PSEUDO-CIRCLES AND UNIVERSAL CIRCULARLY CHAINABLE CONTINUA

BY

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In 1922 Knaster [7] constructed the first hereditarily indecomposable continuum. In 1948 Moise [13] described an indecomposable, chainable continuum which was homeomorphic to each of its subcontinua. Because of this property, he called it a pseudo-arc. Bing [2] proved in 1951 that every two hereditarily indecomposable, chainable continua were homeomorphic and hence pseudo-arcs.

In the same year Bing [2] constructed an hereditarily indecomposable continuum which was topologically different from a pseudo-arc. Since this continuum was circularly chainable, Bing called it a pseudo-circle. It seems natural to the author to extend the term pseudo-circle to include any hereditarily indecomposable circularly chainable continuum which is not chainable. The purpose of this study is to investigate the class of pseudo-circles.

The author constructs c topologically distinct pseudo-circles. In fact, there exists a collection of c pseudo-circles with the property that no one can be mapped continuously onto another.

There exists a pseudo-circle which can be mapped continuously onto any circularly chainable continuum and onto any chainable continuum. The pseudo-circles described in this study are called universal pseudo-circles (in this paper, the term "universal" does not refer to embeddings; it refers instead to a particular inverse limit). Exactly one of the universal pseudo-circles is planar. The author proves that it can be mapped continuously onto any planar circularly chainable continuum and onto any chainable continuum.

These pseudo-circles are the "only" ones in the sense that any other one which is universal for a suitable class of continua is homeomorphic to one already described. Each constructed pseudo-circle can be mapped continuously onto any planar pseudo-circle, and no planar pseudo-circle can be mapped continuously onto any non-planar pseudo-circle. Each constructed pseudo-circle can be mapped continuously onto the pseudo-arc, and the pseudo-arc cannot be mapped continuously onto any pseudo-circle.

Since the pseudo-arc, and hence all chainable continua, are continuous images of these pseudo-circles, it is natural to inquire what characterizes the largest class of continua for which one of these pseudo-circles is universal. A very natural characterization is given in Section III in terms of the simple concept of q-chainability. Through different techniques, L. Fearnley has ob-

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tained independently this characterization for planar pseudo-circles [15]. In correspondence with Fearnley, the author has also learned that he has independently obtained a Uniformization Theorem by much different methods and has used it to obtain many of the theorems of the second section.

In this study a continuum is a nondegenerate, compact, connected subset of metric space. A map is a continuous single-valued function. If a term is left undefined, the definition is that of [4].

Uniformization theorem

1. Structure of double graphs in the unit square. Let f and g be two maps of a topological space X onto itself. The subset of X^2

$$[f, g] = \{(x, y) : f(x) = g(y)\}$$

is called the double graph of f and g.

Let I denote the unit interval, and let T denote the class of simplicial (piecewise linear) maps of I onto I. If f and g are in T, then $[f, g]$ is the union of a finite number of isolated points, straight line segments, and rectangular disks. Distinguishing certain subintervals of I will make it possible to analyze the structure of the double graph of two functions f and g belonging to T.

Let $A = [a_1, a_2]$ be a subinterval of I such that $f(A) = I$ but $f(\text{Int } (A))$ does not contain 0 or 1. Denote such an interval by $A+$ if $f(a_1) = 0$ and $f(a_2) = 1$ and by $A-$ if $f(a_1) = 1$ and $f(a_2) = 0$.

Let $B = [b_1, b_2]$ be a maximal subinterval of I such that $f(B) \neq I$ and $f(b_1) = 0 = f(b_2)$ or $f(b_1) = 1 = f(b_2)$. Let $B-$ denote the former condition and $B+$ the latter.

Denote similar intervals for g by $A'+$, $A'-$, $B'+$, and $B'-$. These designations form rectangles in I^2 of the form $A \times A'$, $A \times B'$, $B \times A'$, and $B \times B'$. The following four lemmas, which characterize the behavior of $[f, g]$ in these rectangles, were formulated by Mioduszewski [12].

