APPLICATION OF DE BRANGES SPACES OF INTEGRAL FUNCTIONS TO THE PREDICTION OF STATIONARY GAUSSIAN PROCESSES

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Usage

Greek letters α, β, γ, etc. stand for complex numbers. * indicates conjugation. $f^\times$ means $[f(\gamma^*)]^*$. $\lg^+ x$ is the logarithm of the bigger of 1 and x, and $\lg^- x = \lg^+ (1/x)$ so that $\lg x = \lg^+ x - \lg^- x$. $\int$ indicates integration over the line R^1, as in

$$\int (1 + \gamma^2)^{-1} \lg^+ |f| = \int_{-\infty}^{+\infty} (1 + \gamma^2)^{-1} \lg^+ |f(\gamma)| \, d\gamma.$$

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$\hat{f}(t) = (2\pi)^{-1/2} \int e^{-i\gamma t} f$ is the customary (inverse) Fourier transform. Do not confuse it with the eigendifferential transforms $\hat{f}_{\text{odd}}$ and $\hat{f}_{\text{even}}$ of 4.4. Δ is a non-negative, even, summable weight function on R^1. Δ^+ is the jacked-up weight function $\pi^{-1}\Delta/(1 + \gamma^2)$. $R^{2+}(R^{2-})$ is the open upper (lower) half-plane. Given closed submanifolds $\mathbf{A} \subset \mathbf{B}$ of a Hilbert space, $\mathbf{B}/\mathbf{A}$ stands for the annihilator of $\mathbf{A}$ in $\mathbf{B}$.

1. Introduction

The purpose of this paper is to exploit the connection between *stationary Gaussian processes and (de Branges) spaces of integral functions*, continuing the work of Levinson-McKean [11] in which the Markovian-ness of such processes was described in the language of Hardy functions. De Branges spaces provide us with an eigendifferential transform in the space $\mathbf{Z}(\Delta) \equiv L^2(\Delta, R^1)$, Δ being the spectral weight of the process, patterned after the customary Fourier sine-cosine transform for $\mathbf{Z}(1) = L^2(1, R^1) = L^2(R^1)$, and this transform turns out to be just the tool for predicting the future of the Gaussian process on the basis of a *bounded* segment of the past. The set-up and the results are explained in the rest of part 1. The reader familiar with the papers of de Branges [2]–[6] will see that most of parts 3 and 4 is due to him. But our situation is more special than his, so the results are more concrete and the proofs are simpler. Besides, his proofs are often hard to follow as they are spread out over a long series of papers, and it is our hope that the present paper will make this beautiful development available to a wider public.

1.1. Stationary Gaussian processes. Consider a stationary Gaussian process with *sample paths* $t \to \mathfrak{x}(t) \in R^1$, *universal field* $\mathbf{F}$, *probabilities* $P(B)$, and *expectations* $E(f)$, let $E(\mathfrak{x}) = 0$, and let $\mathbf{M}$ be the Hilbert space formed by closing up sums

$$f = c_1 \mathfrak{x}(t_1) + \cdots + c_n \mathfrak{x}(t_n)$$

with complex coefficients under the norm $f \equiv \sqrt{E(|f|^2)}$. Given a closed submanifold $\mathbf{M}_1 \subset \mathbf{M}$, define $\mathbf{F}_1$ to be the smallest Borel subfield of $\mathbf{F}$ over which $\mathbf{M}_1$ is measurable. Using the fact that perpendicular submanifolds of $\mathbf{M}$ have independent fields, it is easy to see that $\mathbf{M}_1$ can also be described as the biggest closed submanifold of $\mathbf{M}$ that is measurable over $\mathbf{F}_1$; *esp.*, the projection upon $\mathbf{M}_1$ is just the conditional expectation $E(\cdot \mid \mathbf{F}_1)$. This geometrical principle establishes a perfect correspondence between Borel subfields of $\mathbf{F}$ and closed submanifolds of $\mathbf{M}$ and permits us to express a wide class of problems concerning the Gaussian process $\mathfrak{x}$ in geometrical language. Of special interest are the submanifolds

$$\mathbf{M}^{ab} = \textstyle\bigcap_{n\geq 1} \text{span}\,[\mathfrak{x}(t) : a - 1/n \leq t \leq b + 1/n]$$
$$\mathbf{M}^{-} = \mathbf{M}^{-\infty 0} = \textit{the past}$$
$$\mathbf{M}^{-\infty} = \textstyle\bigcap_{t<\infty} \mathbf{M}^{-\infty t} = \textit{the remote past}$$
$$\mathbf{M}^{+} = \mathbf{M}^{0\infty} = \textit{the future}$$

$\mathbf{M}^{+/-}$ = *the projection of the future upon the past* = *the splitting manifold*
$\mathbf{M}^0 = \mathbf{M}^{00}$ = *the germ*
$\mathbf{M}_\infty = \text{span}\,[\partial^n \mathfrak{x}(0) : n = 0, 1, 2, etc.]$[2]

and the corresponding fields

$\mathbf{F}^{ab} = \bigcap_{n\geq 1} \text{field}\,[\mathfrak{x}(t) : a - 1/n \leq t \leq b + 1/n]$
$\mathbf{F}^- = \mathbf{F}^{-\infty 0}$ = *the past*
$\mathbf{F}^{-\infty} = \bigcap_{t<\infty} \mathbf{F}^{-\infty t}$ = *the remote past*
$\mathbf{F}^+ = \mathbf{F}^{0\infty}$ = *the future*
$\mathbf{F}^{+/-}$ = *the splitting field*
$\mathbf{F}^0 = \mathbf{F}^{00}$ = *the germ*
$\mathbf{F}_\infty = \text{field}\,[\partial^n \mathfrak{x}(0) : n = 0, 1, 2, etc.]$.

$\mathbf{F}^{+/-}$ (= the splitting field) calls for a little explanation. Given 2 fields such as $\mathbf{F}^-$ and $\mathbf{F}^+$, there is always a *smallest* subfield $\mathbf{F}^{+/-}$ of $\mathbf{F}^-$ which *splits* $\mathbf{F}^-$ and $\mathbf{F}^+$ in the sense that $\mathbf{F}^-$ is independent of $\mathbf{F}^+$ conditional on the knowledge of $\mathbf{F}^{+/-}$:

$$P(A^- \cap A^+ \mid \mathbf{F}^{+/-}) = P(A^- \mid \mathbf{F}^{+/-}) P(A^+ \mid \mathbf{F}^{+/-}) \quad \text{for } A^\pm \in \mathbf{F}^\pm.$$

Because of the correspondence between submanifolds and subfields explained above, this (smallest) splitting field is simply the field of the projection $\mathbf{M}^{+/-}$ of the future upon the past, as is plain from the perpendicular splitting

$$\mathbf{M} = \mathbf{M}^-/\mathbf{M}^{+/-} \oplus \mathbf{M}^{+/-} \oplus \mathbf{M}^+.$$

$\mathbf{F}^{+/-}$ is closely connected with the prediction problem of Kolmogorov-Szegö-Wiener which is to compute the projection of $\mathfrak{x}(t)$ upon the past for fixed $t > 0$. $\mathbf{F}^{+/-}$ can also be viewed as a measure of how non-Markovian the process is: for instance, the case $\mathbf{F}^{+/-} = \mathbf{F}^-$ may be termed *anti-Markovian*, while the case $\mathbf{F}^{+/-} = \mathbf{F}^0$, in which the process splits over its germ, may be regarded as increasingly favorable as the dimension of $\mathbf{M}^0$ gets smaller, the simplest case being the Ornstein-Uhlenbeck process for which $\dim \mathbf{M}^0 = 1$.

1.2. Trigonometrical approximation with a Hardy weight. Because $\mathfrak{x}$ is Gaussian, its distribution in function space is completely determined by the knowledge of the inner products $E[\mathfrak{x}(t_1)\mathfrak{x}(t_2)] \equiv Q(|t_1 - t_2|)$. Q is the Fourier transform of an even non-negative mass distribution Δ on R^1 of total mass $E[\mathfrak{x}(0)^2] < \infty$:

$$E[\mathfrak{x}(t_1)\mathfrak{x}(t_2)] = \int e^{i\gamma(t_1 - t_2)}\Delta.$$

This formula provides us with a new geometrical picture of the process $\mathfrak{x}$: it states that the map $\mathfrak{x}(t) \to \exp(i\gamma t)$ provides an isomorphism (inner products included) between $\mathbf{M}$ and the Hilbert space $\mathbf{Z} = \mathbf{Z}(\Delta) \equiv L^2(\Delta, R^1)$, and this fact permits a wide class of problems concerning the process $\mathfrak{x}$ to be

[2] ∂ stands for differentiation with respect to time. $\partial^n \mathfrak{x}(0)$ may not be defined for all $n > 0$.

stated in the language of trigonometrical approximation in **Z**, *esp.*, the following submanifolds of **Z** come to the fore:

$\mathbf{Z}^{ab} = \bigcap_{n\geq 1} \text{span}\,[e^{i\gamma t} : a - 1/n \leq t \leq b + 1/n]$
$\mathbf{Z}^{-} = \mathbf{Z}^{-\infty 0}$
$\mathbf{Z}^{-\infty} = \bigcap_{t<\infty} \mathbf{Z}^{-\infty t}$
$\mathbf{Z}^{+} = \mathbf{Z}^{0\infty}$
$\mathbf{Z}^{+/-}$ = *the projection of* $\mathbf{Z}^{+}$ *upon* $\mathbf{Z}^{-}$
$\mathbf{Z}^{0} = \mathbf{Z}^{00}$
$\mathbf{Z}_{\infty} = \text{span}\,[(i\gamma)^{n} : n = 0, 1, 2, etc.]$.[3]
$\mathbf{Z}^{-\infty\infty} = \mathbf{Z}$ since, if $f \in \mathbf{Z}$ is perpendicular to $\mathbf{Z}^{-\infty\infty}$, then $f^{2}\Delta \in L^{1}(R^{1})$ and $(f\Delta)^{\wedge} = 0$, so that $f\Delta = 0$.

Szegö's alternative can now be stated; for simplicity, it is supposed that Δ has a density relative to Lebesgue measure, denoted by the same letter. The problem is to describe the remote past $\mathbf{Z}^{-\infty}$ (alias $\mathbf{M}^{-\infty}$, alias $\mathbf{F}^{-\infty}$), and this comes out in the following striking way:

$$either \quad \int \frac{lg\,\Delta}{1+\gamma^2} > -\infty, \quad \mathbf{Z}^{-\infty} = 0, \quad and \quad \mathbf{Z}^{-} \neq \mathbf{Z}$$

$$or \quad \int \frac{lg\,\Delta}{1+\gamma^2} = -\infty \quad and \quad \mathbf{Z}^{-\infty} = \mathbf{Z}.$$ [4]

The reader will note that $\int (1+\gamma^2)^{-1}\,\lg^{+}\Delta \leq \int \Delta < \infty$ so that $\int (1+\gamma^2)^{-1}\lg\Delta$ can only diverge to $-\infty$. Δ is said to be a *Hardy weight* if $\int (1+\gamma^2)^{-1}\lg\Delta > -\infty$; the reason will appear in a moment. The reader will also notice that perfect prediction of the future on the basis of the whole past is possible precisely in the non-Hardy case. The dividing line between the Hardy and the non-Hardy case is $\Delta = \exp(-|\gamma|)$, approximately. The reader will note that the polynomials *cannot* span **Z** in the Hardy case $[\mathbf{Z}_{\infty} \subset \mathbf{Z}^{-}]$.

Δ is now assumed to be a Hardy weight, and a theorem of Paley-Wiener [13] is invoked to express Δ as $|h|^2$, in which $h \in L^2(R^1)$ is extensible to an outer Hardy function of class H^{2+} on the open upper halfplane R^{2+},[5] subject to $h^* = h(-\gamma)$ on R^1. Using the language of Hardy functions, the principal results of Levinson-McKean [11] on the Markovian-ness of the process $\mathfrak{x}$ can be stated in terms of the manifolds

$$\mathbf{Z}^{-} \supset \mathbf{Z}^{+/-} \supset \mathbf{Z}^{-} \cap \mathbf{Z}^{+} \supset \mathbf{Z}^{0} \supset \mathbf{Z}_{\infty}$$

as follows:

(1) $\mathbf{Z}^{-} \neq \mathbf{Z}^{+/-}$ *iff* h/h^* *agrees on* R^1 *with the ratio of* 2 *inner Hardy functions.*

(2) $\mathbf{Z}^{+/-} = \mathbf{Z}^{-} \cap \mathbf{Z}^{+}$ *iff* h/h^* *agrees on* R^1 *with an inner Hardy function.*

[3] n takes only such values as make $\int \gamma^{2n}\Delta < \infty$.
[4] Levinson-McKean [11, 103–4] gives a rapid proof of Szegö's alternative.
[5] The reader who is unfamiliar with this terminology will find an explanation in 2.1.

(3) $\mathbf{Z}^{+/-} = \mathbf{Z}^0$ *iff* $1/h$ *is an integral function of minimal exponential type* (see 5.2 for a new proof).
(4) $\mathbf{Z}^0 \neq \mathbf{Z}_\infty$ *is possible* (see 3.5 for an example).

The situation is especially simple (and also well known) if Δ, and with it h, is a rational function:[6]

(5) $\dim \mathbf{Z}^{+/-} = n < \infty$ *iff* Δ *is a rational function of degree* $2n$.

The following easy tests can be applied in this case:

(6) $\mathbf{Z}^0 = \mathbf{Z}^- \cap \mathbf{Z}^+$ *iff* $h \neq 0$ *on* R^1
(7) $\mathbf{Z}^- \cap \mathbf{Z}^+ = \mathbf{Z}^{+/-}$ *if* $h \neq 0$ *on* R^{2-}
(8) $\mathbf{Z}^0 = \mathbf{Z}^{+/-}$ *iff* $h \neq 0$ *on* R^2
(9) $\mathbf{Z}^0 = \mathbf{Z}_\infty$ *automatically.*

The special case that $\mathfrak{x}$ splits over its germ [(8)] is of particular interest since the following statements are the same:

(10) $\mathbf{Z}^0 = \mathbf{Z}^{+/-}$ *is of dimension* n.
(11) $1/h$ *is a polynomial of degree* n.
(12) $D[\mathfrak{x}]$ *is a white noise for some differential operator* D *of degree* n *with constant coefficients* $[D = (2\pi)^{-1/2}h^{-1}(-i\partial)]$.[7]
(13) $\mathbf{F}^{-\infty a}$ *and* $\mathbf{F}^{b\infty}$ *are independent, conditional upon the knowledge of* $\mathfrak{x}(t_k) : k \leq n$ *for any* $-\infty < a = t_1 < \cdots < t_n = b < \infty$.

1.3. de Branges spaces of integral functions. To see more deeply into the structure of $\mathbf{Z}$, it is necessary to introduce the apparatus of de Branges spaces of integral functions. This is done in 3 [de Branges subspaces of $\mathbf{Z}$] and 4 [eigendifferentials] after some simple preparations have been disposed of in 2.

Given an integral function E subject to $|E| > |E^\times|$ on R^{2+},[8] define $\mathbf{B}(E)$ to be the class of integral functions f of $\gamma = a + ib \in R^2$ subject to

$$\text{(1a)} \quad \| f \|^2 = \int | f/E |^2 < \infty,$$

$$\text{(1b)} \quad \| f(\gamma) \|^2 \leq \| f \|^2 \frac{| E(\gamma) |^2 - | E^\times(\gamma) |^2}{4\pi b} \quad (b \neq 0).$$

An elaborate study of such spaces has been made by de Branges [2]–[6], so it is just to call $\mathbf{B}(E)$ the de Branges space based upon E. A wide class of spaces $L^2(\Delta, R^1)$ permit a spectral decomposition [eigendifferential transform] expressed in terms of an increasing family of such spaces. The model for this

[6] Levinson-McKean [11, 122].
[7] ∂ stands for differentiation with respect to time.
[8] $E^\times = [E(\gamma^*)]^*$.

decomposition is the Fourier sine-cosine transform for

$$\mathbf{Z}(1) = L^2(R^1) = L^2(1, R^1):$$

$f \in L^2(R^1)$ is first split into even and odd parts and these are expressed as

(2a) $$f_{\text{even}} = \pi^{-1} \int_0^\infty \cos \gamma T \hat{f}_{\text{even}}\, dT$$

(2b) $$f_{\text{odd}} = \pi^{-1} \int_0^\infty \sin \gamma T \hat{f}_{\text{odd}}\, dT.$$

A theorem of Paley-Wiener [13] tells us that the map

(3a) $$f_{\text{even}} \rightarrow \pi^{-1} \int_0^R \cos \gamma T \hat{f}_{\text{even}}\, dT$$

(3b) $$f_{\text{odd}} \rightarrow \pi^{-1} \int_0^R \sin \gamma T \hat{f}_{\text{odd}}\, dT$$

is the projection onto the class of integral functions of exponential type $\leq R$, and it turns out that this is just the de Branges subspace of $L^2(R^1)$ based upon $E = \exp(-i\gamma R)$.

The eigendifferential transform for $\mathbf{Z}(\Delta]$ *will now be described.* $\mathbf{Z}$ is first split into 3 pieces,

$$\mathbf{Z} = \mathbf{Z}_\infty \oplus \mathbf{Z}^0/\mathbf{Z}_\infty \oplus \mathbf{Z}/\mathbf{Z}^0,$$

and 3 corresponding spectra are introduced: *a discrete spectrum* (for $\mathbf{Z}_\infty$), *a singular spectrum* (for the so-called gap $\mathbf{Z}^0/\mathbf{Z}_\infty$), and a *type spectrum* (for $\mathbf{Z}/\mathbf{Z}^0$). The statement made below about the singular spectrum is merely conjectured, though part of the desired picture could be extracted from de Branges [2]–[5]; the rest is *proved*, subject to a technical condition on the type spectrum to be discussed at the end of this section. The discrete spectrum is visualized as a countable series of points T_n ($n = 0, 1, 2$, etc.), one to each moment $\int \gamma^{2n}\Delta < \infty$, placed as follows:

$$-\infty < T_0 < T_1 < T_2 < \text{etc.} \leq T_\infty \leq 0.$$

$T_\infty = \sup_{n \geq 0} T_n$ is placed to the left of 0 if a gap $\mathbf{Z}^0/\mathbf{Z}_\infty \neq 0$ is present; $T_\infty = 0$ otherwise. The singular spectrum is visualized as the interval $(T_\infty, 0]$ and the *type* spectrum as the interval $(0, \infty)$. On the union of these spectra is placed a pair of positive mass distributions $Q^\pm$. On the discrete spectrum, Q^+ lives only on the even-indexed points, Q^- only on the odd-indexed ones. On the singular spectrum, $Q^\pm$ is jump-free and $\sqrt{dQ^- dQ^+} = 0$, explaining the adjective *singular*, while on the type *spectrum*, $Q^\pm$ is still jump-free but now $\sqrt{dQ^- dQ^+} = dT$.

