseteq C_H(T) \lhd H$. Since $Q^H = Q^G$, $Q^G \subseteq C_H(T)$, and therefore $T \cap Q^G \subseteq Z(Q^G)$. Since the center of a P-group is 1, $T \cap Q^G = Z(Q^G)$, by (4c). Hence $T \cap Q^G \lhd G$, and therefore $T \cap Q^G = 1$, by (1). So finally, (4c) yields that Q^G is a P-group.
Hence we have shown that, regardless whether $q = 2$ or not, we have

(5) Q^G is a P-group of order $p^n q$, p a prime, $p > q$.

We want to show that Q^G is a Hall subgroup of G. By the Frattini argument we have $G = Q^G \cdot N_G(Q)$; furthermore $Q^G \cap N_G(Q) = N_{Q^G}(Q) = Q$, by (5). Hence $|G : Q^G| = |N_G(Q):Q|$, i.e. (by (3)) we have only to show that p does not divide $|N_G(Q)|$. So assume

(6) p divides $|N_G(Q)|$.

Let X be a subgroup of order p of $N_G(Q)$, and let $L = X \cdot Q^G$. Then

(6a) $|L| = p^{n+1} \cdot q$, $n \geq 1$.

Clearly $L \cap N_G(Q) = N_L(Q) = QX$. Since $M \subseteq N_G(Q)$, $M \cap L \subseteq QX$, i.e. $M \cap L = Q$ or $M \cap L = QX$. Now $M \cap L = Q$ (i.e. $Q \mathrel{\mathfrak{m}} L$) gives a contradiction: by Theorem 1, either $Q \lhd_q L$ or $L = Q^L \times K$, $(|Q^L|, |K|) = 1$. In the first case $Q \lhd L$ (by (3)), hence $Q \lhd Q^G$, contradicting (5); in the second case $K = 1$ (by (6a)), and so $Q^L = L \supset Q^G$, which is obviously not the case.—Hence $M \cap L = QX$, i.e.

(6b) p divides $|M|$.

Let P be the p-Sylow subgroup of M; by (1) and (6a), Lemma 4 implies that $P \lhd_q G$. Hence P is normalized by every conjugate of Q, i.e. by Q^G. Since M is nilpotent, P centralizes Q, i.e. $P \cap Q^G \subseteq C_{Q^G}(Q) = Q$. Hence $P \cap Q^G = 1$, and so $P \subseteq C_G(Q^G)$. Since $C_G(Q^G) \lhd G$, also $P^G \subseteq C_G(Q^G)$. Finally $P^G \cap Q^G \subseteq Z(Q^G) = 1$, hence

(6c) $P^G Q^G = P^G \times Q^G$.

P^G is a p-group (since $P \lhd_q G$). We want to show that the subgroup $\Omega(P^G)$ generated by the elements of order p in P^G is contained in P. This will contradict (1).

So assume, there is an element $a \in P^G$, $a \notin P$, $o(a) = p$. Let $b \in Q^G$ such that

$o(b) = p$, and let $Q = \{x\}$. By (5) and (6c) we have that $a^x = a$, $b^x = b^k$, where $k \not\equiv 1\ (p)$, $k^q \equiv 1\ (p)$. Let $c = ab$. Then $c^x = ab^k$, and hence

(6d) $\{c\} \cup Q = \{a, b, x\}$.

Now $M \cap P^G Q^G = PQ$, and therefore $PQ \mathrel{\mathfrak{m}} P^G Q^G$. So PQ is maximal in $PQ \cup \{c\}$, since (by (6c)) $|\{c\}| = p$. But since $a \notin P$ and $b \notin P^G$,

$$PQ \subset PQ \cup \{a\} \subset PQ \cup \{a, b\} = PQ \cup \{c\}$$

(by (6d)), which is impossible.—Hence we have shown

(6e) $\Omega(P^G) \subseteq P$.

But since $P^G \lhd G$, (6e) contradicts (1). This contradiction shows that (6) is false, i.e. that

(7) Q^G is a (normal) Hall subgroup of G.