LEMMA 1. *$[f, g] \cap \text{Fr } (A \times A')$ contains two opposite vertices of $A \times A'$, which are joined in $[f, g] \cap (A \times A')$ by a continuum. The vertices are the upper right and lower left if the signs of A and A' are the same and the upper left and lower right if the signs are different.*

LEMMA 2. *$[f, g] \cap \text{Fr } (A \times B')$ contains two adjacent vertices of $A \times B'$, which are joined in $[f, g] \cap (A \times B')$ by a continuum. The vertices lies on the right side or on the left side of $A \times B'$ according as the signs of A and B' are the same or different.*

LEMMA 3. *$[f, g] \cap \text{Fr } (B \times A')$ contains two adjacent vertices of $B \times A'$, which are joined in $[f, g] \cap (B \times A')$ by a continuum. The vertices lie on the upper side or on the lower side of $B \times A'$ according as the signs of B and A' are the same or different.*

LEMMA 4. *$[f, g] \cap \mathrm{Fr}\ (B \times B')$ contains all or none of the vertices of $B \times B'$ according as the signs of B and B' are the same or different. Furthermore if the signs of B and B' are plus and the maximum attained by f on B is greater than or equal to the maximum attained by g on B', then the vertices on the left (right) side of $B \times B'$ are joined by a continuum in $[f, g] \cap (B \times B')$. Similar results are true if the signs of B and B' are negative or if g is greater than f over the respective intervals.*

The preceding lemmas state that some pairs of vertices of a certain rectangle may be joined by continua lying in $[f, g]$ and in the rectangle. For each such pair of vertices, pick an arbitrary arc, lying in $[f, g]$ and in the rectangle, which joins the vertices. The collection of such arcs, one for each appropriate pair of vertices of each of the rectangles into which we have partitioned I^2, is said to be the collection of *admissible* arcs of $[f, g]$. A continuum contained in $[f, g]$ is said to be a *path* if it is the union of admissible arcs.

2. Uniformization theorem for circles. Let K be a category and f and g be morphisms of K. A pair α and β of morphisms of K is said to be a uniformization of f and g if

$$f \circ \alpha = g \circ \beta.$$

The purpose of this chapter is to investigate the uniformization problem for the category of simplicial maps of the unit circle onto itself.

Double graphs are useful in the investigation of this problem because of the following property, the proof of which is obvious:

PROPERTY 1. *Let C denote the unit circle and let f and g be maps of C onto itself. The curve $x = \alpha(t)$, $y = \beta(t)$, $t \in C$, lies in $[f, g]$ if and only if $f\alpha = g\beta$.*

Since a uniformization is possible for each pair of simplicial maps of I onto itself [12], it is tempting to conjecture that the same result holds for simplicial maps of C onto itself. That this is not the case is shown by

Example 1. Let C be realized as I with the points 0 and 1 identified. Let maps of C onto C be defined by straight line segments in I^2 as follows: Let g be defined by segments joining $(0, 0)$ and $(\frac{1}{2}, 1)$ and joining $(\frac{1}{2}, 0)$ and $(1, 0)$. Let f be defined by segments joining $(0, 0)$ and $(\frac{1}{2}, 1)$ and joining $(\frac{1}{2}, 0)$ and $(1, 1)$. As one may see from $[f, g]$, no possible uniformization curve in $[f, g]$ has parametric equations $x = \alpha(t)$, $y = \beta(t)$, $t \in C$, such that α maps *onto* C.

The map g, however, has degree zero (intuitively the degree of a map is the "winding number"). Maps of degree zero are somewhat pathological in that they are homotopic to a constant map and thus, in some sense, are not really onto. Theorem 1 will show that a uniformization is possible for pairs of maps in the class U of simplicial maps of C onto C which have nonzero degree.

Orient C once and for all so that there exists a definite sense of rotation. Let p be a fixed point in C and let $f \in U$. Let $z_0, z_1, \cdots, z_k$ be the boundary points of $f^{-1}(p)$ ordered by positive rotation. For each i, $0 \leq i < k$, let $[z_i, z_{i+1}]$ denote the arc of C such that z_i is the initial point when ordered by positive rotation. If $f \mid [z_i, z_{i+1}]$ is onto, call $[z_i, z_{i+1}]$ an $A+$ or $A-$ according as the image of $f \mid [z_i, z_{i+1}]$ emanates from p in the positive or negative direction. Denote the other intervals by $B-$ or $B+$ according as the image of f restricted to such an interval emanates from p in the positive or negative direction. It is obvious that this definition of A and B is similar to the definition of A and B for the unit interval and that Lemmas 1–4 are applicable to the double graph of maps in U.

The proofs of the following lemmas are standard.