Given a fixed number $\gamma \in R^2$, let $A = A^T(\gamma)$ and $B = B^T(\gamma)$ be the solution of

(4a) $dA = -\gamma B\, dQ^-$
(4b) $dB = +\gamma A\, dQ^+$

with $A \equiv 1$ and $B \equiv 0$ to the left of T_0, taking them continuous from above at the jumps of $Q^{\pm}$. A and B are the so-called *eigendifferentials* of the title of this section. Then $E = A - iB$ is an integral function with the following properties:

(5a) $E(0) = 1$.
(5b) $E^{\times}(\gamma) = E(-\gamma)$.
(5c) $|E| > |E^{\times}|$ *on* R^{2+}.
(5d) *E is root-free on the closed upper half-plane.*
(5e) *E is of precise exponential type* $\int_{-\infty}^{T} \sqrt{dQ^- \, dQ^+} = \max(0, T)$.
(5f) $[\exp(i\gamma \max(0, T))E]$ *is an outer function of class* H^{2+}, *esp.*,

$$\int (1 + \gamma^2)^{-1} |\lg |E|| < \infty.$$

Define $\mathbf{B}(E)$ to be the de Branges space based upon E. Then *for any* $T < R$, $\mathbf{B}(E^T) \subset \mathbf{B}(E^R) \subset \mathbf{Z}$, *the injections* $\mathbf{B}(E^T) \to \mathbf{B}(E^R) \to \mathbf{Z}$ *are isometric, and* $\mathbf{B}(E^T)$ *becomes dense in* $\mathbf{Z}$ *as* $T \uparrow \infty$. $\mathbf{B}(E)$ has a reproducing kernel expressible as

$$(6) \quad \begin{aligned} J(\alpha, \beta) = J_\alpha(\beta) &= \frac{E(\alpha)^* E(\beta) - E(\alpha^*) E^{\times}(\beta)}{-2\pi i(\beta - \alpha^*)} \\ &= \frac{1}{\pi} \int_{-\infty}^{T+} A(\alpha)^* A(\beta) \, dQ^+ + \frac{1}{\pi} \int_{-\infty}^{T+} B(\alpha)^* B(\beta) \, dQ^-; \end{aligned}$$

this means that $f = (f, J_\bullet)_\Delta = \int f J_\bullet \Delta$ for $f \in \mathbf{B}(E)$. For general $f \in \mathbf{Z}$, the map $f \to (f, J_\bullet)_\Delta$ is the projection upon $\mathbf{B}(E)$, and these projections can be thought of as an *eigendifferential transform pair*,

$$(7a) \quad \hat{f}_{\text{odd}} = \int B f \Delta,$$

$$(7b) \quad \hat{f}_{\text{even}} = \int A f \Delta,$$

mapping $\mathbf{Z}(\Delta)$ onto $L^2(Q^-) \oplus L^2(Q^+)$, with *inverse transform*

$$(8a) \quad f_{\text{odd}} = \pi^{-1} \int B \hat{f}_{\text{odd}} \, dQ^-,$$

$$(8b) \quad f_{\text{even}} = \pi^{-1} \int A \hat{f}_{\text{even}} \, dQ^+,$$

mapping $L^2(Q^-) \oplus L^2(Q^+)$ back onto $\mathbf{Z}(\Delta)$, and a *Plancherel formula*

$$(9a) \quad \pi \| f_{\text{even}} \|_\Delta^2 = \int |\hat{f}_{\text{even}}|^2 \, dQ^+$$

$$(9b) \quad \pi \| f_{\text{odd}} \|_\Delta^2 = \int |\hat{f}_{\text{odd}}|^2 \, dQ^-,$$

expressing the fact that these maps are isometries up to a factor $\sqrt{\pi}$. The projection upon $\mathbf{B}(E)$ is simply the map

$$(10)\quad f \rightarrow \pi^{-1}\int_{-\infty}^{T+} A\hat{f}_{\text{even}}\, dQ^{+} + \pi^{-1}\int_{-\infty}^{T+} B\hat{f}_{\text{odd}}\, dQ^{-};$$

this statement is the analogue of the customary Paley-Wiener theorem for $\mathbf{Z}(1) = L^2(R^1)$. The reader should be advised that while the eigendifferential transform provides an elegant spectral resolution for $\mathbf{Z}(\Delta)$, one does not know what it is the spectral resolution *of*! Undoubtedly, it comes from some important operator which is connected with Δ in a natural way; esp., the mass distributions $Q^{\pm}$ should have some natural probabilistic interpretation, but this has eluded us to date.[9]

The picture can be made more concrete by the actual identification of the spaces $\mathbf{B}(E)$. As T runs over the discrete spectrum, $A(B)$ recapitulates the even (odd) orthogonal polynomials of Δ, and $\mathbf{B}(E)$ runs through the polynomial subspaces

$$\mathbf{Z}_n = \textit{the polynomials of degree } \leq n \textit{ in } \mathbf{Z},$$

ending with

$$\mathbf{Z}_\infty = \textit{the span of polynomials in } \mathbf{Z}.$$

As T runs over the type spectrum $(0, \infty)$, $\mathbf{B}(E)$ runs through the spaces

$$\mathbf{Z}^T = \mathbf{Z}^{-TT} = \bigcap_{n\geq 1} \text{span}\, [e^{i\gamma t} : |t| \leq T + 1/n].$$

$\mathbf{Z}^T$ can also be identified as the class of integral functions $f \in \mathbf{Z}$ of exponential type $\leq T$.[10]

The discussion of (4) on the type spectrum [4.1] requires a hopefully superfluous

TECHNICAL CONDITION. *$\mathbf{Z}^{ab}$ is continuous in the pair ab.*

As is plain from the definition $\mathbf{Z}^{ab} = \bigcap \mathbf{Z}^{cd}$ $(c < a \leq b < d)$, so the meaning of the condition is that $\bigcup \mathbf{Z}^{cd}$ $(a < c < d < b)$ should be dense in $\mathbf{Z}^{ab}$. This is automatic if $a = -\infty$ or $b = +\infty$ since the map $f \rightarrow \exp(i\gamma c)f$ sends $\mathbf{Z}^{ab}$ into $\mathbf{Z}^{a+c,b+c}$ and

$$\lim_{c\downarrow 0} \| \exp(i\gamma c)f - f\|_\Delta = 0.$$

The proof for $-\infty < a < b < \infty$ eludes us, but the following condition is sufficient:

$$\Delta(k\gamma) \leq \textit{constant} \times \Delta(\gamma) \textit{ far out, uniformly as } k \downarrow 1.$$

This covers all examples of any practical significance including the case that $\Delta \in \downarrow$ on $[0, \infty)$ and the case of rational Δ.

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[9] Gelfand-Levitan's paper [7] on the inverse scattering problem is closely related to (4) and (9).

[10] Levinson-McKean [11, 135–142]. A new and simpler proof of this fact will be found in 3.2.

Proof. Clearly, it is enough to deal with the case $-a = b = T > 0$. Pick $f \in \mathbf{Z}^T$ define $f_1 = f(\gamma/k)$, and use the fact, proved in 3.1 that $\mathbf{Z}^T$ is exactly the class of integral functions belonging to $\mathbf{Z}$ of exponential type $\leq T$. By the bound on Δ, the tail of $\int |f_1|^2 \Delta$ is bounded by a constant multiple of the tail of $\int |f|^2 \Delta$ as $k \downarrow 1$. But also f_1 tends to f uniformly on compacts, so f_1 tends to f in $\mathbf{Z}$. At the same time, $f_1 \in \mathbf{Z}^{T/k}$ since it is of exponential type $\leq T/k$. This completes the proof.

1.4. Prediction. The classical prediction problem of Kolmogorov-Szegö-Wiener is to compute the projection upon the past of $\mathfrak{x}(t)$ [alias exp $(i\gamma t)$] fo- fixed $t > 0$. This can be done most elegantly in the language of Hardy funcr tions and Fourier transforms following Karhunen [9] [see 5.1]. Now suppose that only the knowledge of a bounded segment $\mathfrak{x}(s) : -T \leq s \leq 0$ of the past is permitted. In trigonometrical language, the problem is to project exp $(i\gamma t)$ upon $\mathbf{Z}^{-T0}$, or what comes to the same thing, to project exp $[i\gamma(t + T/2)]$ upon the submanifold $\mathbf{Z}^{T/2}$ of integral functions of exponential type $\leq T/2$. The eigendifferential transform described in 1.3 is the appropriate tool for this, and it develops that, *as functions of $a = T/2$ and $b = t + T/2$, the odd and even prediction errors $\sqrt{D^\pm}$ and the total error $\sqrt{D} = \sqrt{D^+ + D^-}$ satisfy*

$$\partial^2 D^+/\partial Q^-(a)\partial Q^+(a) + \partial^2 D^-/\partial Q^+(a)\partial Q^-(a) = \partial^2 D/\partial b^2,$$

under the technical condition $\int \gamma^6 \Delta < \infty$ [see 5.2]. The method also leads to a simple proof of the fact that *the process $\mathfrak{x}$ splits over $\mathbf{F}^{-T0}$, i.e., $\mathbf{F}^{+/-} \subset \mathbf{F}^{-T0}$, iff* exp $(i\gamma T/2)h$ *is the reciprocal of an integral function of exponential type $\leq T/2$.*[11]

The problem of prediction on the basis of a bounded segment of the past has been studied in the electrical literature; see, esp. Ragazzini-Zadeh [14] for the case of rational Δ. Grenander-Szegö [8, pp. 188–191] also studied this problem for stationary chains [see 5.2.9)].

1.5. Germ and gap. From a probabilistic point of view the presence of a gap $\mathbf{Z}^0/\mathbf{Z}_\infty \neq 0$ is something of a conundrum.

$\mathbf{F}^{+/-} = \mathbf{F}^-$ iff h/h^* cannot be expressed on R^1 as the ratio of 2 inner functions; in this case, the minimal splitting field is the whole past and the process may be called anti-Markovian. At the opposite extreme stands the case $\mathbf{F}^{+/-} = \mathbf{F}^0$ in which the process splits over its germ [see 5.2.7]. If no gap intervenes $[\mathbf{F}^{+/-} = \mathbf{F}^0 = \mathbf{F}_\infty]$, the situation is simple enough: *the splitting* field is just the field of $\partial^n \mathfrak{x}(0) : n = 0, 1, 2$, etc. *and* the process

$$\mathfrak{z}(t) = [\mathfrak{x}(t), \partial\mathfrak{x}(t), \partial^2\mathfrak{x}(t), \text{etc.}]$$

is a nice Gaussian diffusion in a possibly infinite number of dimensions. If

[11] Levinson-McKean [11, 121–123] proved this for $T = 0$.

$\dim \mathbf{M}_\infty = n + 1 < \infty$, which is the same as to say that $1/h$ is a polynomial of degree $n + 1$, then

$$\mathfrak{z} = [\mathfrak{x}, \partial\mathfrak{x}, \cdots, \partial^n\mathfrak{x}]$$

is a diffusion on R^{n+1} with (singular) generator

$$\mathbf{G} = x_1 \frac{\partial}{\partial x_0} + x_2 \frac{\partial}{\partial x_1} + \cdots + x_n \frac{\partial}{\partial x_{n-1}} - c_{n+1}^{-1}[c_0 x_0 + \cdots + c_n x_n] \frac{\partial}{\partial x_n} + \frac{1}{2} c_{n+1}^{-2} \frac{\partial^2}{\partial x_n^2},$$

as the reader can easily check using the fact that the differential operator

$$\mathbf{D} = [c_0 + \cdots + c_{n+1}\, \partial^{n+1}] = [\sqrt{2\pi} h(-i\partial)]^{-1}$$

sends $\mathfrak{x}$ into a standard white noise [1.2.12]. But if $1/h$ is a transcendental function, the picture is not so simple. $\mathfrak{z}$ is still a diffusion, but the part $(\frac{1}{2})c_{n+1}^{-2}\, \partial^2/\partial x_n^2$ of $\mathbf{G}$ disappears over the horizon, and it is not clear how $\mathbf{G}$ should be expressed. The situation is even more confusing if a gap is present. Assume for conversation's sake that $\dim \mathbf{M}_\infty = \infty$. Then the germ $\mathfrak{z}$ at time $t \in R^1$ still performs a Gaussian diffusion on R^∞, but it is not fully described by $[\mathfrak{x}, \partial\mathfrak{x}$, etc.] any more: *it is necessary to adjoin an infinite number of local functionals from* $\mathbf{M}^0/\mathbf{M}_\infty$ *to fill out the rest.* The example of 3.5 shows that this peculiar situation can actually occur, but it is an open problem to make a concrete picture of it.

2. Preparations

At this place, it is convenient to list for future use some simple facts about Hardy functions, integral functions of exponential type, and de Branges spaces.

2.1 Hardy functions. Paley-Wiener [13] first proved the basic facts about this class of functions. Bring in the Poisson kernel $p_b(a) = (b/\pi)(a^2 + b^2)^{-1}$; let $p_b \circ f(a)$ be the customary convolution on R^1:

$$\int p_b(a - c) f(c)\, dc.$$

A function $f = f(a + ib)$ defined on R^{2+} is a *Hardy function* there [$f \in H^+$ for short] if

(1a) *f is analytic on R^{2+},*
(1b) $\lim_{b \downarrow 0} f(a + ib) = f(a)$ *exists a.e. on R^1,*
(1c) $\int (1 + a^2)^{-1} \lg^+ |f| < \infty$,
(1d) $\lg |f(a + ib)| \leqq p_b \circ \lg |f|\ (a)$ *on R^{2+}.*

Because of (1c) and (1d), either $f \equiv 0$ or $(1 + a^2)^{-1} \lg |f| \in L^1(R^1)$. According to Beurling's nomenclature, $f \in H^+$ is an *outer function* if $f \not\equiv 0$ and (1d) can be sharpened to

(1e) $\lg |f(a + ib)| = p_b \circ \lg |f|\ (a)$ *on R^{2+},* in which case $1/f$ is likewise an outer function. $f \in H^+$ is an *inner function* if

(2a) $|f| = 1$ *a.e. on* R^1
(2b) $|f| \leq 1$ *on* R^{2+}.

Given $f \in H^+$, the expression

$$(3a) \quad f_{\text{out}}(\alpha) = \exp\left[\frac{1}{\pi i}\int \frac{\alpha\beta + 1}{\beta - \alpha} \lg |f| \frac{d\beta}{1 + \beta^2}\right]$$

is an outer Hardy function on R^{2+} with

$$(3b) \quad \lg |f_{\text{out}}(a + ib)| = p_b \circ \lg |f| \ (a),$$

and $f_{\text{in}} = f/f_{\text{out}}$ is inner. This shows that any $f \in H^+$ is the product of an inner and outer part, and it is easy to see that this splitting is unique up to a multiplicative constant of modulus 1.

H^- denotes the class of Hardy functions on the open lower half-plane R^{2-}. The definition is similar and $f \in H^-$ iff $f^\times \in H^+$.

H^{2+} is the class of functions $f \in H^+$ for which $f_{0+}(a) = \lim_{b \downarrow 0} f(a + ib)$ belongs to $L^2(R^1)$; for such functions, $f_b(a) = f(a + ib)$ also belongs to $L^2(R^1)$ for any $b > 0$, and

$$(4) \quad \|f\| \equiv \|f_{0+}\|_2 = \sup_{b>0} \|f_b\|_2\,;$$

esp., f is completely determined on R^{2+} by the function f_{0+} on R^1, so it is permissible to confuse them. (4) leads to an alternative description of H^{2+} as the class functions f defined on R^{2+} satisfying 1(a) and

$$(5) \quad \sup_{b>0} \|f_b\|_2 < \infty.$$

A third description of H^{2+} is as one-sided Fourier transforms.[12] Define the inverse Fourier transform $\hat{f} = (2\pi)^{-1/2} \int \exp(-iat)f$. Then $f \in L^2(R^1)$ can be extended over R^{2+} to a function of class H^{2+} iff $\hat{f} = 0$ on the left half-line. Because of this and the corresponding statement for H^{2-}, $L^2(R^1) = H^{2-} \oplus H^{2+}$; in fact, this splitting is just the image under the (forward) Fourier transform of the perpendicular splitting $L^2(R^1) = L^2(-\infty, 0] \oplus L^2[0, \infty)$.

A non-negative function $\Delta \in L^1(R^1)$ can be expressed as $|h|^2$ for some outer function $h \in H^{2+}$ iff

$$(6a) \quad \int (1 + \gamma^2)^{-1} \lg \Delta > -\infty,$$

as is clear from the bound

$$(6b) \quad \int (1 + \gamma^2)^{-1} \lg^+ \Delta \leq \int \Delta < \infty$$

and the formula (3a) with $\sqrt{\Delta}$ in place of f. h can be made to satisfy $h^* = h(-\gamma)$ on R^1 if Δ is even; this makes $\hat{h}$ real.

---

[12] Paley-Wiener [13].

A function $h \in H^{2+}$ is outer if (1e) holds, but the condition can also be expressed in any of the following ways:

(7a) $\lg | h(a + ib) | = p_b \circ \lg | h | \ (a)$ *at a single point of* R^{2+}, *for example at* i.
(7b) $[e^{i\gamma t}h : t \geq 0]$ *spans* H^{2+}.
(7c) $| h | \geq | f |$ *on* R^{2+} *for any* $f \in H^{2+}$ *with the same modulus as* h *on* R^1.
(7d) $\int_0^t | \hat{h} |^2 \geq \int_0^t | \hat{f} |^2$ *for* $t \geq 0$ *and any* $f \in H^{2+}$ *with the same modulus as* h *on* R^1.

(7a) is the simplest test in practice. (7b) can be used to make a slick proof of

(8) $\displaystyle \frac{1}{2\pi i} \int \frac{f(\beta)}{\beta - \alpha} \, d\beta = 0$ *or* $f(\alpha)$ *according as* α *belongs to* R^{-2} *or* R^{2+}:

simply check (8) for $f = \exp(i\gamma t)(1 - i\gamma)^{-1}$ $(t \geq 0)$, note that $(1 - i\gamma)^{-1}$ is an outer function so that the functions f span out the whole of H^{2+}, and use the fact that evaluation at $\alpha = a + ib \in R^{2+}$ is a continuous application of H^{2+} thanks to (1d) and the resulting bound

(9) $| f(a + ib) |^2 \leq \exp [p_b \circ \lg | f |^2 \ (a)] \leq p_b \circ | f |^2 \ (a) \leq \| f \|_2^2 / \pi b$.

