SUBALGEBRAS IN A SUBSPACE OF $C(X)$

BY
S. P. LLOYD

1. Introduction

Let $\mathfrak{X}$ be an algebra and let L be a linear subspace of $\mathfrak{X}$. The set $\mathfrak{M}_L$ of (left) multipliers for L is defined as $\mathfrak{M}_L = \{x \in \mathfrak{X} : xL \subset L\}$, that is, left translation by a member of $\mathfrak{M}_L$ leaves subspace L invariant. It is easy to see that $\mathfrak{M}_L$ is a subalgebra of $\mathfrak{X}$ and that if $\mathfrak{X}$ has a unit and L contains the unit then $\mathfrak{M}_L$ is contained in L. We will be concerned with the case where $\mathfrak{X}$ is the continuous real functions on a compact Hausdorff space X and L is a closed subspace of $C(X)$ containing the constants. Now, L is ordered by the order in $C(X)$, so that the conjugate Banach space L^* is ordered; we find it necessary to assume that L^* is a lattice in this ordering. Under this hypothesis we prove that $\mathfrak{M}_L$ is the maximum subalgebra of L; that is, every subalgebra of $C(X)$ contained in L is a subalgebra of $\mathfrak{M}_L$. An example will show that the assumption that L^* is a lattice is not superfluous. Our characterization of $\mathfrak{M}_L$ involves the ideas and methods associated with Choquet's theorem.

2. Generalized harmonic functions

References for Choquet's theorem are Phelps [1] and the Edwards lecture notes [2]. In the present section we obtain a generalization of a result of Bauer [3] concerning generalized harmonic functions.

Let X be a compact Hausdorff space and let $C(X)$ be the algebra of continuous real functions on X. We identify the conjugate Banach space of $C(X)$ with the space $rca(X)$ of signed Radon measures (regular Borel measures) on X. We denote by $rca^+(X)$ the nonnegative members of $rca(X)$, and by prob (X) the probability measures in $rca(X)$. For $x \in X$, $\delta_x \in$ prob (X) will denote the evaluation measure at x. On several occasions we will deliberately confuse x with δ_x, choosing to regard X as a subset of $w^* - rca(X)$.

Let L be a closed subspace of $C(X)$ containing the constants. We assume to begin with that L separates the points of X; at the end we drop this assumption. Sometimes we will treat L as a Banach space in its own right, and we denote by $\kappa : L \to C(X)$ the injection into $C(X)$. The adjoint $\kappa^* : rca(X) \to L^*$ does the following things:

(i) maps $rca(X)$ onto L^*; the mapping is $w^* - w^*$ continuous and preserves order;

(ii) maps prob (X) onto the convex w^* − compact set

$$K = \{\xi \in L^* : \|\xi\| = 1 = (1, \xi)\};$$

Received March 25, 1968.

the set K will always have the relative weak * topology;

(iii) maps $X \subset w^* - rca(X)$ homeomorphically onto a subset X_0 of K;

(iv) maps the Choquet boundary $B \subset X$ homeomorphically onto the set of extreme points of K.

The set K is a base for the nonnegative cone $(L^*)^+$ in L^*. In Theorems 1 and 2 we will make the assumption that L^* is a lattice; this is the same as assuming that K is a Choquet simplex [1, §9].

The equivalence relation $\sim$ in $rca(X)$ is defined by $\mu_1 \sim \mu_2$ iff $(f, \mu_1) = (f, \mu_2)$ for all $f \in L$. It is easy to see that the annihilator $L^\perp \subset rca(X)$ of subspace L can be represented as

$$L^\perp = \{\mu_1 - \mu_2 : \mu_1, \mu_2 \in rca^+(X) \quad \text{and} \quad \mu_1 \sim \mu_2\}.$$

(If $\nu = \nu^+ - \nu^- \in L^\perp$ then $\nu^+ \sim \nu^-$.)