LEMMA 5. *Let* $f, g \in U$. *Then* $\deg (f \circ g) = (\deg f) \cdot (\deg g)$.

LEMMA 6. *Let* $f \in U$. *Then the degree of* f *equals the number of* $A+$*'s of* f *diminished by the number of* $A-$*'s of* f. *The degree of* f *is independent of the point* p.

Lemma 6 motivates the following definition: If $[x_1, x_2]$ is an arc in C such that $f(x_1) = p = f(x_2)$, then the degree of $[x_1, x_2]_f$, or simply the degree of $[x_1, x_2]$, is the number of $A+$'s of $f \mid [x_1, x_2]$ diminished by the number of $A-$'s of $f \mid [x_1, x_2]$.

Let S denote the class of simplicial maps of C onto itself which have positive degree. In order to uniformize maps of U, it is sufficient to uniformize maps of S. For let f, g be in U, and let $h \in U$ be a simplicial homeomorphism of degree minus one. Assume that f has negative degree and g has positive degree. Then $f \circ h$ and g belong to S; hence there exist $\alpha, \beta \in S$ such that $f \circ h \circ \alpha = g \circ \beta$. Then $h \circ \alpha$ and β is a uniformization of f and g.

For the remainder of Section I, let f, g be two fixed functions of S. The definition of A and B partitions C^2 into rectangles as in the previous section. Points which are vertices of such rectangles are said to be *vertices* of $[f, g]$.

LEMMA 7. *There exists an even number of admissible arcs emanating from each vertex of* $[f, g]$.

Proof by computation.

The definition of A and B also partitions the domain of f into a finite collection $W(f)$ of A's and B's. An enumeration $V = \{v_1, v_2, \cdots, v_n\}$ of the elements of $W(f)$ is a *canonical ordering of* $W(f)$ if the order of V agrees with positive rotation and the degree of any initial segment of V is positive. The initial point of v_1 is said to be the initial point of V.

LEMMA 8. *There exists a canonical ordering of* $W(f)$.

Proof. Let $D_1 = \{d_1, d_2, \cdots, d_n\}$ be an ordering of $W(f)$, which agrees with positive rotation. Suppose that D_1 is not canonical. Then there exists a smallest initial segment $S_1 = [d_1, d_m]$ such that degree $(S_1) \leqq 0$. Let D_2 be the sequence $\{d_{m+1}, \cdots, d_n\}$. If some initial segment of D_2 has nonpositive degree, let S_2 be the smallest such segment and let $D_3 = D_2 - S_2$.

Continue this process as many times as possible. All of D_1 cannot be discarded in this manner because degree (f) is positive. Hence there is a remainder $D = \{d_{l+1}, \cdots, d_n\}$ satisfying the hypothesis of the lemma.

Consider now the ordering V of $W(f)$ given by $\{D, S_1, S_2, \cdots, S_k\}$. V agrees with positive rotation. Suppose that there exists a smallest initial segment Z of V such that degree $(Z) \leqq 0$. Clearly $D \subset Z$. Hence

$$Z = d_{l+1}, \cdots, d_n, d_1, \cdots, d_j\}$$

and d_j belongs to some S_p. If d_j is the last element of S_p, then f has nonpositive degree, and if d_j is not the last element of S_p, then S_p was not chosen to be the smallest such initial segment. This contradiction proves that V is the desired sequence.

Let $U = \{u_1, u_2, \cdots, u_n\}$ and $V = \{v_1, v_2, \cdots, v_m\}$ be canonical orderings of $W(f)$ and $W(g)$ respectively with initial points x_0 and y_0 respectively. Cut C^2 along the circles $x_0 \times C$ and $C \times y_0$; then C^2 may be realized as a rectangle in the plane with opposite sides identified. In particular, let this rectangle be $R = [0, n] \times [0, m]$ with u_i realized as $[i - 1, i]$, v_i realized as $[i - 1, i]$, and (x_0, y_0) realized as the origin. Then the vertices of $[f, g]$ are precisely the lattice points of R. Point sets in C^2 and their realizations in R will be interchanged without mention where no confusion is likely to arise.

Let P be a path in R. The largest integer y such that, for some x, (x, y) is a vertex of P is said to be the *maximum* of P. *The minimum* of P is defined to be the smallest integer y such that, for some x, (x, y) is a vertex of P. The *variation* of P is the maximum of P minus the minimum of P.