Proof of (7). (7a) is plain from (1d) and the fact that a non-negative harmonic function cannot have a root unless it is $\equiv 0$. (7b) is the same as to say that for $f \in H^{2+}$,

$$\text{span } [e^{i\gamma t} f : t \geq 0] = f_{\text{in}} H^{2+},$$

which is easy to check by computing annihilators in $L^1(R^1)$. (7c) is immediate from the fact that $f = f_{\text{out}} \times f_{\text{in}}$. (7d) can be found, stated a little differently in Robinson [15]; see 5.1 for a quick proof.

A couple of simple bounds for outer Hardy functions are deduced from (1e) for future use:

(10a) $\lg | h(a + ib) | \geq -\text{constant} \times (1 + a^2)$ *for fixed* $b > 0$
(10b) $| h(\text{Re}^{i\theta}) | = e^{o(R)}$ *for* $R \uparrow \infty$ *and fixed* $0 < \theta < \pi$.

Proof of (10a). Divide the integral of (1e) into 2 parts, one for $| c | \leqq 2a$, one for $| c | > 2a$, and estimate as follows:

$$\lg | h(a + ib) | \geq -\frac{1 + 4a^2}{\pi b} \int_{|c| \leq 2a} \frac{\lg^- | h |}{1 + c^2} - \frac{b}{\pi} \int_{|c| > 2a} \frac{\lg^- | h |}{c^2/4 + b^2}$$
$$\geq -\text{constant} \times (1 + a^2).$$

Proof of (10b). $| \text{Re}^{i\theta} - c |^2 \geq (1 - | \cos \theta |)(R^2 + c^2)$, so

$$R^{-1} | \log | h | (\text{Re}^{i\theta}) | \leq \frac{\sin \theta}{\pi} \int \frac{\lg | h |}{| \text{Re}^{i\theta} - c |^2}$$
$$\leq \frac{\sin \theta}{\pi(1 - | \cos \theta |)} \int \frac{| \log | h | |}{R^2 + c^2}$$
$$= o(1).$$

2.2. Integral functions of exponential type. Boas [1] is the best general reference for this subject. Boas [1, p. 92] proves that if $f \not\equiv 0$ is an integral function of exponential type, i.e.,

(1) $T = \overline{\lim}_{R\uparrow\infty} R^{-1} \max_{0\le\theta<2\pi} \lg |f(\mathrm{Re}^{i\theta})| < \infty,$

and if

(2a) $\int (1+\gamma^2)^{-1} \lg^+ |f| < \infty,$

then

(3a) $\lg |f(a+ib)| \le p_b \circ \lg |f|\ (a) + kb \text{ on } R^{2+}$

with

(3b) $k = \lim_{R\uparrow\infty} R^{-1} \dfrac{2}{\pi} \displaystyle\int_0^\pi \lg |f(\mathrm{Re}^{i\theta})| \sin\theta\, d\theta,$

esp.,

(2b) $\int (1+\gamma^2)^{-1} \lg^- |f| < \infty.$

If f is root-free on R^{2+}, then (3a) can be improved to

(4a) $\lg |f(a+ib)| = p_b \circ \lg |f|\ (a) + kb \text{ on } R^{2+}$

with

(4b) $k = \lim_{b\uparrow\infty} b^{-1} \lg |f(ib)|.$

A useful consequence is the following fact:

(5) *if* $\Delta = |h|^2$ *on* R^1 *for some outer function* $h \in H^{2+}$ *and if f ian integrals function of exponential type* $\le T$ *which belongs to* $\mathbf{Z}(\Delta) = L^2(\Delta, R^1)$, *then* $\exp(i\gamma T) fh \in H^{2+}$.

Proof of (5). $f \in \mathbf{Z}(\Delta)$ and $\int (1+\gamma^2)^{-1} \lg^- |h| < \infty$ so

$$\lg^+ |f|^2 \le \lg^+ |fh|^2 + \lg^- |h|^2 \le |f|^2 \Delta + \lg^- |h|^2$$

is summable on R^1 relative to the weight $(1+\gamma^2)^{-1}$, and (2a) holds. Because $k =$ (3b) cannot exceed the type T, it follows from (3a) that

$$\lg |e^{i\gamma T} f| \le p_b \circ \lg |f|\ (a) \text{ for } \gamma = a + ib \in R^{2+}.$$

But also h is outer, so

$$\lg |e^{i\gamma T} fh| \le p_b \circ \lg |fh|\ (a),$$

and $\exp(i\gamma T) fh \in H^+ \cap L^2(R^1) = H^{2+}$, as stated.

A convenient form of the Phragmen-Lindelöf principle is stated for future use:

(6) *if an integral function f is of exponential type in a sector* $D : \alpha \le \theta \le \beta$ *of opening* $\beta - \alpha < \pi$, *i.e., if*

$$\overline{\lim}_{R\uparrow\infty} R^{-1} \max_{\alpha\le\theta\le\beta} \lg |f(\mathrm{Re}^{i\theta})| < \infty,$$

then

$$\sup_D |f| \le \sup_{\partial D} |f|.$$

2.3. de Branges spaces. Given an integral function e with $|e| > |e^{\times}|$ on R^{2+} and $e^{\times} = e(-\cdot)$, define the associated de Branges space $\mathbf{B}(e)$ to be the class of integral functions f of $\gamma = a + ib$ such that

$$(1) \quad \|f\|^2 = \int |f/e|^2 < \infty$$

and

$$(2a) \quad |f(\gamma)|^2 \le \|f\|^2 \frac{|e(\gamma)|^2 - |e^{\times}(\gamma)|^2}{4\pi b} \qquad (b \ne 0).$$

The principal task of this section is to prove that $\mathbf{B}(e)$ *is a Hilbert space with reproducing kernel*

$$(3) \quad J_\alpha(\beta) = J(\alpha, \beta) = \frac{e(\alpha)^* e(\beta) - e(\alpha^*) e^{\times}(\beta)}{-2\pi i(\beta - \alpha^*)}.$$

This means that $J_\alpha \in \mathbf{B}(e)$ for $\alpha \in R^2$, and

$$(4) \quad f = (f, J_\bullet) = \int f J_\bullet^* |e|^{-2};$$

esp., for any $\gamma = a + ib \in R^2$,
(5) $\max |f(\gamma)|^2$ *for* $f \in \mathbf{B}(e)$ *with* $\|f\| \le 1$

$$= J(\gamma, \gamma)$$

$$= \frac{|e(\gamma)|^2 - |e^{\times}(\gamma)|^2}{4\pi b} \qquad (b \ne 0)$$

$$= \frac{|e(\gamma)|^2}{\pi} \theta^{\bullet} \qquad (b = 0)$$

in which $\theta = -\text{phase } e$ and $\theta^{\bullet} = \partial\theta/\partial a$. Under the condition (1), (2a) is the same as the milder looking bound

$$(2b) \quad |f(\gamma)|^2 \le \text{constant} \times \frac{|e(\gamma)|^2 - e^{\times}(\gamma)|^2}{4\pi b} \quad (b \ne 0).$$

The results of this section can all be found in de Branges [2]–[5].

Proof of (4) *from* (1) *and* (2b). Given an integral function f subject to (1) and (2b) and a point $\alpha \in R^{2+}$, the estimate

$$\int_0^\pi \left|\frac{f}{e}\right| (\text{Re}^{i\theta})\, d\theta \leq \text{constant} \times \int_0^\pi \frac{d\theta}{\sqrt{R \sin\theta}} = o(1)$$

justifies the evaluation

$$\frac{f}{e}(\alpha) = \frac{1}{2\pi i} \int \frac{f}{e} \frac{d\beta}{\beta - \alpha}.$$

A similar proof justifies

$$0 = \frac{1}{2\pi i} \int \frac{f}{e^*} \frac{d\beta}{\beta - \alpha},$$

and (4) follows for $\alpha \in R^{2+}$. The reader will check that the same formula holds on the rest of R^2.

Proof that (2a) = (2b) *under the condition* (1). The same line of proof shows that J reproduces itself: $J(\alpha, \beta) = (J_\alpha, J_\beta)$, so (4) now implies

$$|f(\alpha)|^2 = |(f, J_\alpha)|^2 \leq \|f\|^2 \|J_\alpha\|^2 = \|f\|^2 J(\alpha, \alpha),$$

and this is just the bound (2a). $J_\alpha \in \mathbf{B}(e)$ follows at no extra cost from

$$|J_\alpha(\beta)|^2 = |(J_\alpha, J_\beta)|^2 \leq \|J_\alpha\|^2 J(\beta, \beta),$$

which is just (2a) for $f = J_\alpha$.

Proof that $\mathbf{B}(e)$ *is a Hilbert space.* The only troublesome point is to see why $\mathbf{B}(e)$ is closed under addition, and that is plain from (2b).

Proof of (5) should be plain.

The following additional facts about de Branges spaces will be of general use:

(6) $\mathbf{B}(e)$ *is the same as* $\mathbf{B}(e_1)$ *(inner products included) iff*

$$e_1 = k(e^\times + e)/2 - k^{-1}(e^\times - e)/2$$

for some real number $k \neq 0$, *esp.,* e *is completely determined by* $\mathbf{B}(e)$ *under the additional condition* $e(0) = 1$. $J = (3)$ *is unaffected by this transformation.*

(7) *A de Branges space is closed under the map* $f \to f^\times$; *it is also closed under the map*

$$f \to f_\alpha(\beta) = (f(\beta) - f(\alpha))/(\beta - \alpha)$$

for any $\alpha \in R^2$, *if* $1 \in \mathbf{B}(e)$.

Proof of (6). Define e_1 as in (6) for $k \neq 0$. Then $e_1^\times = e_1(-\,\cdot\,)$, and it is easy to check that the function $J = (3)$ based upon e_1 is the same as that based upon e, esp., $4\pi b J(\gamma, \gamma) = |e_1|^2 - |e_1^\times|^2 > 0$ on R^{2+}, so that e_1 defines a de

Branges space $\mathbf{B}(e_1)$. But then $\mathbf{B}(e)$ and $\mathbf{B}(e_1)$ have the same reproducing kernel, esp., $J_\omega : \omega \in R^2$ spans them both, and the stated identification follows from

$$J(\alpha, \beta) = (J_\alpha, J_\beta) = \int J_\alpha J_\beta^* |e|^{-2} = (J_\alpha, J_\beta)_1 = \int J_\alpha J_\beta^* |e_1|^{-2}.$$

The converse is even simpler to prove.

Proof of (7). $\mathbf{B}(e)$ is closed under the map $f \to f^\times$ since the bound (2a) is symmetrical about R^1. Now consider the map $f \to f_\alpha$ for fixed $\alpha \in R^2$. f_α is an integral function, $\|f_\alpha\| < \infty$, and, if $1 \in \mathbf{B}(e)$, then $1 \le \|1\|^2 J(\beta, \beta)$ and $|f_\alpha| \le$ constant $\times$ $J(\beta, \beta)$, which is just the bound (2b) for f_α. But (2a) $=$ (2b) in the presence of (1), so $f_\alpha \in \mathbf{B}(e)$, as stated.

3. De Branges Subspaces of Z

Given an even non-negative (spectral) weight $\Delta \in L^1(R^1)$ subject to the Hardy condition $\int (1+\gamma^2)^{-1} \lg\Delta > -\infty$, put $\Delta = |h|^2$ for some outer function $h \in H^{2+}$ with $h^* = h(-\gamma)$ on R^1 and introduce the space $\mathbf{Z} = \mathbf{Z}(\Delta) \equiv L^2(\Delta, R^1)$ of measurable functions f defined on R^1 with

$$\|f\| = \|f\|_\Delta = \sqrt{\int |f|^2\Delta} < \infty.$$

The principal task of this part is the identification of

$$\mathbf{Z}^T = \mathbf{Z}^T(\Delta) = \cap_{n\ge 1} \text{span}\,[e^{i\gamma t} : |t| \le T + 1/n]$$

for $T \ge 0$, first as a de Branges space [3.1], and second as the class of all integral functions f belonging to $\mathbf{Z}$ of exponential type $\le T$ [3.2]. The fine structure of $\mathbf{Z}^0$ is also examined, with special attention to the gap $\mathbf{Z}^0/\mathbf{Z}_\infty$.

3.1. $\mathbf{Z}^T$ as a de Branges space. The purpose of this section is to prove that $\mathbf{Z}^T$ is a de Branges space based upon an integral function E satisfying (1.3.5):

(1) $E(0) = 1$.
(2) $E^\times(\gamma) = E(-\gamma)$.
(3) $|E| > |E^\times|$ *on* R^{2+}.
(4) E *is root-free on the closed upper half-plane.*
(5) E *is of precise exponential type* T.
(6) $[\exp(i\gamma T)E]^{-1}$ *is an outer function of class* H^{2+}, *esp.*,
$$\int (1+\gamma^2)^{-1} |\lg|E|| < \infty.$$

(1) is imposed to make $A = 2^{-1}(E^\times + E)$ and $B = (2i)^{-1}(E^\times - E)$ satisfy a pair of coupled first order differential equations in $T > 0$ for fixed $\gamma \in R^2$, as will be proved in 4.1 under an extra technical assumption. $E = A - iB$ stands in the same relation to $\mathbf{Z}^T = \mathbf{B}(E)$ as $\exp(-i\gamma T) = \cos\gamma T - i \sin\gamma T$ does to

$$\mathbf{Z}^T(1) = \left(\int_{|t|\le T} e^{i\gamma t} f(t)\,dt : \int_{|t|\le T} |f|^2 < \infty\right) = \mathbf{B}(e^{-i\gamma T}).$$

The reader will recognize this identity as a statement of the Paley-Wiener theorem for $\mathbf{Z}(1) = L^2(R^1)$.

The proof is made by identifying $\mathbf{Z}^T$ as the de Branges space based upon the function

(7) *e = the projection of* $[\exp(i\gamma T)h]^{-1}$ *upon* $\mathbf{Z}^T(\Delta^+)$, *divided by the* Δ^+ *norm of this projection,*

Δ^+ being the jacked-up weight $\pi^{-1}\Delta/(1 + \gamma^2)$, and then adjusting e by means of (2.3.6) to get (1). The reader will deduce (2)–(6) from the corresponding properties of e; only (6) offers any difficulty, and this is easily overcome using (2.2.4). The actual proof is divided into 8 steps. Anticipating future use, it is convenient to prove a little more than is needed for the present purpose [step 8].

STEP 1. *$\mathbf{Z}^T(\Delta)$ is inhabited by integral functions of exponential type $\leq T$, esp., the map $f \to f(\omega)$ is a continuous application of $\mathbf{Z}^T(\Delta)$ for any $\omega \in R^2$, and there is a bound $|f(\omega)| \leq$ constant $\times \|f\|_\Delta$ with a constant not depending upon f which is uniform for ω confined to a compact figure of R^2, esp., $e \in \mathbf{Z}^T(\Delta^+)$ is an integral function of exponential type $\leq T$.*

STEP 2. *$\mathbf{Z}^T(\Delta)$ is closed under the maps $f \to f^\times$ and*

$$f \to f_\alpha(\beta) = (f(\beta) - f(\alpha))/(\beta - \alpha);$$

in fact, $f_\alpha \in \mathbf{Z}^T(\Delta)$ for any $f \in \mathbf{Z}^T(\Delta^+)$.

STEP 3. $\int e^* f \Delta^+ = (f/e)(i)$ *for any $f \in \mathbf{Z}^T(\Delta^+)$.*

STEP 4.

$$J_\alpha(\beta) = J(\alpha, \beta) = \frac{e(\alpha)^* e(\beta) - e(\alpha^*) e^\times(\beta)}{-2\pi i(\beta - \alpha^*)}$$

is the reproducing kernel for $\mathbf{Z}^T(\Delta)$, i.e., $J_\alpha \in \mathbf{Z}^T$ for $\alpha \in R^2$, and $f = (f, J_\bullet)_\Delta$ for any $f \in \mathbf{Z}^T$, esp.,
max $|f(\gamma)|^2$ *for $f \in \mathbf{Z}^T(\Delta)$ with $\|f\| \leq 1$*

$$= J(\gamma, \gamma) = \frac{|e(\gamma)|^2 - |e^\times(\gamma)|^2}{4\pi b} \quad (b \neq 0)$$

$$= \frac{|e(\gamma)|^2}{\pi} \theta^\bullet \quad (b = 0)$$

in which $\gamma = a + ib$, $\theta = -$phase e, and $\theta^\bullet = \partial\theta/\partial a$.

STEP 5. *$e^\times = e(-\gamma)$, $|e| > |e^\times|$ on R^{2+}, and e is root-free on the closed upper half-plane; also, e is of precise exponential type T.*

STEP 6. $[\exp(i\gamma T)e]^{-1}$ *is an outer Hardy function of class H^{2+}.*

STEP 7. *$\mathbf{Z}^T(\Delta) = \mathbf{B}(e)$ inner products included, esp., for any $f \in \mathbf{Z}^T(\Delta)$,*

(8a) $\|f\|_\Delta^2 = \int |f/e|^2;$

also,

(8b) $\|f\|^2_{\Delta^+} = \int |f|^2[\pi(1+\gamma^2)|e|^2]^{-1}.$

STEP 8. $\pi \int J(\gamma,\gamma)\Delta^+ = \pi^{-1}\int (1+\gamma^2)^{-1}\theta^\bullet < \infty.$

WARNING. (8a) holds with E in place of e, but step 3 and (8b) do not.