The order relation $\prec$ in $rca(X)$ is defined as follows. Let $\hat{L} \subset C(X)$ be the set of all functions $\kappa g_1 \wedge \cdots \wedge \iota g_m$ for all m and all $g_1, \cdots, g_m \in L$. Then $\mu \prec \nu$ iff $(f, \mu) \geq (f, \nu)$ for all $f \in \hat{L}$. The $\prec$ relation refines the $\sim$ relation; that is, $\mu \prec \nu$ implies $\mu \sim \nu$. The $\prec$ relation is equivalent to the one defined by Bishop and de Leeuw [4].

If $x \in X$ is given, a measure $\mu_x \in$ prob (X) with the property $\mu_x \sim \delta_x$ is said to be a representing measure for x. It is known that if $\mu_x \in$ prob (X) and $\mu_x \sim \delta_x$ then $\delta_x \prec \mu_x$ [2].

For given $\mu \in rca^+(X)$ the functional $p_\mu(f), f \in C(X)$, is defined by

$$p_\mu(f) = \inf_g \{(g, \mu) : g \in \hat{L} \text{ and } g \geq f\}$$

$$(-p_\mu(-f) = \sup_g \{(g, \mu) : g \in -\hat{L} \text{ and } g \leq f\}.)$$

By $p_x(f)$ we will mean $p_{\delta_x}(f)$, $x \in X$. It is known that $p_\mu(f)$ has equivalent representations

$$p_\mu(f) = \sup_{\mu'} \{(f, \mu') : \mu' \in rca^+(X) \text{ and } \mu \prec \mu'\}$$

$$= \int p_x(f)\mu(dx).$$

$$(-p_\mu(-f) = \inf_{\mu'} \{(f, \mu') : \mu' \in rca^+(X) \text{ and } \mu \prec \mu'\}$$

$$= \int [- p_x(-f)]\mu(dx).)$$

For each $\mu \in rca^+(X)$ the functional $p_\mu(f)$ has the following properties [2]:

(i) $p_\mu(f + g) \leq p_\mu(f) + p_\mu(g), f, g \in C(X)$,

(ii) $p_\mu(cf) = cp_\mu(f), c \geq 0, f \in C(X)$,

(iii) $|p_\mu(f)| \leq \|\mu\| \|f\|, f \in C(X)$,

(iv) $p_\mu(f) \geq (f, \mu), f \in C(X)$, with equality when $f \in \hat{L}$,

(v) $p_\mu(f) = -p_\mu(-f) = (f, \mu), f \in L$,

(vi) $p_\mu(f) = (f, \mu)$ for all $f \in C(X)$ iff μ is maximal in the ordering $\prec$; in particular,

(vii) $p_x(f) = f(x)$ for all $f \in C(X)$ iff $x \in B$.

The functions $f \in C(X)$ for which $p_x(f) = -p_x(-f)(=f(x))$ for every $x \in X$ are called L-harmonic by Bauer [3]. We denote by $\mathcal{H}_L$ the set of L-harmonic functions; $\mathcal{H}_L$ is a closed linear subspace of $C(X)$, and $L \subset \mathcal{H}_L$ by (v). It is of interest to determine when $\mathcal{H}_L = L$. We remark that the annihilator of $\mathcal{H}_L$ is w^*-spanned by the set

$$\{\delta_x - \mu_x : \mu_x \in \text{prob } (X) \text{ and } \delta_x \prec \mu_x ,\, x \in X\},$$

while the annihilator of L is w^*-spanned by

$$\{\mu_1 - \mu_2 : \mu_1 ,\, \mu_2 \in \text{prob } (X) \text{ and } \mu_1 \sim \mu_2\};$$

the problem of whether or not $\mathcal{H}_L = L$ is equivalent to the problem of whether or not such measures $\mu_1 - \mu_2$ are w^*-spanned by measures $\delta_x - \mu_x$ in $(\mathcal{H}_L)^\perp$.