LEMMA 9. *If $x_1 \leqq x_2$ and P is a path from (x_1, y) to (x_2, y), then the degree of $[x_1, x_2]$ is zero.*

Proof by induction on the variation. The case $n = 0$ is obvious, since $[x_1, x_2]$ consists solely of B's. Assume now that the lemma is true for variations $n, 0 \leqq n \leqq k$, and let P be a path of variation $k + 1$. Let $z_1 = x_1, z_2, z_3, \cdots, z_m = x_2$ be the sequence of points, ordered by their occurrence in P, such that (z_i, y) is in P. In order to prove the lemma, it suffices to prove that the degree of $[z_i, z_{i+1}]$ is zero.

Assume that $z_i < z_{i+1}$ and that y is the minimum of that part of P between (z_i, y) and (z_{i+1}, y) (denoted by $P \mid [z_i, z_{i+1}]$). The case where (z_i, y) and (z_{i+1}, y) are the only vertices of $P \mid [z_i, z_{i+1}]$ reduces to the case $n = 0$; so assume there exist other vertices. Let $(a, y + 1)$ be the vertex immediately following (z_i, y) in the order of P and let $(b, y + 1)$ be the vertex immediately preceding (z_{i+1}, y). Then $P \mid [a, b]$ is a path with variation less than $k + 1$. Hence the degree of $[a, b]$ is zero.

There are now four cases, depending on whether the symbol associated with $[y, y + 1]$ is $A+$, $A-$, $B+$, or $B-$. Assume the symbol is $A+$; the other cases are similar. In order to go from the level $y + 1$ to the level y, the symbol associated with the interval between z_i and a must be $A+$ or $A-$ according as z_i is less than or greater than a. Accordingly, by adding $P \mid [z_i, a]$ to $P \mid [z, b]$, we either add an $A+$ or subtract an $A-$. In either case the net effect is to make the degree of $[z_i, b]$ equal to one. But by adjoining $P \mid [b, z_{i+1}]$, we either add an $A-$ or subtract an $A+$. Thus one end nullifies the other, and the degree of $[z_i, z_{i+1}]$ is zero.

The same method proves

LEMMA 10. *If $y_1 \leqq y_2$ and P is a path from (x, y_1) to (x, y_2), then the degree of $[y_1, y_2]$ is zero.*

Consider a process by which paths in $[f, g]$ may be associated with paths in the first quadrant Q of the plane. Fill up Q with copies of R. These copies may be described as $[ln, (l+1)n] \times [km, (k+1)m]$, where l, k are natural numbers. Define an equivalence relation on Q by $(x, y) \sim (x', y')$ if and only if $x = x' \pmod{n}$ and $y = y' \pmod{m}$.

The canonical ordering of $W(f)$ and $W(g)$ provides an admissible arc joining $(0, 0)$ and $(1, 1)$. Consider an arbitrary path P constructed by adding at each step an admissible arc not equivalent to one used in a previous stage. Lemma 7 asserts the existence of such different admissible arcs for vertices not equivalent to the origin. Terminate the path P when it reaches a vertex equivalent to the origin, and call such a vertex the terminal point of P. P has a terminal point because there exist less than $4mn$ distinct admissible arcs. Lemmas 9 and 10 say that P intersects the coordinate axes only at the origin.

For such a path P, let P' denote the set of points in R equivalent to points of P. In C^2 realized as R, P' is not only a subset of $[f, g]$ but a simplicial image of C in $[f, g]$. Furthermore if $x = \alpha(t)$, $y = \beta(t)$, $t \in C$, denote parametric equations of the realization of P' and if (l, k) denotes the terminal point of P, then the degree of α is l and the degree of β is k. In particular, α and β are both onto because k and l are positive. These remarks prove

THEOREM 1. *If f and g are simplicial maps of C onto C of nonzero (positive) degree, then there exists simplicial maps α and β of C onto C of nonzero (positive) degree such that $f \circ \alpha = g \circ \beta$.*

LEMMA 11. *If P is a path in R which originates at $(0, 0)$ and terminates at (n, h), then the degree of $[0, h]_g$ equals the degree of $[0, n]_f$.*

Proof. Define a larger rectangle by assigning symbols to intervals in $[n, n+h]$ on the x-axis. Namely, if the symbol in $[j, j+1]_g$, $0 \leqq j < h$, is an A, assign an A of the opposite sign to $[j+n, j+1+n]$, and if the symbol is a B, assign a B of the same sign. Hence in each rectangle

$$[n+j, n+j+1] \times [j, j+1]$$

there exists an admissible arc joining the upper left and lower right vertices.