Proof of Step 1 (adapted from Levinson-McKean [11, p. 114]). Given $f \in \mathbf{Z}^T(\Delta)$, choose a trigonometrical sum $f^n = k_1 \exp(i\gamma t_1) +$ etc. with $|t_1|$, etc. $\le T + 1/n$ and $\|f - f^n\|_\Delta \le 1/n$. Pick $c_1 > T$ and n so large that $c_1 > T + 1/n$. Then $\exp(i\gamma c_1)f^n h \in H^{2+}$, so by (2.1.9),

(9a) $|e^{i\gamma c_1}f^n h|^2\,(a+ib) \le \|f^n h\|_2^2/\pi b = \|f^n\|_\Delta^2/\pi b \quad (b > 0);$

similarly,

(9b) $|e^{-i\gamma c_1}f^n h^*|^2\,(a+ib) \le \|f^n\|_\Delta^2/\pi\,|b| \quad (b < 0)$

since $\exp(-i\gamma c_1)f^n h^* \in H^{2-}$. Recall the bounds (2.1.10). By (9) and (2.1.10a),

(10) $|\exp(-c_2\,\gamma^2)f^n| \le c_3\,\|f^n\|_\Delta \quad (b = \pm 1)$

with constants c_2 and c_3 not depending upon f or upon n. Because the left side of (10) tends to 0 at the ends of the strip $b \le 1$, it lies under the same bound in the whole strip, and esp.,

$$|f^n| \le c_4\,\|f^n\|_\Delta \quad \textit{in the disc } |\gamma| \le 1.$$

An application of (2.1.10b) to (9) now implies that for any $c_5 > 0$ and $n \uparrow \infty$,

(11a) $|e^{i\gamma T}f^n|\,(\mathrm{Re}^{i\theta}) \le c_6\,\|f^n\|_\Delta\,e^{c_5 R} \quad (\theta = \pi/4, 3\pi/4)$
(11b) $|e^{-i\gamma T}f^n|\,(\mathrm{Re}^{i\theta}) \le c_6\,\|f^n\|_\Delta\,e^{c_5 R} \quad (\theta = 5\pi/4, 7\pi/4)$

with a constant c_6 depending upon c_5 alone, and now the Phragmen-Lindelöf principle (2.2.6) applied to each of the 4 sectors between $\theta = \pi/4, 3\pi/4, 5\pi/4$, and $7\pi/4$ shows that for $n \uparrow \infty$,

(12) $|f^n| \le c_8\,\|f^n\|_\Delta\,e^{c_7 R}$ *on the whole of* R^2

for any $c_7 > T$ and some c_8 depending upon c_7 alone. Now pick $n = n_1 < n_2 <$ etc. $\uparrow \infty$ so as to make f^n converge locally uniformly on R^2 to an integral function f^∞. $f = f^\infty$ a.e. on R^1, so they may be identified, and f^∞ can be viewed as an integral extension of f. (12) applies to this extension with the same constants, and that finishes the proof of Step 1.

Proof of Step 2. $f \to f^\times$ maps $\mathbf{Z}^T(\Delta)$ into itself since $\exp(i\gamma t)^\times = \exp(-i\gamma t)$. As to the map $f \to f_\alpha$, pick a trigonometrical sum $f^n \in \mathbf{Z}^{T+1/n}(\Delta^+)$ approximating $f \in \mathbf{Z}^T(\Delta^+)$ as in the proof of Step 1: $\|f - f^n\|_{\Delta^+} < 1/n$. Using the fact that

$$(e^{i\beta t})_\alpha = \frac{e^{i\beta t} - e^{i\alpha t}}{\beta - \alpha} = i\int_0^t e^{i\alpha(t-s)}e^{i\beta s}\,ds \in \mathbf{Z}^t(\Delta),$$

and the bound of Step 1, one finds that $\| f_\alpha - f_\alpha^n \|_\Delta \le$ constant $\times \| f - f^n \|_{\Delta^+}$.

$$f_\alpha \in \bigcap_{n \ge 1} \mathbf{Z}^{T+1/n}(\Delta) = \mathbf{Z}^T(\Delta)$$

follows, and the proof is complete.

Proof of Step 3. By the definition of e,

$$e^* = k \times \textit{the projection of } e^{i\gamma T}/h^* \textit{ upon } \mathbf{Z}^T(\Delta^+).$$

with a constant $k > 0$ chosen so that $\| e \|_{\Delta^+} = 1$. This definition is possible since the projection is $\not\equiv 0$ in view of (2.1.8) applied to exp $(i\gamma T)h/(\gamma + i) \in H^{2+}$:

$$\int \frac{e^{i\gamma T}}{h^*} \Delta^+ = \int \frac{e^{i\gamma T} h}{\pi(1 + \gamma^2)} = \bar{e}^T h(i) \neq 0.$$

A second application of (2.1.8) to exp $(i\gamma T)fh/(\gamma + i) \in H^{2+}$ [see 2.2.5] gives

$$\int e^* f \Delta^+ = k \int \frac{e^{i\gamma T} fh}{\pi(1 + \gamma^2)} = k\bar{e}^T fh(i),$$

and the stated formula follows upon putting $f = e$ and eliminating $k = \| e \|_{\Delta^+}^2 e^T/eh(i)$.

Proof of Step 4. Given $f \in \mathbf{Z}^T(\Delta)$ and $\alpha = a + ib \in R^2$, $(1 + \beta^2)f_\alpha \in \mathbf{Z}^T(\Delta^+)$, as the reader will deduce from Step 2. Because this function has a root at $\beta = i$, Step 3 gives

$$0 = \int e^*(1 + \beta^2) f_\alpha \Delta^+ = \int \frac{e^*}{\pi} \frac{f(\beta) - f(\alpha)}{\beta - \alpha} \Delta.$$

Replacing f_α by $f_{\alpha^*}^\times$ and conjugating the result gives

$$0 = \int e(1 + \beta^2) f_\alpha \Delta^+ = \int \frac{e}{\pi} \frac{f(\beta) - f(\alpha)}{\beta - \alpha} \Delta,$$

and combining these two formulas, it develops that

(13) $(f, J_\alpha)_\Delta = f(\alpha)(1, J_\alpha)_\Delta$.

Now $J_\alpha \in \mathbf{Z}^T(\Delta)$ by Step 2; also, by Step 1, since $e \not\equiv 0$, $\| J_\bullet \|_\Delta$ has at most a countable number of roots. An application (13) with J_α in place of f gives

$$\| J_\alpha \|_\Delta^2 = J(\alpha, \alpha)(1, J_\alpha)_\Delta \le J(\alpha, \alpha) \| 1 \|_\Delta \| J_\alpha \|_\Delta .$$

This shows that the integral function $(1, J_\bullet)_\Delta$ is non-negative; as such, it is constant, and this constant may be evaluated as 1 by putting $\alpha = i$ and using Step 3 to compute

$$(1, J_i)_\Delta = \int \frac{e(i)e^* - e(i)^\times e}{2\pi i(\gamma - 1)} \Delta$$

$$= \frac{e(i)}{2i} \int e^*(\gamma + i)\Delta^+ - \frac{e(i)^\times}{2i} \int e(\gamma + i)\Delta^+$$

$$= 1 - 0 = 1.$$

The rest of Step 4 is obvious.

Proof of Step 5. $e^{\times} = e(-\gamma)$ follows from the fact that $f = [\exp(i\gamma T)h]^{-1}$ satisfies $f^{\times} = f(-\gamma)$ on R^1, since the projection upon $\mathbf{Z}^T(\Delta^+)$ commutes with the map $f \to f^{\times}$. The rest follows from the evaluation of $J(\gamma, \gamma)$ in Step 4 and from the bounds

$$1 \leq \| 1 \|_\Delta^2 J(\gamma, \gamma) \quad \text{and} \quad e^{2bT} \leq \| e^{-i\gamma T} \|_\Delta^2 J(ib, ib) \leq | e(ib) |^2 \ (b \uparrow \infty).$$

Proof of Step 6. $e \in \mathbf{Z}^T(\Delta^+)$, so $\exp(i\gamma T)eh/(1 - i\gamma) \in H^{2+}$ by (2.2.5), and since $h/(1 - i\gamma)$ is an outer Hardy function, $\exp(i\gamma T)e$ is itself a Hardy function. Now the additional fact that e is of exponential type and root-free on the closed upper half-plane permits the application of (2.2.4) to the function $\exp(i\gamma T)e$:

$$\lg | e^{i\gamma T} e | \, (a + ib) = p_b \circ \lg | e | \, (a) + kb \text{ on } R^{2+}$$

with k as in (2.2.4b),

$$k = \lim_{b \uparrow \infty} b^{-1} \lg | e^{i\gamma T} e | \, (ib).$$

By the final bound of the proof of Step 5, $k = 0$, and the proof of Step 6 is completed by anticipating (8a), which states that $\| e^{-1} \|_2 = \| 1 \|_\Delta < \infty$.

Proof of Step 7. Step 5 shows that a de Branges space $\mathbf{B}(e)$ may be based upon e. By Step 4, $\mathbf{Z}^T(\Delta)$ and $\mathbf{B}(e)$ have the *same* reproducing kernel, so they are identical (inner products included), and (8a) follows. As to (8b), simply note from Step 2 that $f_1 = (\gamma - i)^{-1}[f(i)e - e(i)f]$ belongs to $\mathbf{Z}^T(\Delta)$ and compute $\| f_1 \|_\Delta$ using (8a), $\| e \|_{\Delta^+} = 1$, and the formula of Step 3.

Proof of Step 8. The fact that $| e | > | e^{\times} |$ on R^{2+} and $e^{\times} = e(-\gamma)$ permits us to express e as a Hadamard product of the special form[13]

$$e = e(0)e^{ik\gamma} \prod_{n=1}^{\infty} \left(1 - \frac{\gamma}{ic_n}\right) \prod_{n=1}^{\infty} \left(1 - \frac{\gamma}{a_n + ib_n}\right)\left(1 - \frac{\gamma}{-a_n + ib_n}\right)$$

with

$$k \in R^1, \quad a_n > 0, \quad b_n < 0, \quad c_n < 0, \quad \sum c_n^{-1} > -\infty, \quad \sum b_n/(a_n^2 + b_n^2) > -\infty.$$

But this means that, on R^1,

$$\theta^{\bullet} = \frac{\partial}{\partial b} \lg | e | = -k + \sum_{e(\omega)=0} | \text{imag}\, \omega | \, | \gamma - \omega |^{-2},$$

and since each term under the summation is positive,

$$\pi \int J(\gamma, \gamma)\Delta^+ = \int | e |^2 \theta^{\bullet} \Delta^+$$

can be computed term-wise:

$$\int | e |^2 \theta^{\bullet} \Delta^+ = -k \| e \|_{\Delta^+}^2 + \sum_{e(\omega)=0} | \text{imag}\, \omega | \int \left| \frac{e}{\gamma - \omega} \right|^2 \Delta^+.$$

[13] Boas [1, 129].

By Step 2, $e/(\gamma - \omega) \in \mathbf{Z}^T(\Delta)$, so (8b) can be applied with the result that

$$\int |e|^2\theta^\bullet\Delta^+ = -k + \sum_{e(\omega)=0} |\text{imag}\ \omega| \int \frac{1}{|\gamma - \omega|^2\pi(1+\gamma^2)}$$
$$= \int \frac{\theta^\bullet}{\pi(1+\gamma^2)} = \int \frac{\partial}{\partial b} \lg |e| \frac{1}{\pi(1+\gamma^2)}.$$

But it is also easy to see from the Hadamard product that $|e(a + ib)|$ is an increasing function of $b \geq 0$, so by Step 6 and Fatou's lemma,

$$\int \frac{\partial}{\partial b} \lg |e| \frac{1}{\pi(1+\gamma^2)} = \frac{\partial}{\partial b} \int \frac{\lg |e(\gamma + ib)|}{\pi(1+\gamma^2)}\ \textit{evaluated at}\ b = 0$$
$$= \frac{\partial}{\partial b} \lg |e|(i) < \infty,$$

and this finishes the proof of Step 8. Actually, $\partial \lg e/\partial b = p_b \circ \theta^\bullet(a)$ on R^{2+}, as a little more attention to the proof will show.

3.2. $\mathbf{Z}^T$ As integral functions of type $\leq T$ belonging to Z. Levinson-McKean [11: 135] proved that $\mathbf{Z}^T(\Delta)$ *can be identified as the class of integral functions of exponential type* $\leq T$ *belonging to* $\mathbf{Z}(\Delta)$. A much simpler proof of this fact can now be made using the machinery of 3.1. Because $\bigcap_{R>T} \mathbf{Z}^R(\Delta) = \mathbf{Z}^T(\Delta)$, it is enough to check that any integral function $f \in \mathbf{Z}(\Delta)$ of type $<T$ belongs to $\mathbf{Z}^T(\Delta)$.

Proof. Given such f of type $T_1 < T$, (2.2.5) tells us that $\exp\ (i\gamma T_1) fh \in H^{2+}$. Because $\exp\ (i\gamma T)e$ is an outer Hardy function, $|f/e|\ (\text{Re}^{i\theta})$ tends to 0 exponentially fast as $R \uparrow \infty$ for fixed $0 < \theta < \pi$ [2.1.10b]. A similar appraisal of $|f/e^\times|\ (\text{Re}^{i\theta})$ holds for $\pi < \theta < 2\pi$. The rest of the proof is adapted with simplifications, from de Branges [3, p. 131]. Define

$$Q[e](\omega) = \int \frac{e(\omega)f(\gamma) - e(\gamma)f(\omega)}{\gamma - \omega} k\Delta$$

with $k \in \mathbf{Z}(\Delta)$ perpendicular to $\mathbf{Z}^T(\Delta)$. The plan is to check that $Q \equiv 0$ and to deduce that $f \in \mathbf{Z}^T(\Delta)$. $Q[e]$ is an integral function of exponential type, as is the corresponding function $Q[e^\times]$ based upon $e^\times$, and

$$eQ[e^\times] - e^\times Q[e] = f(\omega) \int \frac{e^\times(\omega)e(\gamma) - e(\omega)e^\times(\gamma)}{\gamma - \omega} k\Delta$$
$$= -2\pi i f(\omega)(k, J_\omega) = 0,$$

so that $j = Q[e]/e = Q[e^\times]/e^\times$ is also integral. Besides, j is of exponential type, as can be deduced from the estimate (based on the Poisson formula for the circle)

$$\lg^+ |j(\gamma)| \leq c_1 \int_{-\pi}^{\pi} \lg^+ |j(\text{Re}^{i\theta})|\ d\theta\ \textit{for}\ |\gamma| \leq R/2,$$

using the bound

$$1 \leq \| 1 \|_\Delta^2 \, J(\mathrm{Re}^{i\theta}, \mathrm{Re}^{i\theta}) \leq c_2 \,|\, e(\mathrm{Re}^{i\theta}) \,|^2 / R \sin \theta$$

on R^{2+} to justify

$$R^{-1} \int_0^\pi \lg^+ |\, j(\mathrm{Re}^{i\theta}) \,|\, d\theta \leq c_3 + R^{-1} \int_0^\pi \lg^+ |\, e(\mathrm{Re}^{i\theta}) \,|^{-1} \, d\theta$$

$$\leq c_4 + R^{-1} \int_0^\pi \lg^+ (R \sin \theta)^{-1/2} \, d\theta$$

$$\leq c_5 (R \uparrow \infty)$$

and making a similar appraisal of $\int_{-\pi}^0 \lg^+ |\, j(\mathrm{Re}^{i\theta}) \,|$. By the estimates of f/e and $f/e^\times$ at the beginning of the proof, $j(\mathrm{Re}^{i\theta}) = o(1)$ as $R \uparrow \infty$ for $\theta = \pi/4$, $3\pi/4$, $5\pi/4$, $7\pi/4$, and an application of the Phragmen-Lindelöf principle [2.2.6] shows that j is bounded on the whole of R^2. But this means that j is constant $(=0)$, and so

$$f_1 \equiv \frac{-\omega[e(\omega)f(\gamma) - f(\omega)e(\gamma)]}{e(\omega)(\gamma - \omega)} \in \mathbf{Z}^T(\Delta)$$

for any $\omega \in R^2$. The proof is now completed by putting $\omega = ib$ and noticing that

$$\| f - f_1 \|_\Delta^2 \leq 2 \int \frac{\gamma^2 \,|\, f \,|^2}{\gamma^2 + b^2} \Delta + 2 \left| \frac{f}{e} \right|^2 (ib) \int \frac{b^2 \,|\, e \,|^2 \Delta}{\gamma^2 + b^2}$$

tends to 0 as $b \uparrow \infty$ since $|f/e|\ (ib)$ is exponentially small far out.

3.3. $\mathbf{Z}^0$ and the gap $\mathbf{Z}^0/\mathbf{Z}_\infty$. $\mathbf{Z}^0(\Delta)$ turns out to be quite a complicated object. To begin with, one may distinguish the polynomial subspaces

$$\mathbf{Z}_n(\Delta) = \textit{the polynomials of degree } \leq n \textit{ belonging to } \mathbf{Z}(\Delta),$$

and their closure

$$\mathbf{Z}_\infty(\Delta) = \textit{the span of all polynomials in } \mathbf{Z}(\Delta).$$

$\mathbf{Z}_n(\Delta)$ $(n \leq \infty)$ is a de Branges space based upon the function

e_{n+1} = *the projection of* $1/h$ *upon* $\mathbf{Z}_{n+1}(\Delta^+)$ *divided by the* Δ^+ *norm of this projection.*

e_{n+1} is a polynomial of degree $n + 1$ if $n < \infty$ and $\int \gamma^{2n}\Delta < \infty$, and an integral function of minimal exponential type if $n = \infty$; it may be adjusted as in (2.3.6) to obtain a function E_{n+1} satisfying (1.3.5) for $T = 0$, such that $\mathbf{Z}_n(\Delta) = \mathbf{B}(E_{n+1})$. The proof is exactly the same as in 3.1. $\mathbf{Z}_\infty = \mathbf{Z}^0$ in nice cases, but in general there is a gap $\mathbf{Z}^0/\mathbf{Z}_\infty \neq 0$ between them, as an example of Levinson-McKean [11, p. 130] shows. 3.5 is devoted to this point. The presence of a gap sounds implausible if one recalls that any $f \in \mathbf{Z}^0$ is an integral function of minimal exponential type, and as such has a nice power series. But what makes the gap is that this power series *does not converge in* $\mathbf{Z}$! The actual structure of $\mathbf{Z}^0/\mathbf{Z}_\infty$ is open to conjecture [see 1.3]. The rest of the present section is devoted to tests for the nonexistence of the gap. An alternative discussion of the functions E_{n+1} ($n \leq \infty$) is the subject of 3.4.

Define e_∞ as above, so that $\mathbf{Z}_\infty(\Delta) = \mathbf{B}(e_\infty)$, and e as in 3.1 so that $\mathbf{Z}^0(\Delta) = \mathbf{B}(e)$. Because e is of minimal exponential type, *either it is a polynomial of degree* $0 \neq n + 1 < \infty$, *in which case the bound* $|f|^2 \leq \|f\|^2 J(\gamma, \gamma)$ *implies that* $\mathbf{Z}^0(\Delta) = \mathbf{Z}_\infty(\Delta) = \mathbf{Z}_n(\Delta)$, *or it is a transcendental function with an infinite number of roots.* Confining attention to the second alternative, the objective is to prove the following facts:

(1) $\dim \mathbf{Z}^0(\Delta) = \infty$, *esp., there is a gap unless* $\dim \mathbf{Z}_\infty(\Delta) = \infty$.
(2) $\mathbf{Z}_\infty(\Delta) = \mathbf{Z}^0(\Delta)$ *iff* $e \in \mathbf{Z}_\infty(\Delta^+)$.
(3) $\dim \mathbf{Z}^0(\Delta)/\mathbf{Z}_\infty(\Delta) = 0$ *or* $+\infty$, *i.e., either* $\mathbf{Z}^0(\Delta) = \mathbf{Z}_\infty(\Delta)$, *or* $\mathbf{Z}^0(\Delta)$ *is of infinite dimension over* $\mathbf{Z}_\infty(\Delta)$.
(4) $\mathbf{Z}_\infty(\Delta) = \mathbf{Z}^0(\Delta)$ *iff* $1 = \inf \| p \|_{\Delta+}$ *for* $p \in \mathbf{Z}_\infty(\Delta^+)$ *with* $p(i) = e(i)$.
(5) $\mathbf{Z}_\infty(\Delta) = \mathbf{Z}^0(\Delta)$ *if* $|e|^2 = \sum_{n=0}^{\infty} k_n \gamma^{2n}$ *on* R^1 *with* $k_0 > 0 \leq k_1, k_2$, *etc., esp., this is the case if all the roots of e lie in the sector* $-3\pi/4 \leq \theta \leq -\pi/3$;[14] *a finite number of exceptional roots does not spoil the nonexistence of the gap.*

Proof of (1). e has an infinite number of distinct roots ω, and the functions $(\gamma - \omega)^{-1}e$ are independent.