It will be convenient to consider also the functional $q_\mu(f), f \in C(X)$, defined for given $\mu \in rca^+(X)$ by

$$q_\mu(f) = \sup_{\mu'}\{(f, \mu') : \mu' \in rca^+(X) \text{ and } \mu \sim \mu'\}$$

$$(-q_\mu(-f) = \inf_{\mu'} \{(f, \mu') : \mu' \in rca^+(X) \text{ and } \mu \sim \mu'\}.)$$

This functional has the properties

(i) $q_\mu(f + g) = q_\mu(f) + q_\mu(g), f, g \in C(X)$,

(ii) $q_\mu(cf) = cq_\mu(f), c \geq 0, f \in C(X)$,

(iii) $|q_\mu(f)| \leq \|\mu\|\|f\|, f \in C(X)$,

(iv) $p_\mu(f) \leq q_\mu(f), f \in C(X)$,

(v) $q_\mu(f) = -q_\mu(-f)$ for all $\mu \in rca^+(X)$ iff $f \in L$.

Properties (i)–(iv) are straightforward or obvious. To see (v), note first that if $f \in L$ then $q_\mu = -q_\mu(-f)$, clearly. On the other hand, if $f \notin L$ then $(f, \mu - \mu') \neq 0$ for some $\mu, \mu' \in rca^+(X), \mu' \sim \mu$, whence $q_\mu(f) > -q_\mu(-f)$.

THEOREM 1. *Let L be a closed linear subspace of $C(X)$ which contains the constants and separates the points of X. Assume that L^* is a lattice; that is, the base K of $(L^*)^+$ is a Choquet simplex. Then $\mathcal{H}_L = L$; that is, for $f \in C(X)$ the condition $p_x(f) = -p_x(-f)$ for every $x \in X$ is necessary and sufficient for $f \in L$.*

Proof. We will show that $p_x(f) = -p_x(-f)$ for all $x \in X$ implies $q_\mu(f) = -q_\mu(-f)$ for all $\mu \in rca^+(X)$. Suppose to the contrary that $f \in C(X)$ is such that $p_x(f) = -p_x(-f)$ for all $x \in X$ but that $q_\mu(f) > -q_\mu(-f)$ for some $\mu \in rca^+(X)$. Since q is positive homogeneous in μ, we may assume that $\mu \in \text{prob } (X)$. Let $\mu_1 \in rca(X)$ be such that $-q_\mu(-g) \leq (g, \mu_1) \leq q_\mu(g)$ for all $g \in C(X)$ and also $(f, \mu_1) = q_\mu(f)$ (Hahn-Banach). Since $-q_\mu(-g) \geq 0$ for $g \geq 0$, we have $\mu_1 \geq 0$. Since for $g \in L$ it is true that $-q_\mu(-g) = (g, \mu_1)$

$= q_\mu(g) = (g, \mu)$, we have $\mu_1 \sim \mu$; the case $g = 1$ shows that $\mu_1 \in \text{prob } (X)$. Using $-f$ in the same arguments, we find $\mu_2 \in \text{prob } (X)$ such that $\mu_2 \sim \mu$ and $(f, \mu_2) = -q_\mu(-f)$.

The assumption that $p_x(f) = -p_x(-f)$ for all $x \in X$ entails $p_{\mu_1}(f) = -p_{\mu_1}(-f)$, whence $(f, \mu') = (f, \mu_1)$ for every $\mu' \in rca^+(X)$ such that $\mu_1 \prec \mu'$. Let $\mu_1' \in \text{prob } (X)$ be a maximal measure [2] such that $\mu_1 \prec \mu_1'$; then $(f, \mu_1') = q_\mu(f)$. Similar arguments give a maximal $\mu_2' \in \text{prob } (X)$ such that $\mu_2 \prec \mu_2'$ and $(f, \mu_2') = -q_\mu(-f)$.

