THE NUMBER OF HALL π-SUBGROUPS OF A FINITE GROUP

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This note gives a theorem on the number of Hall π-subgroups of a finite group which includes a recent result of Marshall Hall on the number of Sylow subgroups as well as the classical theorem of Philip Hall on the number of Hall π-subgroups of a solvable group (cf. [1] and [2]).

We shall consider groups G which satisfy the following proposition for a given set of primes π.

A_π. Given any π-subgroups P_i of G for $i = 1, 2$, there are Hall π-subgroups H_i of G so that $H_i \geq P_i$ and an automorphism α of G so that $H_1 \alpha = H_2$.

If a group satisfies proposition A_π we shall call it an A_π-group. It is clear that a group satisfying the well-known proposition D_π of Philip Hall (cf. [3]) is an A_π-group. But the class of A_π-groups is larger than the class of groups satisfying D_π since for instance the projective group PSL(2, 7) of order 168 has two classes of subgroups isomorphic to the symmetric group S_4 which are conjugate in the automorphism group of PSL(2, 7). It is also clear that an A_π-group satisfies proposition E_π of [3] and that there are E_π-groups not A_π-groups; for instance, PSL(2, 11) of order 660 which has two non-isomorphic groups of order 12.

Before stating the main theorem it will be convenient to have the following lemma whose easy proof is omitted.

LEMMA. *Let the group D be the direct product of groups G_i for $i = 1, 2, \cdots, n$, where each G_i is isomorphic to a given group G. Then a Hall π-subgroup of D is the product of Hall π-subgroups H_i of G_i and D is an A_π-group if and only if G is.*

The main theorem is as follows.

THEOREM. *Let G be a finite A_π-group for a certain set of primes π: then the number $n_\pi(G)$ of Hall π-subgroups of G is a product of integers such that each integer is either the number of Hall π-subgroups of a simple A_π-group or is a prime power congruent to 1 modulo a prime of π.*

The proof is by induction on $|G|$ the order of G. If G is a direct product of isomorphic simple groups the theorem follows from the above lemma. Accordingly we consider the case where G has a proper non-trivial characteristic subgroup K, which we shall assume to be minimal. We let $\bar{G}$ denote G/K.

Case I. K is a π-group. It is easy to see then that there is a one-one cor-

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respondence between the Hall π-subgroups of G and those of $\bar{G}$, and that $\bar{G}$ is an A_π-group if and only if G is. Thus $n_\pi(G) = n_\pi(\bar{G})$ and the theorem follows from the induction assumption since $|\bar{G}| < |G|$.

Case II. K is neither a π-group nor a π'-group. Since $|K|$ is not a prime power, it follows from the minimality of K that K is a direct product of simple groups X_i. For each Hall π-subgroup H of G, $H \cap K$ is a Hall π-subgroup of K (cf. p. 220 of [4]) and each of the Hall π-subgroups of G which contains $H \cap K$ is contained in $N = \mathfrak{N}(H \cap K)$ the normalizer of $H \cap K$. We shall show now that N is an A_π-group. Let S be a maximal π-subgroup of N. Then $S \geq H \cap K$ since $H \cap K$ is a normal π-subgroup of N. Since G is an A_π-group, there is a Hall π-subgroup H_1 of G so that $S \leq H_1$. Since $H_1 \geq S \geq H \cap K$ and since $H \cap K$ is a Hall π-subgroup of K, $H_1 \cap K = H \cap K$. Thus H_1 normalizes $H \cap K$ and $H_1 \leq N$. Since S is maximal, $S = H_1$. Since G is an A_π-group there is an automorphism α of G so that $H\alpha = H_1$. Thus $(H \cap K)\alpha = H_1 \cap K = H \cap K$. Hence α stabilizes $H \cap K$ and consequently its normalizer N. Thus α induces an automorphism of N and $H\alpha = S$. Hence N is an A_π-group as we wished to show.