Proof of (2). $\mathbf{Z}_\infty(\Delta) = \mathbf{Z}^0(\Delta)$ implies that $(\gamma - \omega)^{-1}e \in \mathbf{Z}_\infty(\Delta)$ for any root ω of e. But then it is possible to find a polynomial p such that $\| (\gamma - \omega)^{-1}e - p \|_\Delta < 1/n$, and $e \in \mathbf{Z}_\infty(\Delta^+)$ follows from the appraisal

$$\| e - (\gamma - \omega)p \|_{\Delta+}^2 = \int \left| \frac{e}{\gamma - \omega} - p \right|^2 \frac{|\gamma - \omega|^2}{\pi(1 + \gamma^2)} \Delta \leq 2 \frac{1 + |\omega|^2}{\pi n^2}.$$

Conversely, $e \in \mathbf{Z}_\infty(\Delta^+)$ implies that $e_\infty = e$, esp., the reproducing kernel for $\mathbf{Z}^0(\Delta)$ is the same as that for $\mathbf{Z}_\infty(\Delta)$, so that these 2 spaces are identical.

Proof of (3). $\dim \mathbf{Z}^0(\Delta)/\mathbf{Z}_\infty(\Delta) < \infty$ implies a dependence over $\mathbf{Z}_\infty(\Delta)$ among the functions $(\gamma - \omega)^{-1}e$:

$$\left[\frac{c_1}{\gamma - \omega_1} + \frac{c_2}{\gamma - \omega_2} + \cdots + \frac{c_n}{\gamma - \omega_n} \right] e \in \mathbf{Z}_\infty(\Delta).$$

But this means that $e = pf$ for some $p \in \mathbf{Z}_\infty(\Delta)$ and some rational function $f = p_+/p_-$ with $\deg p_+ = \deg p_- + 1$. $e \in \mathbf{Z}_\infty(\Delta^+)$ follows easily, and an application of (2) does the rest.

Proof of (4). Use the formula of Step 3 of 3.1

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[14] Levinson-McKean [11, 125]. The present proof is simpler; see Levinson-McKean [11, 127] for a sharper test.

$$\|p - e\|^2_{\Delta^+} = \|p\|^2_{\Delta^+} - 2\,\mathrm{Re}\int e^* p\Delta^+ + 1$$
$$= \|p\|^2_{\Delta^+} + 1 - 2\,\mathrm{Re}\,(p/e)(i)$$

and apply (2).

Proof of (5). A partial sum $\sum_{n \le m} k_n \gamma^{2n}$ can be expressed as $|p|^2$ on R^{1}' p being a polynomial which is root-free on the closed upper half-plane. Now p is an outer Hardy function on R^{2+}, so that

$$\lg\left|\frac{p}{e}\right|(i) = \frac{1}{\pi}\int \frac{\lg|p/e|}{1+\gamma^2} \uparrow 0 \quad (m \uparrow \infty).$$

But also, by (3.1.8b),

$$\|p\|^2_{\Delta^+} = \int \left|\frac{p}{e}\right|^2 \frac{d\gamma}{\pi(1+\gamma^2)} \uparrow 1 \quad (m \uparrow \infty),$$

so the formula in the proof of (4) can be applied to complete the proof of the first statement. The rest follows from the Hadamard product for e [see the proof of Step 8 of 3.1].

3.4. Alternative discussion of $\mathbf{Z}_\infty$. Another proof that $\mathbf{Z}_n(\Delta)$ $(n \le \infty)$ is a de Branges space will now be explained under the condition $\int \gamma^{2n}\Delta < \infty$ $(n \ge 0)$; the modifications needed for the general case will be self-evident. The formulas (1), (2), and (3) will be familiar to students of Szegö [17].

Define $A_{2n}(B_{2n+1})$ to be even (odd) orthogonal polynomials of degree $2n(2n+1)$ for the weight Δ, subject to a special normalization to be imposed in a moment, and put $A_{2n+1} = A_{2n}$, $B_{2n+2} = B_{2n+1}$, $E_n = A_n - iB_n$, and

$$(1) \qquad J_\alpha(\beta) = J(\alpha, \beta) = \frac{E_{n+1}(\alpha)^* E_{n+1}(\beta) - E_{n+1}(\alpha^*) E^{\times}_{n+1}(\beta)}{-2\pi i(\beta - \alpha^*)}$$

for $n < \infty$. Given $f \in \mathbf{Z}_n(\Delta)$, f_α is a polynomial of degree $<n$, and as such it is perpendicular to the numerator of J_α. Thus, $(f, J_\alpha)_\Delta = f(\alpha)(1, J_\alpha)_\Delta$; esp.,

$$0 \not\equiv \|J_\alpha\|^2_\Delta = J(\alpha, \alpha)(1, J_\alpha)_\Delta,$$

from which one infers that $(1, J_\alpha)_\Delta$ is constant $\not\equiv 0$, as in the proof of Step 4 of 3.1. Computing this constant for $\alpha = 0$ gives

$$(1, J_0)_\Delta = A_{n+1}(0)\int \frac{B_{n+1}}{\pi\gamma} \ne 0,$$

and so it is permissible to take $A_{n+1}(0) = 1$ and $\int (\pi\gamma)^{-1} B_{n+1}\Delta = 1$ $(n \ge 0)$, with the result that

(2) $(f, J_\alpha)_\Delta = f(\alpha)$ *for any* $f \in \mathbf{Z}_n(\Delta)$.

Now E_{n+1} satisfies (1.3.5) for $T = 0$, as the reader will check using

(3) $|f(\gamma)|^2/\|f\|^2_\Delta \le J(\gamma, \gamma) = (|E_{n+1}(\gamma)|^2 - |E^{\times}_{n+1}(\gamma)|^2)/4\pi b \quad (b \ne 0)$

as in the proof of Step 5 of 3.1. This permits us to base a de Branges space $\mathbf{B}(E_{n+1})$ upon E_{n+1}, and it is evidently the same as $\mathbf{Z}_n(\Delta)$ since it has the same reproducing kernel. The map $f \to (f, J_\bullet)_\Delta$ can now be identified as *the projection upon* $\mathbf{Z}_n(\Delta)$. As $n \uparrow \infty$, this tends to the projection upon $\mathbf{Z}_\infty(\Delta)$, and it is easy to deduce that E_{n+1} converges in $\mathbf{Z}_\infty(\Delta^+)$ to some integral function E_∞ of minimal exponential type. The point at issue is to verify that $\mathbf{Z}_\infty(\Delta) = \mathbf{B}(E_\infty)$. Define J as in (1) with $n = \infty$. Then (2) holds for $f \in \mathbf{Z}_\infty(\Delta)$, and the rest of the proof is just the same as for $n < \infty$.

3.5. Example of a gap. This section contains an example of a gap $[\mathbf{Z}^0/\mathbf{Z}_\infty \neq 0]$ adapted from Levinson-McKean [11, p. 130].

Consider the weight $\Delta = |e|^{-2}$ based upon the Hadamard product

$$e = \prod_{n\neq 0} (1 - \gamma/\omega_n)$$

with

$$\omega_n = a_n + ib_n, \quad a_n = \text{sign}(n) \times n^2 \quad \textit{and} \quad b_n = -e^{-\pi|n|} \quad (n \neq 0).$$

The problem is to check that

(1) Δ *is a Hardy weight.*
(2) $\dim \mathbf{Z}_\infty(\Delta) = \infty$.
(3) $\mathbf{Z}^0(\Delta) \neq \mathbf{Z}_\infty(\Delta)$, *i.e., there is a gap.*

Bring in the function

$$f = \prod_{n\neq 0} (1 - \gamma \,\text{sign}(n)/n^2) = \frac{\sin \pi\sqrt{\gamma}\,\sinh \pi\sqrt{\gamma}}{\pi\gamma},$$

and note for future use the following simple bounds:

(4a) $0 < c_1 \leq |f/e| \leq c_2 < \infty$ *on* $\bigcap_{n\geq 1} (|\gamma \pm n^2| \geq 1)$
(4b) $0 < c_3 \leq |f/e| \, |(\gamma - \omega_{\pm n})/(\gamma \mp n^2)| \leq c_4 < \infty$ *on* $|\gamma \mp n^2| \leq 1$.

Proof of (1). As can be seen from the product, e is an integral function of minimal exponential type, and $|e(a + ib)|$ is an increasing function of $b \geq 0$. Thus

$$\int |e(a + ib)|^{-2}\, da \leq \int \Delta,$$

and $e^{-1} \in H^{2+}$ will follow from the fact that $\int \Delta < \infty$, which is proved in the next step.

Proof of (2). As n runs through the integers ≥ 2, the intervals

$$A = [(n - \tfrac{1}{2})^2, n^2 - 1), \quad B = [n^2 - 1, n^2 + 1), \quad C = [n^2 + 1, (n + \tfrac{1}{2})^2)$$

provide a covering of $[9/4, \infty)$, and Δ is estimated on each of these. Beginning on the interval A, one finds

$$|\sin \pi\sqrt{\gamma}| \geq |\sin \pi(\sqrt{n^2-1}-n)| \geq c_5/n,$$

$$\sinh \pi\sqrt{\gamma} \geq \sinh \pi(n-\tfrac{1}{2}) \geq c_6\, e^{\pi n},$$

so by (4a),

$$\Delta \leq c_7\, n^6 \exp(-2\pi n).$$

A similar bound holds on C, while on B,

$$|\sin \pi\sqrt{\gamma}| = \left|\sin \pi n \sqrt{1+\frac{\gamma-n^2}{n^2}}\right| \geq c_8 n^{-1}|\gamma - n^2|,$$

$$\sinh \pi\sqrt{\gamma} \geq \sinh \pi\sqrt{n^2-1} \geq c_9\, e^{\pi n},$$

so by (4b),

$$\Delta \leq c_{10}\frac{n^6 e^{-2\pi n}}{(\gamma-n^2)^2+e^{-2\pi n}},$$

and

$$\int_B \Delta \leq c_{10} n^6 e^{-2\pi n}\int 1/(\gamma^2+e^{-2\pi n}) = \pi c_{10} n^6 e^{-\pi n}.$$

(2) is immediate from this.

Proof of (3).

$$|\gamma|^{2n} \leq \|\gamma^n\|_\Delta^2\, |e|^2/4\pi b \quad on\ R^{2+}$$

since $\gamma^n \in \mathbf{Z}^0$, so

$$\int_0^{\pi/2} |e(\mathrm{Re}^{i\theta})|^{-1}\, d\theta \leq c_{11} R^{-n}\int_0^{\pi/2}\frac{d\theta}{\sqrt{R\sin\theta}}$$

is a rapidly decreasing function of R. (4a) permits us to carry this estimate over to $e^{\times}$ for $R = m^2+2$ ($m \uparrow \infty$), and this justifies the use of Cauchy's formula to compute the integral

$$Q = \int_{-\infty}^{+\infty}\frac{p^2}{e^{\times}}(ib)\, db = -2\pi\sum_{n=1}^{\infty}\frac{p^2}{e^{\times\prime}}(\omega_n^*)$$

for any polynomial p. Now suppose $\mathbf{Z}^0 = \mathbf{Z}_\infty$. Then it would be possible to find polynomials p which are real on iR^1 and tend to $(i\gamma)^{-1}(e-1)$ in $\mathbf{Z}(\Delta)$, and this would make Q diverge since $e^{\times} > 0$ on iR^1, so that by (4a),

$$\underline{\lim}\, Q \geq \int \underline{\lim}\frac{p^2}{e^{\times}}(ib)\, db = \int\frac{(e-1)^2}{e^{\times}}(ib)\frac{db}{b^2} \geq c_{12}\int_1^{\infty} f(ib)\frac{db}{b^2} = \infty.$$

The proof is completed by checking the contradictory over-estimate: $\overline{\lim}\, Q < \infty$. Apply the bound $|p(\gamma)|^2 \leq \|p\|_\Delta^2\, J(\gamma,\gamma)$ at a root ω_n^* of $e^{\times}$, using (4b) and the fact that $\|p\|_\Delta$ is bounded:

$$|p(\omega_n^*)|^2 \leq \|p\|_\Delta^2\, |e(\omega_n^*)|^2/(4\pi|b_n|) \leq c_{13}\, e^{\pi n}\, |f(\omega_n^*)|^2.$$

A second application of (4b) justifies the bound

$$|e^{\chi\prime}(\omega_n^*)| = \lim_{\gamma\to\omega_n^*} |e^{\chi}(\gamma)/(\gamma - \omega_n^*)|$$
$$\geq c_4^{-1} |f(\omega_n^*)/(\omega_n^* - n^2)| = c_4^{-1}e^{\pi n} |f(\omega_n^*)|.$$

But this means that

$$|p^2/e^{\chi\prime}|(\omega_n^*) \leq c_{14} |f(\omega_n^*)|$$
$$\leq c_{15}\, n^{-2} |\sin \pi[n + O(e^{-\pi n}/n)]| \sinh \pi[n + O(e^{-\pi n}/n)]$$
$$\leq c_{16}\, n^{-3},$$

and now the final contradiction is easily obtained:

$$\overline{\lim}\, Q \leq \overline{\lim}\, 2\pi \textstyle\sum_{n=1}^{\infty} |p^2/e^{\chi}|\ (\omega_n^*) \leq c_{17} \sum_{n=1}^{\infty} n^{-3} < \infty.$$

4. Eigendifferentials

The object of this part is to establish the differential equations (1.3.4) on the discrete spectrum and on the type spectrum, and to discuss the eigendifferential transform (1.3.7). 4.1 deals with the type spectrum $(0, \infty)$ under the extra technical condition discussed in 1.3: $\bigcup_{R<T} \mathbf{Z}^R$ *is dense in* $\mathbf{Z}^T$. 4.2 deals with the discrete spectrum. The eigendifferential transform is assembled in 4.3, *assuming the singular spectrum to be as conjectured in* 1.3.

4.1. Type spectrum for $\mathbf{Z}/\mathbf{Z}^0$. Think of the A, B, E, J associated with $\mathbf{Z}^T(\Delta) = \mathbf{B}(E)$ in 3.1 as functions A^T, B^T, E^T, J^T of $T \geq 0$. The purpose of this section is to prove that under the technical condition stated above there exists a pair of jump-free positive mass distributions $Q^{\pm}$ defined on the type interval $(0, \infty)$ such that

(1a) $Q^-(I) = (1/\pi) \|\gamma^{-1}A^I\|_\Delta^2 = -(1/\pi)\int \gamma^{-2}A^I\Delta$,[15]
(1b) $Q^+(I) = (1/\pi) \|\gamma^{-1}B^I\|_\Delta^2 = \pi J^I(0, 0)$,
(2a) $dA^T = -\gamma B^T\, dQ^-$,
(2b) $dB^T = +\gamma A^T\, dQ^+$,
(3) $\pi J^I(\alpha, \beta) = \int_I A^T(\alpha)^* A^T(\beta)\, dQ^+ + \int_I B^T(\alpha)^* B^T(\beta)\, dQ^-$,
(4) $\sqrt{dQ^-\, dQ^+} = dT$.[16]

The explanation of (4) is that while E^T is of exact exponential type T, this type can also be computed from (2) as the integral

$$\int_0^T \sqrt{dQ^-\, dQ^+}.$$

There is evidence from the examples of 4.4 for thinking that $dQ^{\pm} = p^{\pm 1}\, dT$ with $p = -i \lim_{b\uparrow\infty} (B/A)(ib)$ for a wide class of weights Δ, but the proof escapes us. The proof of (1)–(4) is divided into 8 easy steps.

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[15] I is a half-open interval $0 \leq T_1 < T \leq T_2 \leq \infty$. f^I stands for $f^{T_2} - f^{T_1}$.

[16] (4) means that $\int_I \sqrt{p^- p^+}\, dQ$ = *the length* of I, in which $Q = Q^- + Q^+$ and $p^{\pm} = dQ^{\pm}/dQ$.

STEP 1. *A^T and B^T are continuous relative to the distance in* $\mathbf{Z}(\Delta^+)$.

Proof. $\mathbf{Z}^T(\Delta^+)$ is continuous from above by definition and from below by the technical condition so the projection upon $\mathbf{Z}^T(\Delta^+)$ is jump-free. The statement is immediate from this using the projection recipe (3.1.7).

STEP 2. *The interval functions* $Q^\pm$ = (1) *define jump-free non-negative mass distributions on* $(0, \infty)$.

Proof. Because of Step 1, it suffices to prove that $Q^\pm$ is additive. Step 3 of 3.1 implies that for $0 \le t \le T$,

$$(5) \quad \int \gamma^{-1}A^T \sin \gamma t\Delta = \text{constant} \times \int e^{T}*\frac{\sin \gamma t}{\gamma}\pi(1+\gamma^2)\Delta^+ = 0,$$[17]

and it follows from Step 1 that

$$(6) \quad \int \gamma^{-1}A^T f\Delta = 0 \textit{ for any odd } f \in \mathbf{Z}^T(\Delta).$$

This shows that $\gamma^{-1}A^I$ is perpendicular to $\gamma^{-1}A^K$ in $\mathbf{Z}(\Delta)$ if $I \cap K = \emptyset$, esp., it shows that $Q^-(I) = \pi^{-1}\| \gamma^{-1}A^I \|_\Delta^2$ is additive. The second expression for Q^- is also an easy consequence. The additivity of Q^+ follows from the fact that the map

$$f \to (f, J_0^I)_\Delta = (f, B^I/\pi\gamma)_\Delta$$

is projection onto $\mathbf{Z}^{T_2}/\mathbf{Z}^{T_1}$ followed by evaluation at 0, so that

$$(1/\pi)\| \gamma^{-1}B^I \|_\Delta^2 = \pi[J^{T_2}(0, 0) - 2J^{T_1}(0, 0) + J^{T_1}(0, 0)] = \pi J^I(0, 0),$$

as in the second part of (1b).