We have assumed that K is a Choquet simplex. We invoke the Choquet-Meyer uniqueness theorem in the form that is established in Lemma 1, following: the set $\{\mu' \in \text{prob } (X) : \mu' \sim \mu\}$ contains one and only one maximal measure. That is, $\mu_1' = \mu_2'$, whence $q_\mu(f) = (f, \mu_1') = (f, \mu_2') = -q_\mu(-f)$, contrary to supposition. It must be the case then that $q_\mu(f) = -q_\mu(-f)$ for all $\mu \in rca^+(X)$. As we have seen, however, this is equivalent to $f \in L$. □

Theorem 1 is a generalization of Satz 6 of [3]; there it is assumed that L itself is a lattice, equivalent to assuming that K is a Choquet simplex and that B is closed. (*Added in proof*: Reference [9] has been brought to the author's attention by Professor Brauer. Theorem 1 above is implied by [9, Theorem 4.1], given the Choquet-Meyer uniqueness theorem in the form of Lemma 1, following.)

LEMMA 1. *Let L be a closed subspace of $C(X)$ which contains the constants and separates the points of X. Assume that the base K of $(L^*)^+$ is a Choquet simplex. Then for each $\mu \in \text{prob } (X)$, the set*

$$\{\mu' \in \text{prob } (X) : \mu' \sim \mu\}$$

contains a unique maximal measure.

Proof. Let $\iota : L \to C(K)$ denote the injection of L into $C(K)$; the members of ιL are continuous affine on K and ιL separates the points of K and contains the constants. Let $(L_K)^\wedge \subset C(K)$ be the set of all functions of the form $\iota g_1 \wedge \cdots \wedge \iota g_m$ for all m and all $g_1, \cdots, g_m \in L$. The members of $(L_K)^\wedge$ are concave on K and $(L_K)^\wedge - (L_K)^\wedge$ is a lattice which is norm dense in $C(K)$. The relation $\sim$ in $rca(K)$ is defined by $\theta \sim \varphi$ iff $(\iota f, \theta) = (\iota f, \varphi)$ for all $f \in L$; the relation $\prec$ in $rca(K)$ is defined by $\theta \prec \varphi$ iff $(h, \theta) \geq (h, \varphi)$ for all $h \in (L_K)^\wedge$. The $\prec$ relation refines the $\sim$ relation, and it is known that for $\xi \in K$ and $\theta \in \text{prob } (K)$, if $\delta_\xi \sim \theta$ then $\delta_\xi \prec \theta$ [1, p. 25].

Since K contains a compact set X_0 homeomorphic to X, every measure in $rca(X)$ transfers in the natural way to a measure on $X_0 \subset K$; we denote by $\Theta : rca(X) \to rca(K)$ this injection into $rca(K)$. For given $g_1, \cdots, g_m \in L$ and $\lambda \in rca(X)$ we have

$$(\kappa g_1 \wedge \cdots \wedge \kappa g_m, \lambda) = (\iota g_1 \wedge \cdots \wedge \iota g_m, \Theta\lambda).$$

If we consider $m = 1$ only we obtain $\lambda_1 \sim \lambda_2$ in $rca(X)$ iff $\Theta\lambda_1 \sim \Theta\lambda_2$ in $rca(K)$;

if we consider all values of m we obtain $\lambda_1 \prec \lambda_2$ in $rca(X)$ iff $\Theta\lambda_1 \prec \Theta\lambda_2$ in $rca(K)$.

Let $\mu \in \text{prob}\ (X)$ be given. This measure represents a point $\xi = \kappa^*\mu$ in K. Let μ'' be a maximal measure in the set $\{\mu' \in \text{prob}\ (X) : \mu' \sim \mu\}$. Then μ'' also represents ξ, and it is straightforward that $\delta_\xi \sim \Theta\mu''$ and hence $\delta_\xi \prec \Theta\mu''$ in prob (K). Let $\varphi \in \text{prob}\ (K)$ be a maximal measure such that $\Theta\mu'' \prec \varphi$. The closed support of φ is contained in the closure of κ^*B (=the set of extreme points of K), a fortiori in X_0. Thus $\varphi = \Theta\nu$ for some $\nu \in \text{prob}\ (X)$. From $\Theta\mu'' \prec \Theta\nu$ we have $\mu'' \prec \nu$ and hence $\nu = \mu''$, μ'' being maximal. Thus $\varphi = \Theta\mu''$, so that $\Theta\mu''$ is maximal in $rca(K)$. We use now our assumption that K is a Choquet simplex. By the Choquet-Meyer uniqueness theorem [1, p. 66], the set $\{\theta \in \text{prob}\ (K) : \delta_\xi \prec \theta\}$ contains a unique maximal measure. Thus $\Theta\mu''$ is unique, as is then μ''. □