It is clear that the Hall π-subgroups of K are conjugates under the automorphism group of G and that K is also an A_π-group. Hence $n_\pi(G) = n_\pi(K)n_\pi(N)$. If $N = G$ then $H \cap K$ is a characteristic π-subgroup of G and the theorem follows from Case I above. If $N < G$, then by the induction assumption $n_\pi(N)$ is an integer of the prescribed form while

$$n_\pi(K) = \prod_i n_\pi(X_i)$$

with the X_i simple A_π-groups by the lemma and hence $n_\pi(K)$ is also of the prescribed form.

Case III. K is a π'-group. Then $n_\pi(G) = n_\pi(\bar{G})\cdot n_\pi(KH)$. It is easy to check that $\bar{G}$ is an A_π-group. Since $(|K|, |H|) = 1$, KH is an A_π-group by Theorem VII.2.j of [4]. Hence the theorem follows from the induction assumption unless $KH = G$. Thus we assume that $KH = G$ and wish to show that $|G:\mathfrak{N}(H)|$ is a number of the form prescribed by the theorem. Now $K \lhd G = KH$ and therefore $\mathfrak{N}(H) = H\cdot\mathfrak{C}_K(H)$ with $\mathfrak{C}_K(H)$ denoting the centralizer of H in K and $|G:\mathfrak{N}(H)| = |K:\mathfrak{C}_K(H)|$. For $j = 1, \cdots, t$, let P_j be Sylow p_j-subgroups of K for the different primes p_j dividing $|K|$ chosen so that $\mathfrak{C}_{P_j}(H)$ is a Sylow p_j-subgroup of $\mathfrak{C}_K(H)$ and so that P_j is normalized by H. This is possible for the following reason. We begin with C_j a Sylow p_j-subgroup of $\mathfrak{C}_K(H)$ and consider a maximal p_j-subgroup M_j containing C_j and normalized by H. If M_j is not a Sylow p_j-subgroup of K then the Frattini argument applied to the normalizer of M_j gives a contradiction to the maximality in the choice of M_j. Then

$$|K:\mathfrak{C}_K(H)| = \prod_{j=1}^{t} |P_j:\mathfrak{C}_{P_j}(H)|$$

and the theorem follows from the induction assumption unless K is a p-group for some prime p, and in fact an abelian group since K is minimal characteristic.

Thus we finally consider the case $G = PH$ with P an abelian p-group normal in PH and will show that $|P:\mathfrak{C}_P(H)| \equiv 1$ mod a prime of π. If $|H|$ is even then p is odd since p is a π'-number and $|P:\mathfrak{C}_P(H)| = 1 \bmod 2$. If $|H|$ is odd, then H is solvable by the Feit-Thompson theorem and hence has a minimal normal q-subgroup Q. Now $PQ \lhd G = PH$ and hence $\mathfrak{C}_P(Q) \lhd G$ since $\mathfrak{C}_P(Q)$ is the intersection of P and the center of PQ. Thus $\mathfrak{C}_P(Q) = 1$; for otherwise G has a non-trivial normal π-subgroup and the theorem follows from Case I. It follows that $n_\pi(G) = |P|$. But

$$PQ = Q \cup Qx_1 Q \cup Qx_2 Q \cup \cdots$$

with the number of right cosets of Q in $Qx_i Q$ a positive power of q since it is the index $(Q:Q \cap Q^{x_i})$. Thus $|P| = 1 \bmod q$ and the theorem is proved.

It should be remarked that when the Hall π-subgroup is solvable as in the theorems of Marshall Hall and Philip Hall, then the reference to the Feit-Thompson theorem is unnecessary.

It should also be pointed out (I am indebted to Professor M. Suzuki for this) that the simple groups of the theorem are composition factors of G provided the Hall π-subgroups are nilpotent. A proof can be given by following the proof of the theorem. Since H is nilpotent, $H \cap K$ is nilpotent and hence intravariant in K. Then $G = KN$ so that $N/N \cap K \cong G/K$, and in the induction argument the relevant composition factors of N are now composition factors of G. I have been unable to prove the above assertion if the Hall π-subgroups are not nilpotent.

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