STEP 3. *The interval function* $M(I) = \pi \int J^I(\gamma, \gamma)\Delta^+ < \infty$ *defines a jump-free non-negative mass distribution on* $(0, \infty)$.

Proof. Use Step 1, Step 8 of 3.1, and the fact that J^T on the diagonal is an increasing function of $T \ge 0$.

STEP 4. $\| \gamma^{-1}A^I \|_{\Delta^+}$ *and* $\| \gamma^{-1}B^I \|_{\Delta^+}$ *are bounded above by*

$$Q(I) \equiv Q^-(I) + Q^+(I) + M(I).$$

Proof. By (6), $f = \gamma^{-1}A^I$ is perpendicular to $\mathbf{Z}^{T_1}(\Delta)$, so that $f = (f, J_\cdot^I)_\Delta$, *esp.*,

$$|\gamma^{-1}A^I(\gamma)|^2 \le \|f\|_\Delta^2 J^I(\gamma, \gamma) = \pi Q^-(I)J^I(\gamma, \gamma),$$

and so

$$\| \gamma^{-1}A^I \|_{\Delta^+} \le \sqrt{\pi Q^-(I) \int J^I(\gamma, \gamma)\Delta^+} = \sqrt{Q^-(I)M(I)} \le Q(I),$$

as stated. The estimation of $\| \gamma^{-1}B^I \|_{\Delta^+}$ is even simpler: just use the bound

[17] e^T is the function (3.1.7).

$$|J^I(\alpha, \beta)|^2 = |(J^I_\alpha, J^I_\beta)|^2 \leq J^I(\alpha, \alpha)J^I(\beta, \beta)$$

to check

$$\|\gamma^{-1}B^I\|^2_{\Delta^+} = \pi^2 \int |J^I_0|^2\Delta^+ \leq \pi J^I(0, 0)\pi \int J^I(\gamma, \gamma)\Delta^+$$

$$= Q^+(I)M(I) \leq Q(I)^2.$$

STEP 5. *The following formulas hold for functions belonging to* $\mathbf{Z}^T(\Delta^+)$:[18]

(7a) $\int \gamma^{-1}A^T f\Delta^+ = -if(i)/e^T(0)e^T(i)$ (*f odd*)

(7b) $\int A^T f\Delta^+ = f(i)/e^T(0)e^T(i)$ (*f even*)

(7c) $\int \gamma^{-1}B^T f\Delta^+ = f(0) - f(i)e^T(0)/e^T(i)$ (*f even*)

(7d) $\int B^T f\Delta^+ = -if(i)e^T(0)/e^T(i)$ (*f odd*).

Proof. Step 1 shows that it is enough to check (7) for $f = \cos \gamma t$ and/or $\sin \gamma t$ and $0 \leq t < T$. (7a) is immediate from Step 3 of 3.1:

$$\int \gamma^{-1}A^T \sin \gamma t\, \Delta^+ = \int \frac{e^{T*}}{e^T(0)}\gamma^{-1} \sin \gamma t\, \Delta^+ = \frac{\sinh t}{e^T(0)e^T(i)},$$

and differentiating this with respect to t proves (7b). As to (7d), Step 3 of 3.1 gives

$$\int B^T \sin \gamma t\, \Delta^+ = -ie^T(0) \int e^{T*} \sin \gamma t\, \Delta^+ = \sinh t\, e^T(0)/e^T(i),$$

and (7c) follows upon introducing $C(t) \equiv \int \gamma^{-1}B^T \cos \gamma t\, \Delta^+$ and noticing that $-C'$ is the integral just evaluated, while

$$C - C'' = \int (B^T/\pi\gamma) \cos \gamma t\, \Delta = \int J^T_0 \cos \gamma t\, \Delta = 1.$$

STEP 6. *Check* (2) *and* (3).

Proof. By Step 4 and (7b) with $f = 1$,

$$|[e^{T_2}(0)e^{T_2}(i)]^{-1} - [e^{T_1}(0)e^{T_1}(i)]^{-1}| \leq \int |\gamma^{-1}A^I|\, |\gamma|\Delta^+ \leq Q(I)\|\gamma\|_{\Delta^+}.$$

Q is jump-free, so $d[e^T(0)e^T(i)]^{-1}/dQ$ exists at almost all points $T \geq 0$ relative to Q and can be integrated back. Pick such a T and use the bound of Step 4 to select $\varepsilon = \varepsilon_1 > \varepsilon_2 >$ etc. $\downarrow 0$ so as to make

$$Q(T, T + \varepsilon]^{-1}\gamma^{-1}[A^{T+\varepsilon} - A^T]$$

converge weakly in $\mathbf{Z}(\Delta^+)$ to an (odd) function $A^\bullet \in \mathbf{Z}^T(\Delta^+)$. By (7a)

$$\int A^\bullet f\Delta^+ = -if(i)\, d[e^T(0)e^T(i)]^{-1}/dQ \quad \textit{for odd } f \in \mathbf{Z}^T(\Delta^+),$$

[18] e^T is the function (3.1.7).

esp., the *full* weak differential coefficient

$$A^\bullet = \lim_{\varepsilon\downarrow 0} Q(T, T+\varepsilon]^{-1}[A^{T+\varepsilon} - A^T]$$

exists, and comparison with (7d) reveals that

$$A^\bullet = \frac{e^T(i)}{e^T(0)}\,\frac{d[e^T(0)e^T(i)]^{-1}}{dQ}\,B^T.$$

Because evaluation at $\gamma \in R^2$ is a continuous application of $Z^T(\Delta^+)$, this also holds pointwise, together with a bound

$$|\gamma^{-1}A^I(\gamma)| \le \text{constant}\, \|\gamma^{-1}A^I\|_{\Delta+} \le \text{constant} \times Q(I),$$

permitting us to integrate back with the result that

$$\text{(8a)}\quad A^I(\gamma) = \gamma \int_I B^T(\gamma)p^-\,dQ \quad \textit{with } p^- = -\frac{e^T(i)}{e^T(0)}\,\frac{d[e^T(0)e^T(i)]^{-1}}{dQ}.$$

Similarly, (7b) and (7c) may be used to verify that

$$\text{(8b)}\quad B^I(\gamma) = \gamma \int_I A^T(\gamma)p^+\,dQ \quad \textit{with } p^+ = -e^T(0)e^T(i)\,\frac{de^T(0)/de^T(i)}{dQ}.$$

J^I can then be expressed as

$$\text{(9)}\quad \pi J^I(\alpha,\beta) = \int_I A^T(\alpha)^*A^T(\beta)p^+\,dQ + \int_I B^T(\alpha)^*B^T(\beta)p^-\,dQ,$$

as the reader will check by computing dJ^T with the aid of (8), and it remains only to check that $dQ^\pm = p^\pm\,dQ$. But, by (1b) and (9)

$$Q^+(I) = \pi J^I(0,0) = \int_I p^+\,dQ,$$

while, by (1a) and (8a),

$$Q^-(I) = -\frac{1}{\pi}\int \gamma^{-2}A^I\Delta = \int \Delta \int_I \frac{B^T}{\pi\gamma}p^-\,dQ = \int_I p^-(1, J_0^T)_\Delta\,dQ = \int_I p^-\,dQ,$$

as advertised.

STEP 7. $Q^\pm$ *is positive.*

Proof. (3) shows that if $Q^+(I) = 0$, then the even part of J^I vanishes. But this means that the even part of $Z^T(\Delta)$ is unchanged as T runs over I, and that is false. $Q^-(I) = 0$ leads to a similar contradiction.

STEP 8. $\sqrt{dQ^-\,dQ^+} = dT$.

Proof (adapted from de Branges [3, pp. 147–149]). Define $p^\pm = dQ^\pm/dQ$ as above, $p = \begin{pmatrix} 0 & -p^- \\ p^+ & 0 \end{pmatrix}$, and $C = \begin{pmatrix} A \\ B \end{pmatrix}$. Then (2) can be expressed as $dC = \gamma p\, dQC$ and for any constant $k \in SL(2, R^1)$, $d(kC) = \gamma(kpk^{-1})\,dQ(kC)$. Now integrate to get $d(kC) = \gamma(kpk^{-1})\,dQkC$, i.e.,

$$kC^{T_2} = kC^{T_1} + \gamma \int_{T_1}^{T_2} (kpk^{-1})\, dQ(kC).$$

Now iterate to obtain

$$kC^{T_2} = \sum_{n=0}^{\infty} \gamma^n \int_{T_1}^{T_2} kpk^{-1}\, dQ \int_{T_1}^{\bullet} kpk^{-1}\, dQ \cdots \int_{T_1}^{\bullet} kpk^{-1}\, dQ(n\text{-fold}) \times kC^{T_1},$$

make the crude appraisal

$$|kC^{T_2}| \le \exp\left[|\gamma| \int_I |kpk^{-1}|\, dQ\right] |kC^{T_1}|,$$[19]

and compute the exponential type of both sides:

$$T_2 - T_1 \le \int_I |kpk^{-1}|\, dQ.$$

Both sides of this bound are additive in I so it still holds if k is replaced first by a step function from I to $SL(2, R^1)$ and then by a Borel function of the form

$$k = \begin{bmatrix} l & 0 \\ 0 & l^{-1} \end{bmatrix}$$

with l bounded from 0 and ∞. But then

$$|kpk^{-1}| = \textit{the norm of} \begin{bmatrix} 0 & -p^-l^2 \\ p^+/l^2 & 0 \end{bmatrix} = \textit{the larger of } p^-l^2 \textit{ and } p^+/l^2 \equiv m$$

so that

$$T_2 - T_1 \le \int_{I\cap(p^->0=p^+)} p^-l^2\, dQ + \int_{I\cap(p^+>0=p^-)} p^+/l^2\, dQ + \int_{I\cap(p^-p^+>0)} m\, dQ,$$

and to obtain $dT \le \sqrt{dQ^-\, dQ^+}$, one has only to make $l \downarrow 0$ on the first region, $l \uparrow \infty$ on the second, and to make l approximate $\sqrt[4]{p^+/p^-}$ on the third. To prove the opposite inequality, use

$$\begin{aligned} \pi\, dJ^T(ib, ib) &= (|A^T(ib)|^2p^+ + |B^T(ib)|^2p^-)\, dQ \\ &\ge 2|A^TB^T(ib)| \sqrt{p^-p^+}\, dQ \\ &= 2\pi b J^T(ib, ib) \sqrt{p^-p^+}\, dQ \end{aligned}$$

to infer that $d \lg J^T(ib, ib) \ge 2b\sqrt{p^-p^+}\, dQ$. This leads easily to the described bound:

$$\begin{aligned} 2(T_2 - T_1) &= \lim_{b\uparrow\infty} b^{-1} \lg \frac{J^{T_2}(ib, ib)}{J^{T_1}(ib, ib)} \\ &= \lim_{b\uparrow\infty} b^{-1} \int_I d \lg J^T(ib, ib) \ge 2 \int_I \sqrt{p^-p^+}\, dQ \end{aligned}$$

in view of the overestimate

$$J^T(ib, ib) \le |E^T(ib)|^2/4\pi b \le \exp[2b(T + o(1))]$$

[19] $|k^{-1}pk|$ is the conventional norm of $k^{-1}pk$ as an application of R^2.

and the underestimate

$$e^{2bT} \leq \| e^{i\gamma T} \|_\Delta^2 J^T(ib, ib).$$

4.2. Discrete spectrum for Z_∞. As in 3.4, it is now supposed that $\int \gamma^{2n}\Delta < \infty$ $(n \geq 0)$; the modifications needed for the general case will be self-evident. The content of this section is that the orthogonal polynomials A_n and B_n of 3.4, augmented by $B_{-1} = B_0 = 0$ and $A_0 = 1$, can be obtained from difference equations similar to (4.1.2); this fact will be familiar to students of Szegö [17]. The precise statement is that if

$$\text{(1a)} \quad Q^-(2n+1) = \pi \|B_{2n+1}\|_\Delta^{-2} = \frac{1}{\pi} \left\| \frac{A_{2n+2} - A_{2n}}{\gamma} \right\|_\Delta^2, \quad Q^-(2n) = 0$$

and if

$$\text{(1b)} \quad Q^+(2n) = \pi \| A_{2n} \|_\Delta^{-2} = \frac{1}{\pi} \left\| \frac{B_{2n+1} - B_{2n-1}}{\gamma} \right\|_\Delta^2, \quad Q^+(2n+1) = 0,$$

then

$$\text{(2a)} \quad A_{2n+2} - A_{2n} = -\gamma B_{2n+1} Q^-(2n+1),$$

$$\text{(2b)} \quad B_{2n+1} - B_{2n-1} = +\gamma A_{2n} Q^+(2n),$$

$$\text{(3)} \quad \pi J_n(\alpha, \beta) = \sum\nolimits_{\text{even } k \leq n} A_k(\alpha)^* A_k(\beta) Q^+(k) + \sum\nolimits_{\text{odd } k \leq n} B_k(\alpha)^* B_k(\beta) Q^-(k),$$ [20]

$$\text{(4)} \quad \sum\nolimits_{n \geq 0} [Q^-(2n+1) + Q^+(2n)] \leq \pi J_\infty(0,0) + (J_\infty)_{12}(0,0)/\pi J_0(0,0)^2 < \infty.$$ [21]

Proof. Because A_0, B_1, A_2, etc., are the orthogonal polynomials of Δ, it is evident that (3) is a correct expression of the facts for

$$Q^-(2n+1) = \pi \| B_{2n+1} \|_\Delta^{-2} \quad \text{and} \quad Q^+(2n) = \pi \| A_{2n} \|_\Delta^{-2}.$$

J_n can also be expressed as in (3.4.1), or, what is the same,

$$J_n(\alpha, \beta) = \frac{A_{n+1}(\alpha)^* B_{n+1}(\beta) - B_{n+1}(\alpha)^* A_{n+1}(\beta)}{\pi(\beta - \alpha^*)}.$$

A comparison of (3) with this expression gives

$$\begin{aligned} &\pi(\beta - \alpha)(J_{2n+1} - J_{2n})(\alpha^*, \beta) \\ &= (\beta - \alpha) B_{2n+1}(\alpha) B_{2n+1}(\beta) Q^-(2n+1) \\ &= A_{2n+2}(\alpha) B_{2n+2}(\beta) - A_{2n+1}(\alpha) B_{2n+1}(\beta) \\ &\qquad - B_{2n+2}(\alpha) A_{2n+2}(\beta) + B_{2n+1}(\alpha) A_{2n+1}(\beta) \\ &= [A_{2n+2}(\alpha) - A_{2n}(\alpha)] B_{2n+1}(\beta) + \textit{an odd function of } \alpha, \end{aligned}$$

and (2a) follows by fixing β and matching even parts. (2b) is even easier:

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[20] J_n is the reproducing kernel of $Z_n(\Delta)$.

[21] J_{12} means $\partial^2 J/\partial\alpha\partial\beta$.

just look at the two expressions for $(J_{2n} - J_{2n-1})(0, \gamma)$ and it drops out. The remainder of (1) now follows from (2):

$$\frac{1}{\pi}\left\|\frac{A_{2n+2} - A_{2n}}{\gamma}\right\|_\Delta^2 = \frac{Q^-(2n+1)^2}{\pi}\| B_{2n+1}\|_\Delta^2 = Q^-(2n+1)$$

$$\frac{1}{\pi}\left\|\frac{B_{2n+1} - B_{2n-1}}{\gamma}\right\|_\Delta^2 = \frac{Q^+(2n)^2}{\pi}\| A_{2n}\|_\Delta^2 = Q^+(2n),$$

and (4) is evident from (3) and the fact that $B'_{n+1}(0) = \pi J_n(0,0) \geq \pi J_0(0,0) = Q^+(0)$:

$$\pi J_\infty(0, 0) = \sum_{n=0}^\infty | A_{2n}(0)|^2 Q^+(2n) = \sum_{n=0}^\infty Q^+(2n)$$

$$\pi(J_\infty)_{12}(0,0) = \sum_{n=0}^\infty | B'_{2n+1}(0)|^2 Q^-(2n+1) \geq \pi^2 J_0(0,0)^2 \sum_{n=0}^\infty Q^-(2n+1).$$

Step 8 of 3.1 should be compared with

(5) $$\pi \int J_{n-1}(\gamma, \gamma)\Delta = \theta(\infty) - \theta(-\infty) = 2n\pi \quad (\theta = -\text{phase } E_n).$$

Proof. $\pi J_{n-1}(\gamma,\gamma) = |E_n|^2\theta^\bullet = A_n B'_n - A'_n B_n$ is positive on R^1 and of degree $2(n-1)$. As such, it is expressible on R^1 as $|p|^2$ for some polynomial of degree $n - 1$, so that

$$\pi\int J_{n-1}(\gamma,\gamma) = \int |p|^2\Delta = \int |p/E_n|^2 = \theta(\infty) - \theta(-\infty)$$

$$= \int (A_n B'_n - A'_n B_n)\Delta.$$

But for an even degree $n = 2m$, (2a) tells us that

$$\textit{top coefficient } A'_{2m} = -2mQ^-(2m-1) \times \textit{top coefficient } B_{2m-1},$$

so

$$\int (A_{2m} B'_{2m} - A'_{2m} B_{2m})\,\Delta = -\int A'_{2m} B_{2m-1}\,\Delta$$

$$= 2mQ^-(2m-1)\int \textit{top coefficient } B_{2m-1} \times \gamma^{2m-1} B_{2m-1}\,\Delta$$

$$= 2mQ^-(2m-1)\,\| B_{2m-1}\|_\Delta^2$$

$$= 2m\pi.$$

The proof for odd degree is similar.

An aid to computation is the fact that

(6a) $\textit{top coefficient } A_{2n} = (-1)^n Q^-(2n-1)Q^+(2n-2)\cdots Q^-(1)Q^+(0)$

(6b) $\textit{top coefficient } B_{2n+1} = (-1)^n Q^+(2n)Q^-(2n-1)\cdots Q^-(1)Q^+(0).$

This is immediate from (2).

The reader will easily cast (2) into the form suggested in 1.3: simply take points

$$-\infty < T_0 < T_1 < T_2 < \text{etc.} < 0,$$

use the notation $Q^+(T_{2n})[Q^-(T_{2n-1})]$ in place of $Q^+(2n)[Q^-(2n-1)]$, and define

$$A^T = A_{2n} \qquad (T_{2n-1} \leq T < T_{2n+1})$$

$$B^T = B_{2n+1} \quad (T_{2n} \leq T < T_{2n+2}).$$

This makes (2) assume the form (4.1.2) = (1.3.4); it also makes (1) look like (4.1.1).