3. Multipliers

Let $x \in X$ be given, and let $\mu_x \in \text{prob}\ (X)$ be a representing measure for x, that is, $\delta_x \prec \mu_x$. Let $\mathcal{S}(\mu_x)$ denote the closed support of the measure μ_x, and define

$$\mathcal{C}_x = \text{closure of } \bigcup_{\mu_x} \{\mathcal{S}(\mu_x) : \mu_x \in \text{prob}\ (X) \text{ and } \delta_x \prec \mu_x\}$$

THEOREM 2. *Let L be a closed subspace of $C(X)$ which contains the constants and separates the points of X. Assume that the base K of $(L^*)^+$ is a Choque simplex. Then the multipliers $\mathfrak{M}_L$ of L are characterized by the following property: if $f \in C(X)$ then $f \in \mathfrak{M}_L$ iff for each $x \in X$, f is constant $(=f(x))$ on the set $\mathcal{C}_x$. Every subalgebra of $C(X)$ contained in L is contained in $\mathfrak{M}_L$.*

Proof. Suppose $f \in C(X)$ is contained in a subalgebra of $C(X)$ which is contained in L. For each $x \in X$ and each $\mu_x \in \text{prob}\ (X)$ such that $\delta_x \prec \mu_x$ we must have

$$f(x) = \int f(x')\mu_x(dx'), \qquad f^2(x) = \int f^2(x')\mu_x(dx'),$$

whence

$$\int [f(x) - f(x')]^2\mu_x(dx') = 0$$

and thus $f(x') = f(x)$ for $x' \in \mathcal{S}(\mu_x)$. In other words, for each fixed x the closed set $\{x' \in X : f(x') = f(x)\}$ contains $\mathcal{S}(\mu_x)$ for every such μ_x, and hence contains $\mathcal{C}_x$.

On the other hand, let $f \in C(X)$ be such that for each $x \in X$, f is constant on $\mathcal{C}_x$. The constant value is necessarily $f(x)$, since $x \in \mathcal{C}_x$. For every $g \in L$ and every $\mu_x \in \text{prob}\ (X)$ such that $\delta_x \prec \mu_x$ we have

$$\int (fg)(x')\mu_x(dx') = f(x) \int g(x')\mu_x(dx') = f(x)g(x) = (fg)(x),$$

and there follows $p_x(fg) = -p_x(-fg)$ for every $x \in X$. By Theorem 1, $fg \in L$, so that f is a multiplier for L. □

The following example is somewhat trivial, but it illustrates the considerations involved. Let X be the closed unit disk $X = \{\text{complex } x : |x| \leq 1\}$, and let L be the functions in $C(X)$ which are harmonic in the interior of X. The maximal representing measures are the Poisson kernels on the boundary (Choquet, topological) and are unique, so K is a Choquet simplex. The usual Lebesque measure on X is a representing measure for $x = 0$ when normalized, and its closed support is all of X. By Theorem 2, the only subalgebra of $C(X)$ contained in L is the constants.

Theorems 1 and 2 still hold if the assumption that L separates the points of X is dropped. To prove the generalized versions, we apply Theorems 1 and 2 to the quotient space of X determined by the equivalence $\sim$, and then lift to X. The argument is without complications, and we omit it.

4. An example

As we remarked previously, the core of Theorems 1 and 2 is that the measures

$$\{\delta_x - \mu_x : \delta_x \prec \mu_x, x \in X, \mu_x \in \text{prob } (X)\}$$

w^*-span the annihilator of L. The following example shows that the assumption that K is a Choquet simplex is not superfluous.