The addition of these arcs to P forms a path P_1 which originates and terminates on the x-axis. Lemma 9 yields

$$\deg [0, n + h] = \deg [0, n] + \deg [n, n + h] = 0,$$

and hence $\deg [0, n] = -\deg [n, n + h]$. By construction, $\deg [0, h] = -\deg [n, n + h]$. Thus the lemma is proved.

LEMMA 12. *If P is a path in R which originates at* $(0, 0)$ *and terminates at* (k, m), *then the degree of* $[0, k]_f$ *equals the degree of* $[0, m]_g$.

THEOREM 2. *Suppose that* $\deg g = b$. *Let c be the least common multiple of a and b. Then α, β in Theorem* 1 *have degree c/a, c/b respectively.*

Proof. It suffices to consider any path P in the rectangulation of Q by $W(f)$ and $W(g)$ and show that the terminal point of P is $(nc/a, mc/b)$.

Suppose that the terminal point of P is not $(nc/a, mc/b)$. Let L denote the sum of the two rays in Q emanating from $(nc/a, mc/b)$ and parallel to one of the axes. L intersects P.

Suppose $(d, mc/b)$ is the first point of P which is also in L. Let $(nc/a, h)$ be the first point of P which is also in the line $x = nc/a$, and let $(nc/a, i)$ be the last such point before $(d, mc/b)$. By Lemma 10, $\deg [h, i]_g = 0$, and by Lemma 11,

$$\deg [0, h]_g = \deg [0, nc/a] = c.$$

Since $\deg [0, mc/b]_g = c$, $\deg [i, mc/b]_g = 0$.

Consider the rectangle $[nc/a, d] \times [0, mc/b]$ and place it and the part of P between $(nc/a, i)$ and $(d, mc/b)$ in a new copy of Q. Assign to each interval

$$[j, j + 1], \quad mc/b \leqq j < mc/b + d - nc/a,$$

on the y-axis a symbol determined by $[nc/a + j, nc/a + j + 1]$ in the same manner as in the proof of Lemma 11. Exactly as in that proof, it follows that $\deg [mc/b, mc/b + d - nc/a]$ is negative. But the degree of $[i, mc/b + d - nc/a]$ is zero; hence the degree of $[i, mc/b]$ is positive. This contradiction proves the theorem.

Induction extends Theorem 2 to the more general

UNIFORMIZATION THEOREM. *Let* $f_1, f_2, \cdots, f_n \in S$ *and let* $\deg f_i = k_i$. *Then there exist* $\alpha_1, \alpha_2, \cdots, \alpha_n \in S$ *such that for* $1 \leqq j \leqq n$, $f_1 \alpha_1 = f_j \alpha_j$, $\deg \alpha_j =$ *least common multiple of* $\{k_i\}_{i=1}^{n}$ *divided by* k_j, *and each* α_j *has the same triangulation in its domain as* α_1.

Pseudo-circles

1. Construction of pseudo-circles. A chain [14] is a finite sequence $L_1, L_2, \cdots, L_n$ of open sets such that $L_i \cdot L_j \neq \emptyset$ if and only if $|i - j| \leqq 1$. If L_1 is allowed to intersect L_n, the sequence is called a circular chain. Each L_i is called a link. If $\varepsilon > 0$ and the diameter of each link is less that ε, then the sequence is called an ε-chain. A continuum is said to be chainable if for each $\varepsilon > 0$, it can be covered by an ε-chain. Circularly chainable continua are defined similarly.

Mardesic and Segal [9] have shown that circularly chainable continua may be regarded as inverse limits of sequences of unit circles with onto bonding maps. Furthermore, it is obvious that a continuum H is circularly chainable with a defining sequence of circular chains $\{C_i\}$ such that C_{i+1} circles k_i times in C_i (in the sense of [3]) if and only if there exists an inverse limit representation $\{X_i, f_i^{i+1}\}$ for H such that X_i is the unit circle and the degree of f_i^{i+1} is k_i.

A pseudo-arc is an hereditarily indecomposable, chainable continuum. A pseudo-circle is an hereditarily indecomposable, circularly chainable continuum which is not chainable. Mioduszewski [10] used an uniformization theorem for the unit interval to construct a pseudo-arc. The author adopts a similar development whenever convenient in constructing pseudo-circles by use of the Uniformization Theorem.