4.3. Eigendifferential transforms. The contents of 4.1, 4.2, and the results about the singular spectrum conjectured in 1.3. can now be brought together in a uniform notation patterned after that of 4.1, assuming the technical condition stated at the beginning of 4. $Q^{\pm}$, the so-called eigendifferentials A and B, $E = A - iB$, and the kernel J for $\mathbf{B}(E)$ are defined on the full spectrum

$$(T_0 < T_1 < T_2 < \text{etc.}) + [T_\infty, 0] + (0, \infty)$$

so as to have

(1a) $Q^-(I) = \pi^{-1}\| \gamma^{-1}A^I \|_\Delta^2$

(1b) $Q^+(I) = \pi^{-1}\| \gamma^{-1}B^I \|_\Delta^2$

(2a) $T_{2n} - T_{2n-1} = Q^-(T_{2n-1}) > 0 = Q^+(T_{2n-1})$

(2b) $T_{2n+1} - T_{2n} = Q^+(T_{2n}) > 0 = Q^-(T_{2n})$[22]

(3a) $\sqrt{dQ^- \, dQ^+} = 0$ *on* $[T_\infty, 0]$.

(3b) $\sqrt{dQ^- \, dQ^+} = dt$ *on* $(0, \infty)$.

(4a) $dA = -\gamma B \, dQ^-$ *subject to* $A \equiv 1$ *to the left of* T_0,

(4b) $dB = +\gamma A \, dQ^+$ *subject to* $B \equiv 0$ *to the left of* T_0,

(5) $$\pi J(\alpha, \beta) = \frac{E(\alpha)^* E(\beta) - E(\alpha^*) E^\times(\beta)}{-2i(\beta - \alpha^*)} = \int_{-\infty}^{T+} A(\alpha)^* A(\beta) dQ^+ + \int_{-\infty}^{T+} B(\alpha^*) B(\beta) dQ^-$$

(6) $\mathbf{B}(E^T) \to \mathbf{B}(E^R) \to \mathbf{Z}(\Delta)$ *are isometric injections for any* $T < R < \infty$.

(7a) $\mathbf{B}(E) = \mathbf{Z}_n(\Delta)$ *for* $T = T_n$ $(n \leq \infty)$.

(7b) $\mathbf{B}(E) = \mathbf{Z}^T(\Delta)$ *for* $0 \leq T < \infty$.

The fact that the map $f \to (f, J_\bullet)_\Delta$ is projection upon $\mathbf{B}(E)$ can now be expressed in the language of eigendifferential transforms as indicated in 1.3.

Given $f \in \mathbf{Z}(\Delta)$, define a pair of eigendifferential transforms, comparable to the classical sine-cosine transforms for $\mathbf{Z}(1) = L^2(R^1)$, by the recipe

(8a) $\hat{f}_{\text{even}} = \int A f \Delta$

(8b) $\hat{f}_{\text{odd}} = \int B f \Delta$.

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[22] This spacing is adopted merely for definiteness though it is believed to have a natural meaning: see the final remark of 1.3.

Apart from a factor $\sqrt{\pi}$, the mapping $\wedge$ is an isometric application of $\mathbf{Z}_{\text{even}}(\Delta)[\mathbf{Z}_{\text{odd}}(\Delta)]$ onto $L^2(Q^+)[L^2(Q^-)]$ as indicated by the Plancherel formula

$$\text{(9a)} \quad \pi \| f_{\text{even}} \|_\Delta^2 = \int_{-\infty}^{\infty} |\hat{f}_{\text{even}}|^2 dQ^+$$

$$\text{(9b)} \quad \pi \| f_{\text{odd}} \|_\Delta^2 = \int_{-\infty}^{\infty} |\hat{f}_{\text{odd}}|^2 dQ^-,$$

and the inverse mapping is simply

$$\text{(10a)} \quad f_{\text{even}} = (\hat{f}_{\text{even}})^\vee = \pi^{-1} \int_{-\infty}^{\infty} A \hat{f}_{\text{even}}\, dQ^+$$

$$\text{(10b)} \quad f_{\text{odd}} = (\hat{f}_{\text{odd}})^\vee = \pi^{-1} \int_{-\infty}^{\infty} B \hat{f}_{\text{odd}}\, dQ^-.$$

As in the classical case of $\mathbf{Z}(1) = L^2(R^1)$, the integrals in (8) [10] must be interpreted with due caution in $L^2(Q^\pm)[\mathbf{Z}(\Delta)]$. The fact that the map $f \to (f, J_\bullet)_\Delta$ is the projection $\mathfrak{p}$ upon $\mathbf{B}(E)$ can now be expressed as follows:

$$\text{(11a)} \quad \mathfrak{p} f_{\text{even}} = \int_{-\infty}^{T+} A \hat{f}_{\text{even}}\, dQ^+$$

$$\text{(11b)} \quad \mathfrak{p} f_{\text{odd}} = \int_{-\infty}^{T+} \hat{f}_{\text{odd}}\, dQ^-,$$

esp., apart from a factor $\sqrt{\pi}$, $\wedge$ is an isometric mapping of $\mathbf{B}_{\text{even}}(E)[B_{\text{odd}}(E)]$ onto $L^2(Q^+, (-\infty, T])$ $[L^2(Q^-, (-\infty, T])]$. (11) is the counterpart of the customary Paley-Wiener theorem for $\mathbf{Z}(1) = L^2(R^1)$; it provides the basis for the prediction recipes of 5.2.

Proof of (8)–(11). It is most convenient to prove first that the inverse transform $\vee$ defined by (10) is an isometric application of $L^2(Q^+) \oplus L^2(Q^-)$ onto $\mathbf{Z}_{\text{even}} \oplus \mathbf{Z}_{\text{odd}} = \mathbf{Z}$; for this purpose, it will suffice to deal with the even functions. Bring in the *even* inverse transform of the indicator function of a bounded interval $I = (T, R]$:

$$\pi^{-1} \int_I A\, dQ^+ = B^I / \pi\gamma = J_0^I(\gamma).$$

Because of (1b),

$$\pi (J_0^I, J_0^K) = \pi J^{I \cap K}(0, 0) = \pi^{-1} \| \gamma^{-1} B^{I \cap K} \|_\Delta^2 = Q^+(I \cap K),$$

so the mapping $\vee$ applied to such indicator functions extends to an isometry of $L^2(Q^+)$ *into* $\mathbf{Z}_{\text{even}}(\Delta)$ (see (9)). To see that this extended map is actually *onto*, it is enough to notice that the image of $L^2(Q^+)$ contains

$$\pi^{-1} \sum_{k \leq [nT]+1} A_{k/n}(\alpha)^* \int_{(k-1)/n}^{k/n} A(\beta)\, dQ^+,$$

which converges in $\mathbf{Z}(\Delta)$ to the reproducing kernel

$$J_{\text{even}}(\alpha, \beta) = \pi^{-1} \int_{-\infty}^{T+} A(\alpha)^* A(\beta)\, dQ^+$$

for $\mathbf{B}_{\text{even}}(E^T)$, and to use the fact that these functions span out the whole of $\mathbf{Z}_{\text{even}}$ as α runs over R^2 and $T \uparrow \infty$. At the same time, one sees that $\vee$ maps

$$L^2((T, R], Q^+) \quad \textit{onto} \quad \mathbf{B}_{\text{even}}(E^R)/\mathbf{B}_{\text{even}}(E^T)$$

for any $T < R$, and it is this observation that leads to (11). Now for a general function $f \in L^2(Q^+)$, the inverse transforms can be defined as

$$f^\vee = \lim_{T \uparrow \infty} \pi^{-1} \int_{-\infty}^{T} Af\, dQ^+ \quad \textit{in} \quad \mathbf{Z}_{\text{even}}(\Delta),$$

and, for the proof of (10), it remains to check that

$$(f^\vee)^\wedge = \lim_{a \downarrow -\infty, b \uparrow \infty} \int_a^b Af^\vee \Delta \quad \textit{in} \quad L^2(Q^+) = f.$$

Because of (9), it suffices to prove this in case f is the indicator function of a bounded half-open interval I. Using the notation $f^\vee_{ab} = f^\vee \times$ *the indicator function of the interval* ab, it develops that

$$\begin{aligned}
&\left\| f - \int_a^b Af^\vee\Delta \right\|_{Q^+}^2 \\
&= Q^+(I) - 2\int_I dQ^+ \int_a^b Af^\vee\Delta + \int_{-\infty}^{\infty} \left(\int_a^b Af^\vee\Delta\right)^2 dQ^+ \\
&= \pi \| f^\vee \|_\Delta^2 - 2\pi \int_a^b |f^\vee|^2\Delta + \lim_{T\uparrow\infty} \int_a^b f^\vee(\alpha)\Delta(\alpha) \\
&\qquad \cdot \int_a^b f^\vee(\beta)\Delta(\beta) \int_{-\infty}^{T} A(\alpha)A(\beta)\, dQ^+ \\
&= \pi \| f^\vee \|_\Delta^2 - 2\pi \| f^\vee_{ab} \|_\Delta^2 + \lim_{T\uparrow\infty} \pi \int_a^b\int_a^b f^\vee\Delta(\alpha) f^\vee\Delta(\beta) J^T(\alpha, \beta) \\
&= \pi \| f^\vee \|_\Delta^2 - 2\pi \| f^\vee_{ab} \|_\Delta^2 + \lim_{T\uparrow\infty} \pi(f^\vee_{ab}, \textit{the projection of } f^\vee_{ab} \textit{ upon } \mathbf{Z}^T)_\Delta \\
&= \pi \| f^\vee \|_\Delta^2 - \pi \| f^\vee_{ab} \|_\Delta^2,
\end{aligned}$$

and this expression $\downarrow 0$ as $a \downarrow -\infty$ and $b \uparrow \infty$, as desired.

4.4. Some examples. The computation of $Q^\pm$, A, B is very complicated in general. The following simple examples will give the reader some idea of how to go about it.

Example 1 $[\Delta = (1 + \gamma^2)^{-1}]$. $\mathbf{Z}^0(\Delta)$ is simply the space of *constants*, as is easily deduced from the fact that an integral function of minimal exponential type that sits in $\mathbf{Z}(1) = L^2(R^1)$ is $\equiv 0$; also, $1/h = 1 - i\gamma$ is integral and of minimal exponential type, so the projection recipe (3.1.7) gives

$$e^T = \exp(-i\gamma T)(1 - i\gamma) = E^T,$$

the spectrum consists of $(0, \infty)$ plus the point $0 = T_0 = T_\infty$, and

$$A^T = \cos\gamma T - \gamma \sin\gamma T, \qquad B^T = \gamma\cos\gamma T + \sin\gamma T,$$

$$dQ^+ = dT + \textit{a unit mass placed at } T = 0, \qquad dQ^- = dT.$$

The lump of mass $Q^+(0) = 1$ attached to the space of constants $\mathbf{Z}^0 = \mathbf{Z}_0$ comes from the evaluation

$$Q^+(0) = \pi \| A_0 \|_\Delta^{-2} = \pi\left(\int (1 + \gamma^2)^{-1}\right)^{-1} = 1.$$

Example 2 $[\Delta = (1 + \gamma^2)^{-2}]$. $1/h = (1 - i\gamma)^2$ is still an integral function of minimal exponential type, so the projection recipe gives

$$e^T = \exp(-i\gamma T)(1 - i\gamma)^2 = E^T,$$

and for $T > 0$,

$$A^T = (1 - \gamma^2)\cos\gamma T - 2\gamma\sin\gamma T, \qquad B^T = 2\gamma\cos\gamma T + (1 - \gamma^2)\sin\gamma T$$

$$dQ^- = dQ^+ = dT,$$

much as before. But now $Z^0 = Z_1 =$ *polynomials of degree* ≤ 1, and one needs A_0, B_1, and 2 lumps of mass to complete the picture:

$$A_0 = 1, \quad Q^+(T_0) = 2, \quad T_0 = -2,$$

$$B_1 = 2\gamma, \quad Q^-(T_1) = \tfrac{1}{2}, \quad T_1 = T_\infty = 0.$$

Example 3 $[\Delta = \gamma^2(1 + \gamma^2)^{-2}]$. $\mathbf{Z}^0$ is just the space of *constants* as in Example 1, but the computation of A and B is much more complicated. The idea is to express $A^T(B^T)$ as a kind of Gram-Schmidt superposition of cosines (sines) of lower exponential type, as in the paper of Gelfand-Levitan [7]: for example, one could seek to impose the (formal) condition $\int A^T \cos\gamma t\Delta = 0$ $(0 \leq t \leq T)$ upon some (formal) representation $A^T = \int_0^T K(T, t)\cos\gamma t\, dt$ and hope that this would pin down A. As it happens this K is not an honest function, so it is better to put

$$A^T = \cos\gamma T + \gamma^2 \int_0^T K(T, t)\cos\gamma t\, dt$$

with $K(T, \cdot) \in L^2[0, T]$, which can be justified by an application of the standard Paley-Wiener theorem to the function $\gamma^{-2}(A - \cos\gamma T) \in L^2(R^1)$. A is now computed by this plan, assuming that $K(T, \cdot)$ is a polynomial. B can be guessed at in the same fashion, but it is more efficient to put $dA = -\gamma B\, dQ^-$ and to adjust Q^- so as to make $\int (B/\pi\gamma)A = 1$. Q^+ can then be computed on $(0, \infty)$ from $\sqrt{dQ^-\, dQ^+} = dT$. This recipe gives

$$A^T = (T + 2)^{-1}[2\cos\gamma T - \gamma\sin\gamma T + \gamma^{-1}\sin\gamma T],$$

$$B^T = (2T+3)\sin\gamma T + (T+2)\gamma\cos\gamma T - T\gamma^{-1}\cos\gamma T + \gamma^{-2}\sin\gamma T,$$
$$dQ^+ = (T+2)^2\,dT, \qquad dQ^- = (T+2)^{-2}\,dT$$

on the type of spectrum, and it is easy to verify that this is correct. Q^+ is to be augmented by a jump $Q^+(0) = \pi\| A_0 \|_\Delta^{-2} = 2$ to account for the space $\mathbf{Z}^0 = \mathbf{Z}_0$.

Example 4 $[\Delta = \gamma^4(1+\gamma^2)^{-3}]$. The method is the same as for Example 3, and the formulas are as follows:

$$A^T = (3/2)(T^3+9T^2+27T+24)^{-1}[2(3T+8)\cos\gamma T - 2(T+3)\gamma\sin\gamma T$$
$$+ 6(T+2)\gamma^{-1}\sin\gamma T - 2T\gamma^{-2}\cos\gamma T + 2\gamma^{-3}\sin\gamma T]$$
$$B^T = (1/6)(T+3)^{-1}[6(T^3+8T^2+21T+15)\sin\gamma T$$
$$+ 2(T^3+9T^2+27T+24)\gamma\cos\gamma T - 6T(T^2+6T+10)\gamma^{-1}\cos\gamma T$$
$$- 2(T^3-18T-30)\gamma^{-2}\sin\gamma T - 6T(T+3)\gamma^{-3}\cos\gamma T$$
$$+ 6(T+3)\gamma^{-4}\sin\gamma T].$$
$$dQ^+ = \frac{(T^3+9T^2+27T+24)^2}{9(T+3)^2}\,dT + \frac{8}{3}\times \textit{the unit mass at } T = 0$$
$$dQ^- = \frac{9(T+3)^2}{(T^3+9T^2+27T+24)^2}\,dT.$$

5. Prediction

The results of 3 and 4 are now applied to the Gaussian process $\mathfrak{x}$. 5.1 will refresh the reader's memory of the Kolmogorov-Szegö-Wiener problem of prediction using the whole past; the solution is expressed by means of Hardy functions, following Karhunen [9]. 5.2 deals with the same problem when only a bounded segment of the past $\mathfrak{x}(s) : -T \le s \le 0$ is permitted. The eigendifferential transform of 4.3 is the correct tool for this; the reader will note that only the type spectrum comes into play, so this part is independent of any conjectures concerning the singular spectrum. 5.3 is devoted to the germ $\mathbf{B}^0$ and esp. to the significance of the gap $\mathbf{Z}^0/\mathbf{Z}_\infty$.

5.1. Prediction using the whole past. Given a stationary Gaussian process with sample paths $t \to \mathfrak{x}(t)$ and spectral weight Δ, the standard prediction problem is to project $\mathfrak{x}(t)$ for a fixed $t > 0$ upon the past

$$\mathbf{M}^- = \mathbf{M}^{-\infty 0} = \text{span}\,[\mathfrak{x}(s) : s \le 0],$$

or, in trigonometrical language, to project $\exp(i\gamma t)$ upon

$$\mathbf{Z}^- = \mathbf{Z}^{-\infty 0} = \text{span}\,[\exp(i\gamma s) : s \le 0] \textit{ in } \mathbf{Z}(\Delta) = L^2(\Delta, R^1).$$

This problem was solved independently by Kolmogorov [10], Szegö [16], and Wiener [18], but it was Karhunen [9], who first noticed how easily it may be done using Hardy functions.

The first step is to recall Szegö's alternative [see 1.2] which states that

$$\textit{either}\quad \Delta \textit{ is non-Hardy}\left[\int (1+\gamma^2)^{-1}\lg\Delta = -\infty\right] \textit{and } \mathbf{Z} = \mathbf{Z}^- = \mathbf{Z}^{-\infty}$$

$$\textit{or}\quad \Delta \textit{ is Hardy}\left[\int (1+\gamma^2)^{-1}\lg\Delta > -\infty\right] \textit{and } \mathbf{Z} \neq \mathbf{Z}^- \neq \mathbf{Z}^{-\infty} = 0.$$

Δ non-Hardy means that *the prediction is perfect*, so only the Hardy case needs to be discussed. Express Δ as $|h|^2$ with $h \in H^{2+}$ and $h^* = h(-\gamma)$ on R^1 as usual, but do not assume that h is outer. Because

$$\hat{h} = (2\pi)^{-1/2}\int \exp(-i\gamma t)h$$

vanishes on the left half-line, *the process* $\mathfrak{x}$ *can be identified as a filtered white noise* $\hat{h}\circ b^\bullet = \int_{s\le t}\hat{h}(t-s)\,db(s)$, b being a standard Brownian motion and $b^\bullet$ = *white noise* its (formal) derivative with respect to time.

Proof. Using the standard Plancherel formula for $L^2(R^1)$ and the fact that $\hat{h}$ is real, one verifies

$$E[\hat{h}\circ b^\bullet(t_1)\,\hat{h}\circ b^\bullet(t_2)] = \int \hat{h}(t_1-s)\hat{h}(t_2-s)^*\,ds = \int e^{i\gamma(t_1-t_2)}\Delta.$$

This means that $\hat{h}\circ b^\bullet$ and $\mathfrak{x}$ are identical in law, so that the two may be identified.