Let Σ be a left amenable discrete semigroup [5]. We assume given an action $\Sigma \times X \to X$ of Σ on the compact Hausdorff space X. The transform of $x \in X$ by $\sigma \in \Sigma$ will be denoted by $\sigma x \in X$; we have $\sigma_1(\sigma_2 x) = (\sigma_1 \sigma_2)x$, and we require that σx be continuous in x for each fixed $\sigma \in \Sigma$. We will consider only the case where the action is without common fixed points; that is, there is no $x \in X$ such that $\sigma x = x$ for every $\sigma \in \Sigma$.

For each $\sigma \in \Sigma$ let $V(\sigma) : C(X) \to C(X)$ be defined by $[V(\sigma)f](x) = f(\sigma x)$ $x \in X$. Then $V(\sigma)$ is a nonnegative operator of unit norm on $C(X)$, and the V's are an antirepresentation of $\Sigma : V(\sigma_1 \sigma_2) = V(\sigma_2)V(\sigma_1)$, $\sigma_1, \sigma_2 \in \Sigma$. The adjoints $V^*(\sigma) : rca(X) \to rca(X)$ are a representation: $V^*(\sigma_1 \sigma_2) = V^*(\sigma_1)V^*(\sigma_2)$; moreover, the restriction of $V^*(\sigma)$ to $X \subset w^* - rca(X)$ is just a copy of the given action. The set $\mathcal{I}$ of left invariant measures for the action is

$$\mathcal{I} = \{\mu \in rca(X) : V^*(\sigma)\mu = \mu \text{ for all } \sigma \in \Sigma\}.$$

The set $LM(X)$ of left invariant means is defined as $LM(X) = \mathcal{I} \cap \text{prob } (X)$. The set $LM(X)$ is convex, w^*-compact, and since we have assumed Σ is left amenable, nonempty [5], [6].

Σ being left amenable, there exists at least one generalized sequence $\{\varphi_\alpha\}$ of finite means on Σ which converges in norm to left invariance [5]. That is, for each α in the directed indexing set we have $\varphi_\alpha = \sum_\sigma c_{\alpha\sigma} \delta_\sigma$ with $c_{\alpha\sigma} \geq 0$, $\sum_\sigma c_{\alpha\sigma} = 1$, $c_{\alpha\sigma} \neq 0$ for at most finitely many σ depending on α, and $\lim_\alpha \| \sum_\sigma c_{\alpha\sigma}(\delta_{\tau\sigma} - \delta_\sigma) \| = 0$ for each $\tau \in \Sigma$. A function $f \in C(X)$ is left almost convergent (to value k) iff

$$\lim_\alpha \sum_\sigma c_{\alpha\sigma} f(\sigma x) = k \quad \text{uniformly in } x \in X$$

for each such generalized sequence $\{\varphi_\alpha\}$. It is known that for $f \in C(X)$ to be left almost convergent it is necessary and sufficient that $(f, \lambda_1) = (f, \lambda_2)$ for all $\lambda_1, \lambda_2 \in LM(X)$ [6]. The set of left almost convergent functions is a closed linear subspace of $C(X)$ containing the constants; we will show presently that it separates the points of X under our assumption of no fixed points for the action.

The archetypical example is the following. Let Σ be the semigroup N of additive positive integers, let X be the Stone-Čech compactification βN of N, and let the action be determined by $\sigma x = \sigma + x$, $\sigma \in N$, $x \in N \subset \beta N$. With $\{\varphi_n\} = \{(\delta_1 + \cdots + \delta_n)/n\}$ converging in norm to invariance, the (left) almost convergent functions in $C(\beta N)$ are just the almost convergent sequences of G. G. Lorentz extended to $C(\beta N)$. The invariant means are the Banach limits, and the characterization of almost convergence in terms of the invariant means is just the Lorentz theorem [7].

We will need the following result of Choquet [8].