By a theorem of Zassenhaus and Schur,

(8) Q^G has a complement K in G, and all such complements are conjugate in G.

Let T again be the q-complement of M. We want to show that $T \subseteq K_G$, i.e. that T is contained in any complement of Q^G. Let (as before) $H = M \cdot Q^G$, let P_1 be a subgroup of order p of Q^G, and let $H_1 = P_1 M$. Since $P_1 Q$ is non-abelian, $Q \not\lhd H_1$. Hence $M \not\lhd H_1$; let M^x be some conjugate of M in H_1. Since M is a maximal subgroup of H_1, $M \cap M^x$ is maximal in both M and M^x, and hence normal in both groups, i.e. $M \cap M^x \lhd H_1$. Since $Q \not\lhd H_1$, $Q \nsubseteq M \cap M^x$; since $M \cap M^x$ is maximal in M, finally $M \cap M^x = T$. So $T \lhd H_1$, i.e. $P_1 \subseteq N_H(T)$. This is true for any subgroup P_1 of order p of Q^G, and clearly also for Q. Hence by (5),

(9) $T \lhd H = MQ^G$.

Now $T \cap Q^G \subseteq C_{Q^G}(Q) = Q$, and therefore $T \cap Q^G = 1$. So by (9), $H = T \times Q^G$, and hence $T \subseteq C_G(Q^G)$. Now $C_G(Q^G) \cap Q^G = Z(Q^G) = 1$, and therefore by (7) and (8), $C_G(Q^G) \subseteq K_G$. So finally

(10) $T \subseteq K_G$.

Since Q is not contained in K, we have $M \cap K = T$, and therefore $|M : M \cap K| = q$, i.e. $\sigma(M : M \cap K) = 1$. By Lemma 6, $|K \cup M : K|$ is a prime. Since $(q, |K|) = 1$ and $Q \subseteq K \cup M$, we have $K \cup M = K \cup Q$ and $|K \cup Q : K| = q$. Hence $(K \cup Q) \cap Q^G = Q$, i.e. $Q \lhd K \cup Q$. So any complement K to Q^G in G is contained in $\mathrm{N}_G(Q)$, whence $K^G \subseteq N_G(Q)$. So $K^G \cap Q^G \subseteq N_{Q^G}(Q) = Q$, i.e. $K^G \cap Q^G = 1$. Hence $K^G = K$, and therefore $G = Q^G \times K$, $(|Q^G|, |K|) = 1$. By (5) and Lemma 3, Q is modular in G, a final contradiction. This completes the proof of the theorem.

The main theorem announced in the introduction is a trivial consequence of Theorem 3 and Theorem 5. The corollary to the main theorem follows if one observes that if P and Q are two different Sylow subgroups of M, then $P \not\subseteq Q^G$ (because $(|P|, |Q^G|) = 1$, if $Q \triangleleft_q G$, and because $P \cap Q^G \subseteq C_G(Q) \cap Q^G = Q$, if Q is not quasinormal in G), and that therefore $P^G \cap Q^G = 1$. Therefore G is the direct product of all the Q_i^G for which Q_i is not quasinormal in G and the intersection K of all the $p_i q_i$-complements. So G has the structure given in the corollary. The corollaries (1) to (3) of the introduction are also immediate consequences. (1) is just Theorem 5. If M is a minimal modular (but not normal) subgroup of a group G, then $M_G = 1$, and so by Theorem 5, M is a q-group; hence (2) is shown. And if q is a prime dividing $|M : M_G|$, then let Q/M_G be the q-Sylow subgroup of M/M_G. If $Q \triangleleft_q G$, then [4, Satz 1] gives the existence of a normal subgroup N of G with $|G : N| = q$; if Q is not quasinormal in G, then by Theorems 3 and 5, $G/M_G = Q^G/M_G \times K/M_G$, where Q^G/M_G is a P-group of order $p^n q$. But then $KP \subset G$, $|G : KP| = q$, if P is the p-Sylow subgroup of G. Thus also (3) is proved.

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