Because the distribution of $\mathfrak{x} = \hat{h}\circ b^\bullet$ depends upon the modulus $|h| = \sqrt{\Delta}$ only, *it is permissible to pick* h *to be an outer function, and this is precisely the condition that the past*

$$\mathbf{M}^{-\infty 0} = \text{span}\,[\mathfrak{x}(t) : t \le 0]$$

should coincide with the Brownian past span $[b(t) - b(0) : t \le 0]$.

Proof. (2.1.7b) tells us that span $[\hat{h}(t - \cdot) : t \le 0]$ fills up the whole of $L^2(-\infty, 0]$ iff h is outer, so that

$$\text{span}\,[b(t) - b(0) : t \le 0] = [\textstyle\int_{-\infty}^0 k\,db : k \in L^2(-\infty, 0]]$$

is the same as

$$\mathbf{M}^{-\infty 0} = \text{span}\left[\int_{-\infty}^0 \hat{h}(t-\cdot)\,db : t \le 0\right]$$

iff h is outer.

But if h is outer, so that $\mathbf{M}^{-\infty 0}$ matches with the Brownian past, then the projection of $\mathfrak{x}(t)$ upon $\mathbf{M}^{-\infty 0}$ is the *same* as its projection upon the Brownian past, and this is just the expectation of $\mathfrak{x}(t)$, conditional on the Brownian past: $\int_{-\infty}^0 \hat{h}(t-s)\,db(s)$. The prediction error $\sqrt{D}$ (= the norm of the co-projection) is easily evaluated from this:

$$D = \inf_{k \in L^2(-\infty,0]} E\left[\left|\mathfrak{x}(t) - \int_{-\infty}^{0} k\, db\right|^2\right] = \int_0^t |\hat{h}|^2.$$

A simple proof of the test (2.1.7d) for outer H^{2+} functions can be based on the above. Pick $f \in H^{2+}$ with the same modulus on R^1 as the outer function h. Then $\mathfrak{x}_1 = \hat{f} \circ b^\bullet$ has the same statistics as $\mathfrak{x} = \hat{h} \circ b^\bullet$. Define $\mathfrak{p}_1$ [$\mathfrak{p}$] to be the projection upon the past of $\mathfrak{x}_1[\mathfrak{x}]$. Because the past of $\mathfrak{x}$ is the same as the Brownian past, it contains the past of $\mathfrak{x}_1$ as a submanifold, and so $E(|\mathfrak{p}_1 \mathfrak{x}_1|^2) \leq E(|\mathfrak{p}\mathfrak{x}_1|^2)$. But also $\mathfrak{p}_1 \mathfrak{x}_1$ is identical in law to $\mathfrak{p}\mathfrak{x}$, so

$$\int_t^\infty |\hat{h}|^2 = E(|\mathfrak{p}\mathfrak{x}|^2) = E(\mathfrak{p}_1 \mathfrak{x}_1|^2) \leq E(|\mathfrak{p}\mathfrak{x}_1|^2) = \int_t^\infty |\hat{f}|^2,$$

and (2.1.7d) follows from the fact that

$$\| \hat{h} \|_2^2 = E(|\mathfrak{x}|^2) = E(|\mathfrak{x}_1|^2) = \|\hat{f}\|_2^2.$$

5.2. Prediction using a bounded segment of the past. The device of matching fields is not available if it is desired to predict $\mathfrak{x}(t)$ from a bounded segment of the past $\mathfrak{x}(s) : -T \leq s \leq 0$; this is due to the fact that the field $\mathbf{F}^{-T0}$ *never* matches the corresponding Brownian field. The proper tool is the eigen-differential transform of 4.3 associated with the spaces $\mathbf{Z}^T(\Delta) : T \geq 0$. The *best prediction* is the conditional expectation

(1a) $\mathfrak{p}\mathfrak{x} = E[\mathfrak{x}(t) \mid \mathbf{F}^{-T0}]$,

and the *prediction error* $\sqrt{D}$ is the norm of the co-projection:

(2a) $D = E(|(1 - \mathfrak{p})\mathfrak{x}|^2)$.

Now in the trigonometrical language of $\mathbf{Z}(\Delta)$, the best prediction is the projection of $\exp(i\gamma t)$ upon $\mathbf{Z}^{-T0}(\Delta)$:

(1b) $\mathfrak{p}e^{i\gamma t} = e^{-i\gamma T/2} \times$ *the projection of* $e^{i\gamma T/2}e^{i\gamma t}$ *upon* $\mathbf{Z}^{T/2}(\Delta)$,

and the error is

(2b) $\sqrt{D} = \|$ *co-projection of* $e^{i\gamma T/2}e^{i\gamma t}$ *upon* $\mathbf{Z}^{T/2} \|_\Delta$.

To compute (1b), one splits $f = \exp[i\gamma(T/2 + t)]$ into even and odd parts and projects them separately:

(3a) *projection of* $\cos \gamma(T/2 + t)$ *upon* $\mathbf{Z}^{T/2}_{\text{even}} = \pi^{-1}\int_{-\infty}^{T/2} K^+(\cdot, T/2 + t) A\, dQ^+$

(3b) *projection of* $\sin \gamma(T/2 + t)$ *upon* $\mathbf{Z}^{T/2}_{\text{odd}} = \pi^{-1}\int_{-\infty}^{T/2} K^-(\cdot, T/2 + t) B\, dQ^-$,

in which $K^+(\cdot, T) = \int A \cos \gamma T \Delta$ [$K^-(\cdot, T) = \int B \sin \gamma T \Delta$] is the even [odd] transform of $\cos \gamma T$ [$\sin \gamma T$]. Define $a = T/2$ and $b = T/2 + t$. Using the Plancherel formula (4.3.9), the prediction error (2) may be computed, as follows:

$$(4)\quad D = D(a,b) = \pi^{-1}\int_a^b K^+(\cdot,b)^2\,dQ^+ + \pi^{-1}\int_a^b K^-(\cdot,b)^2 dQ^- \equiv D^+ + D^-.$$

The reader who is familiar with *P*. Lévy's idea about Gaussian processes [12, 279–312] will see from the formula

$$E[\mathfrak{x}^\pm(T_1)\mathfrak{x}^\pm(T_2)] = \int_{\sin}^{\cos} \gamma T_1 \, {}^{\cos}_{\sin}\gamma T_2\, \Delta = \pi^{-1}\int_{-\infty}^{\infty} K^\pm(\cdot, T_1)K^\pm(\cdot, T_2)\, dQ^\pm$$

that $K^\pm$ is the kernel of a non-singular representation of the odd [even] part $\mathfrak{x}^-[\mathfrak{x}^+]$ of $\mathfrak{x}$ as a so-called white noise integral:

$$\mathfrak{x}^\pm(T) = \int_{-\infty}^{T} K^\pm(\cdot, T)\, d\mathfrak{y}^\pm.$$

Here, $\mathfrak{y}^\pm$ is a white noise based upon $Q^\pm$, i.e., $\mathfrak{y}^\pm(-\infty, T]$ is a standard Brownian motion run with the clock $Q^\pm(-\infty, T]$; in these representations, the fields match, so they provide explicit prediction formulas much as in the classical case of 5.1.

Example 4. 4.3 provides a simple illustration of the recipe.

$$\Delta = \gamma^2(1+\gamma^2)^{-2}, \quad \text{and for} \quad T \geq R \geq 0,$$

$$K^+(R,T) = \pi e^{-(T-R)}\left[\frac{1-(T-R)}{R+2}\right]$$

$$K^-(R,T) = \pi e^{-(T-R)}[R+2-(T-R)(R+1)]$$

$$D^+ = \pi\int_0^t e^{-2s}(1-s)^2\,ds = \frac{1}{2}\int_0^t |\hat{h}|^2$$

$$D^- = \pi\int_0^t e^{-2s}\left[1 - \frac{s(T/2+t-s+1)}{T/2+t-s+2}\right]^2 ds$$

$$D = \int_0^t |\hat{h}|^2 + T^{-1}\left[4\pi\int_0^t e^{-2s}s(1-s)\,ds + o(1)\right] \quad (T\uparrow\infty).$$

The purpose of this section is to prove, under the extra technical condition $\int\gamma^6\Delta < \infty$, that *the errors* $\sqrt{D^\pm}$ *and* $\sqrt{D}$, *as functions of* $a = T/2$ *and* $b = T/2 + t$, *satisfy*

$$(5)\qquad \frac{\partial^2 D^+}{\partial Q^-(a)\partial Q^+(a)} + \frac{\partial^2 D^-}{\partial Q^+(a)\partial Q^-(a)} = \frac{\partial^2 D}{\partial b^2} \quad (b > a > 0),$$ [23]

subject to the side conditions

$$(6a)\quad \lim_{t\downarrow 0} D = 0$$

[23] $\int\gamma^6\Delta < \infty$ is used to justify differentiation under the integral signs in $K^+(\cdot, T) = \int A\cos\gamma T\Delta$ and $K^-(\cdot, T) = \int B\sin\gamma T\Delta$. The conjecture is that (5) holds in general.

(6b) $D \geq \lim_{T\uparrow\infty} D = \int_0^t |\hat{h}|^2$.

(6b) is attained, i.e.,

(7a) $D = \int_0^t |\hat{h}|^2$ *for any choice of t and T*

and (5) *simplifies to* $\partial^2 D/\partial a^2 = \partial^2 D/\partial b^2$ *iff one of the following equivalent conditions holds*:

(7b) $1/h$ *is an integral function of minimal exponential type.*
(7c) $dQ^{\pm} =$ constant $\times\ dT$ *on* $(0, \infty)$.
(7d) $\mathbf{F}^{+/-} = \mathbf{F}^0$, *i.e.,* $\mathfrak{x}$ *splits over its germ.*[24]

(7) can be amplified as follows. *For any* $T > 0$, *the following conditions are equivalent*:

(8a) $D = \int_0^t |\hat{h}|^2$ *for any* $t \geq 0$.
(8b) $[\exp(i\gamma T/2)h]^{-1}$ *is an integral function of exponential type* $\leq T/2$.
(8c) $dQ^{\pm} =$ constant $\times\ dT$ *on* $[T/2, \infty)$.
(8d) $\mathfrak{x}$ *splits over* $\mathbf{F}^{-T0}$.[25]

Grenander-Szegö [8, pp. 188–191] discuss briefly the prediction of stationary *chains* using a bounded segment of the past. The cheapest of their results can be adapted to the present case as

(9) $\sqrt{D(T/2, T/2 + t) - \int_0^t |\hat{h}|^2}$, *which is the error introduced by using a bounded segment instead of the whole past, is exponentially small for* $T \uparrow \infty$ *if* h *has an* H^{2+} *extension to a strip* $0 > b \geq -k$ *of* R^{2-}, *i.e., if* $h(\cdot - ik) \in H^{2+}$.

Proof of (5). $\int \gamma^6 \Delta < \infty$ permits us to differentiate under the integrals defining $K^{\pm}$: the results are

$$\frac{\partial K^+}{\partial Q^-(a)} = -\frac{\partial K^-}{\partial b}, \qquad \frac{\partial K^-}{\partial Q^+(a)} = -\frac{\partial K^+}{\partial b},$$

$$\frac{\partial^2 K^+}{\partial Q^+(a)\partial Q^-(a)} = \frac{\partial^2 K^+}{\partial b^2}, \qquad \frac{\partial^2 K^-}{\partial Q^-(a)\partial Q^+(a)} = \frac{\partial^2 K^-}{\partial b^2},$$

$K^{\pm}$ and all the displayed partials being continuous on $a \geq b$, and now a straight-forward differentiation of (4) supplies us with the formula

$$\frac{\partial^2 D^+}{\partial Q^-(a)\partial Q^+(a)} + \frac{\partial^2 D^-}{\partial Q^+(a)\partial Q^-(a)} = \frac{2}{\pi}\frac{\partial K^- K^+}{\partial b}.$$

---

[24] (7b) = (7d) was proved in Levinson-McKean [11, 121–123].

[25] Levinson-McKean [11, 142] indicated a proof that if, for fixed $T > 0$, $D = \int_0^t |h|^2$ at a single time $t > 0$, then (7) holds. This is false, but the reader will easily see that for (8) to hold it suffices to have $D = \int_0^t |h|^2$ on an open interval of times $t > 0$.

But also $K^{\pm} = 0$ on the diagonal,[26] so

$$\frac{\partial D}{\partial b} = \frac{1}{\pi}\int_a^b \frac{\partial}{\partial b} K^+(\cdot, b)^2\, dQ^+ + \frac{1}{\pi}\int_a^b \frac{\partial}{\partial b} K^-(\cdot, b)^2\, dQ^-$$
$$= -\frac{2}{\pi}\int_a^b (K^+\, dK^- + K^-\, dK^+)$$
$$= \frac{2}{\pi} K^- K^+(a, b)$$

and (5) follows.

Proof of (6). (6a) is plain. (6b) follows from the fact that, as $T \uparrow \infty$, $\sqrt{D(T/2, T/2 + t)}$ decreases to the error for prediction using the whole past.

Proof of (7). Suppose (7a) holds. Then the projection of exp $(i\gamma t)$ upon $\mathbf{Z}^{-T0}$ is independent of T, *esp.*, the projection upon $\mathbf{Z}^{-\infty 0}$ is the same as the projection upon Z^0, and as such it is an integral function of minimal exponential type. The projection upon $\mathbf{Z}^{-\infty 0}$ can be expressed on R^1 as

$$\frac{1}{h^*} \times \textit{the projection of } e^{i\gamma t}h^* \textit{ upon } H^{2-} = \frac{1}{h^*} \times (2\pi)^{-1/2}\int_{-\infty}^0 e^{i\gamma s}(e^{i\gamma t}h^*)^\wedge ds$$
$$= \frac{1}{h^*\sqrt{2\pi}}\int_{-\infty}^0 e^{i\gamma s}\hat{h}(t - s)\, ds \quad {}^{27}$$
$$= \frac{e^{i\gamma t}}{h^*\sqrt{2\pi}}\int_t^\infty e^{-i\gamma s}\hat{h},$$

and so the conjugate function

$$f_t = e^{-i\gamma t}(h\sqrt{2\pi})^{-1}\int_t^\infty e^{i\gamma s}\hat{h}$$

belongs to $\mathbf{Z}^0$. Pick $t \geq 0$ so that

$$\lim_{\delta \downarrow 0} \delta^{-1}\int_t^{t+\delta} \hat{h} = \hat{h}(t) \neq 0 \textit{ and } \overline{\lim_{\delta \downarrow 0}}\, \delta^{-1}\int_t^{t+\delta} |\hat{h}| < \infty.$$

Then, as the reader will easily check, $\delta^{-1}\| f_{t+\delta} - f_t \|_{\Delta^+}$ stays bounded as $\delta \downarrow 0$, so that the *pointwise limit*

$$\lim_{\delta \downarrow 0} \delta^{-1}(f_{t+\delta} - f_t) = -i\gamma f_t - \hat{h}(t)(h\sqrt{2\pi})^{-1} \in \mathbf{Z}^0(\Delta^+),$$

i.e., $1/h \in \mathbf{Z}^0(\Delta^+)$, which is the statement of (7b).[28] Now suppose (7b) holds. Then e = (3.1.7) = $[\exp(i\gamma T)h]^{-1}$ for $T \geq 0$. A, B, and $Q^{\pm}$ can now be computed and (7c) can be verified:

$$h(0)^2\, dQ^+ = h(0)^{-2}\, dQ^- = dT.$$

[26] (4.1.6) and $\int \gamma^6 \Delta < \infty$ take care of K^+. The reader will cope with K^-.
[27] h is real since $h^* = h(-\cdot)$ on R^1.
[28] This part of the proof is adapted from Levinson-McKean [11, 122].

A routine computation now shows that

$$K^+(a, b) = h(0)^2 K^-(a, b) = \sqrt{\pi/2}\, h(0)\hat{h}(b - a),$$

which implies

$$D^\pm = \frac{1}{\pi}\int_a^b (K^\pm)^2\, dQ^\pm = (1/2)\int_0^t |\hat{h}|^2,$$

so that (7a) and (7b) are the same; also, it is plain that (7a) = (7d), so it remains only to check that (7c) implies (7d). But if $k^{-1}\, dQ^+ = k\, dQ^- = dT$, then

$$A^T = A^0 \cos\gamma T - k^{-1}B^0 \sin\gamma T, \qquad B^T = B^0\cos\gamma T + kA^0 \sin\gamma T$$

by (4.1.2), and for $f \in \mathbf{Z}^T$,

$$\|f\|_\Delta^2 = \int \frac{|f|^2}{(A^T)^2 + (B^T)^2} = \int \frac{|f|^2}{(\sqrt{k}\, A^T)^2 + (B^T/\sqrt{k})^2}$$
$$= \int \frac{|f|^2}{k(A^0)^2 + (B^0)^2/k}$$

by (2.3.6). Because $\bigcup \mathbf{Z}^T$ spans $\mathbf{Z}$,

$$\|f\|_\Delta^2 = \int |f|^2\, |\sqrt{k}A^0 - iB^0/\sqrt{k}|^{-2}$$

for all $f \in \mathbf{Z}$, and since $\sqrt{k}A^0 - iB^0/\sqrt{k}$ is an outer function on R^{2+},

$$1/h = \sqrt{k}A^0 - iB^0/\sqrt{k} \in \mathbf{Z}^0(\Delta^+)$$

up to a factor of modulus 1, i.e., (7b) holds.

The proof of (8) can be left to the reader with the proof of (7) as a model.

Proof of (9). Suppose h possesses an H^{2+} extension to a strip below R^1 of width $k > 0$. Then $\int_0^\infty e^{2kt}\, |\hat{h}|^2 = c < \infty$, and so

$$\begin{aligned} 0 &\le D(T/2, T/2 + t) - \int_0^t |\hat{h}|^2 \\ &\le \|\, \textit{projection } e^{i\gamma t} \textit{ upon } \mathbf{Z}^{-\infty - T}\, \|_\Delta^2 \\ &= \|\, \textit{projection } e^{i\gamma(T+t)} \textit{ upon } \mathbf{Z}^{-\infty 0}\, \|_\Delta^2 \\ &= \int_{T+t}^\infty |\hat{h}|^2 \\ &\le e^{-2kT} c, \end{aligned}$$

as advertised.

Added in proof. After this paper was prepared the authors learned of the deep results of M. G. Kreĭn [IMS-AMS Selected Translations in Math. Stat. and Prob., vol. 4 (1963), pp. 127–131] in which he solved the extrapolation problem by different methods than presented here. An account of his work will appear in a survey article, *Extrapolation and interpolation of stationary gaussian processes*, currently being prepared for publication.

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