LEMMA 2. *$\mathcal{J}$ is a lattice.*

Proof. Let $Q : rca(X) \to rca(X)$ be a nonnegative projection of unit norm onto subspace $\mathcal{J}$ [5], [6]. Let $\nu = \nu^+ - \nu^-$ be the Jordan decomposition of $\nu \in \mathcal{J}$. Then $\nu^+ \geq \nu$ and $\nu^+ \geq 0$, whence $Q\nu^+ \geq Q\nu = \nu$ and $Q\nu^+ \geq 0$, so that $Q\nu^+ \geq \nu^+$. There follows

$$\| Q\nu^+ - \nu^+ \| = (1, Q\nu^+ - \nu^+) = \| Q\nu^+ \| - \| \nu^+ \| \leq 0.$$

Thus $Q\nu^+ = \nu^+$, and $\nu^+, \nu^- \in \mathcal{J}$. □

Let L be the closed subspace of $C(X)$ consisting of the left almost convergent functions. It is clear that L contains the constants. The annihilator $L^\perp$ is w^*-spanned by

$$\{\lambda_1 - \lambda_2 : \lambda_1, \lambda_2 \in LM(X)\}.$$

One sees that if $\nu \in L^\perp$ then $\nu \in \mathcal{J}$ and $(1, \nu) = 0$, the sets being w^*-closed. On the other hand, suppose $\nu \in \mathcal{J} \cap \{\nu : (1, \nu) = 0\}$. If $\nu = \nu^+ - \nu^-$ then ν^+, $\nu^- \in \mathcal{J}$ by Lemma 2, and $(1, \nu) = 0$ requires $\| \nu^+ \| = \| \nu^- \|$, whence $\nu = c(\lambda_1 - \lambda_2)$ for some $c \geq 0$ and some $\lambda_1, \lambda_2 \in LM(X)$. That is, $\nu \in L^\perp$, so

$$L^\perp = \mathcal{J} \cap \{\nu : (1, \nu) = 0\} = \{c(\lambda_1 - \lambda_2) : c \geq 0 \text{ and } \lambda_1, \lambda_2 \in LM(X) \text{ mutually singular}\}.$$

We mention in passing that an extension of the argument shows that $L = M \oplus \{\text{constants}\}$ with M the norm closed span of

$$\{f - V(\sigma)f : f \in C(X), \sigma \in \Sigma\}.$$

THEOREM 3. *L separates the points of X. The Choquet boundary relative to L is all of X, and $\mathcal{H}_L = C(X)$; if $LM(X)$ has more than one element then $L \neq \mathcal{H}_L$.*

Proof. Let $x \in X$, and suppose $\mu_x \in \text{prob }(X)$ is a representing measure for x, that is, $\delta_x \sim \mu_x$. From $\delta_x - \mu_x \in L^\perp$ we obtain $\delta_x - \mu_x = c(\lambda_1 - \lambda_2)$ for some $c \geq 0$ and some mutually singular $\lambda_1, \lambda_2 \in LM(X)$, using the above characterization of $L^\perp$. Thus we have $\mu_x = (\delta_x + c\lambda_2) - c\lambda_1$. Now, our assumption that the action of Σ on X is without common fixed points means that no measure δ_x is in $LM(X)$; every member of $LM(X)$ is either atomless or assigns positive measure to more than one point of X. Thus some part of λ_1 is $(\delta_x + c\lambda_2)$-singular; we cannot have $\mu_x \geq 0$ unless $c = 0$ and hence $\mu_x = \delta_x$.

It follows first that L separates the points of X; if L did not separate the points of X then there would exist $x_1 \neq x_2$ such that $\delta_{x_1} \sim \delta_{x_2}$. We can now assert that the Choquet boundary relative to L is all of X, since for each $x \in X$, the only representing probability for x is δ_x. For each $f \in C(X)$ we have then $p_x(f) = -p_x(-f) = f(x)$ for all $x \in X$, whence $\mathcal{H}_L = C(X)$. If $LM(X)$ has only one element then $L = C(X)$, and our result is vacuous. If $LM(X)$ has more than one element, however, then the inclusion $L \subset \mathcal{H}_L = C(X)$ is proper. We note also that $\hat{L}$ is norm dense in $C(X)$; cf., Satz 4 of [3]. □

Remark. Suppose $\mu \in \text{prob }(X)$ is given, and let $\mu' \in \text{prob }(X)$ be such that $\mu' \sim \mu$. Then $\mu' - \mu = c(\lambda_1 - \lambda_2)$ for some $c \geq 0$ and some mutually singular $\lambda_1, \lambda_2 \in LM(X)$. Thus $\mu' = (\mu + c\lambda_1) - c\lambda_2$, and $\mu' \geq 0$ requires $\mu \geq c\lambda_2$. If $f \notin L$ is given in $C(X)$, then $q_\mu(f) > -q_\mu(-f)$ cannot hold unless μ dominates a positive multiple of a member of $LM(X)$. The functionals $p_\mu(f)$ are of no use in determining L; the functionals $q_\mu(f)$ do determine L, but are effective only for certain μ.

Under the injection Θ: $rca(X) \to rca(K)$, each member of prob (X) becomes a maximal measure in prob (K). In particular, distinct members of $LM(X)$ become distinct maximal members of prob (K). Each member of $\Theta\{LM(X)\}$ is a representing measure for the single point $\xi = \kappa^*\{LM(X)\}$ of K; the uniqueness of maximal measures fails when $LM(X)$ has more than one element. (E.g., for the case $\Sigma = N$, $X = \beta N$ described above, the number of Banach limits is $2^{\mathfrak{c}}$.)

The example N also serves to illustrate the failure of Theorem 2 when K is not a simplex. Consider the elements f', f'' of $C(\beta N)$ determined by the values $f'(n) = 1$, $n \in N$, and $f''(n) = (-1)^n$, $n \in N$. These generate the subalgebra $[f', f'']$ of $C(\beta N)$ consisting of all functions of the form $af' + bf''$, a, b real, and each of these is almost convergent. Let f''' be determined by the values $f''' = (0, 1, 1, 0, 0, 0, 0, 1, \cdots)$ on N; the lengths of the successive blocks are the successive powers of two. It is clear that f''' is not almost convergent, but it is easy to verify that $f''f'''$ is almost convergent. From $f''(f''f''') = f'''$ it follows that f'' is not a multiplier for the almost convergent functions. Thus the subalgebra $[f', f'']$ is contained in the subspace of almost convergent functions but is not contained in the multipliers for that subspace.

REFERENCES

1. R. R. PHELPS, *Lectures on Choquet's theorem*, D. Van Nostrand, Princeton, 1966.
2. D. A. EDWARDS, *Introduction to functional analysis*, Summer Institute lecture notes, Lehigh University, 1964.
3. H. BAUER, *Šilovscher rand und Dirichletsches problem*, Ann. Inst. Fourier (Grenoble), vol. 11 (1961), pp. 89–136.
4. E. BISHOP AND K. DE LEEUW, *The representation of linear functionals by measures on sets of extreme points*, Ann. Inst. Fourier (Grenoble), vol. 9 (1959), pp. 305–331.
5. M. M. DAY, *Amenable semigroups*, Illinois J. Math., vol. 1 (1957), pp. 509–544.
6. S. P. LLOYD, *A mixing condition for extreme left invariant means*, Trans. Amer. Math. Soc., vol. 125 (1966), pp. 461–481.
7. G. G. LORENTZ, *A contribution to the theory of divergent sequences*, Acta Math., vol. 80 (1948), pp. 167–190.
8. G. CHOQUET, *Existence et unicite des representations intégrales au moyen des points extremaux dans les cônes convexes*, Seminaire Bourbaki, December 1956.
9. N. BOBOC AND A. CORNEA, *Convex cones of lower semicontinuous functions on compact spaces*, Rev. Roum. Math. Pures. et Appl., vol. 12, pp. 471–525.

BELL TELEPHONE LABORATORIES, INCORPORATED
MURRAY HILL, NEW JERSEY