Let C' be a division of C into equal segments and let $n(C')$ denote the number of segments of C'. If C'' is another division of this kind, let Map (C', C'') denote the class of simplicial maps of C' onto C'' which have positive degree.

Let S' be a subclass of S and assume that each map of S' has domain C'. The map g in S is said to be a *majorant* for S' if for each pair of maps f, f' in S' there exists α in S such that $f\alpha = f'g$.

LEMMA 13. *There exists a majorant for S'.*

Remark 1. Maps $\alpha_j \beta_i$ belong to Map (C''', C').

Remark 2. If P is a set of primes and if the degree of each f_i is a product of nonnegative powers of elements of P, then so is the degree of g.

Let $C_1, C_2, C_3, \cdots$ be a sequence of unit circles. Consider for every C_n a sequence of subdivisions $\{C_{n,r}\}$, $r \geqq n-1$, of C_n into equal segments. Assume that $C_{n,r+1}$ is a subdivision of $C_{n,r}$ with the property that the length of each segment of $C_{n,r+1}$ is at most a half of that of $C_{n,r}$. The following matrix illustrates this construction.

$$\begin{array}{l}
C_{1,0} \\
C_{1,1} \leftarrow C_{2,1} \\
C_{1,2} \leftarrow C_{2,2} \leftarrow C_{3,2} \\
\vdots \\
C_{1,r} \leftarrow C_{2,r} \leftarrow \cdots \leftarrow C_{r,r} \leftarrow C_{r+1,r} \\
\vdots
\end{array}$$

Arrows between objects indicate that there exist maps of S between the two objects.

Assume that Map $(C_{m,p}, C_{n,p}) \subset$ Map $(C_{m,r}, C_{n,r})$ for $p < r$. Call a map

f *admissible* if $f \in \mathrm{Map}\ (C_{m,m}, C_{n,m})$ for some m. The sequence $\{C_n\}$ and all admissible maps form a category W. W is said to be an A-*category* if

$$(^*) \qquad \lim_{r\to\infty} \frac{n(C_{r,r})}{n(C_{n,r})} = \infty, \qquad n = 1, 2, \cdots.$$

Lemma 13 says that there exists a majorant for

$$\bigcup_{n\leqq r}\{\mathrm{Map}\ (C_{r,r}, C_{n,r})\}.$$

Assume that such a majorant Π_r^{r+1} belongs to the set Map $(C_{r+1,r}, C_{r,r})$. The inverse limit of unit circles with this sequence of majorants as bonding maps is called the *universal circularly chainable continuum* in W and is denoted by $UCC(W)$.

The existence of A-categories satisfying the above assumptions and the hereditary indecomposability of $UCC(W)$ may be proved by techniques similar to those of [10].

If N is a set of natural numbers and if the admissible maps of W are restricted to those the degree of which is a product of nonnegative powers of elements of N, then W is called an $A(N)$-category and $UCC(W)$ is also denoted $U(N)CC(W)$.

2. Properties of pseudo-circles. Define the distance between two points of C to be the length of the shorter of the two arcs into which these points divide C. Let $\varepsilon > 0$.

If f and g are maps define $f =_\varepsilon g$ if $|f(x) - g(x)| \leqq \varepsilon$ for each x in C.

LEMMA 14. *Let $f : C \to C$ be a continuous onto map of positive degree and let $\varepsilon > 0$. Then there exists an integer n such that whenever C' and C'' are divisions of C such that $n(C'')/n(C') \geqq n$ and* mesh $C' \leqq \varepsilon/4$, *there exists a $g \in$* Map (C'', C') *such that $f =_\varepsilon g$.*

Proof. Choose $h \in S$ such that $h =_{\varepsilon/2} f$. It satisfies a Lipschitz condition, so that for every $x, x' \in C$, $|h(x) - h(x')| \leqq K|x - x'|$.

Let $n = [K] + 1$, and let C', C'' satisfy the hypotheses of the lemma. The image of any segment of C'' (under h) lies in two adjacent segments of C'.

Map $g \in \mathrm{Map}\ (C'', C')$ with the property that $g =_{\varepsilon/2} h$ will now be defined. Let $e_0, e_1, \cdots, e_s$ be the division points of C'' ordered by positive rotation. Define a simplicial map g by $g(e_i) = c_k$, where c_k is the least division point in C' which is greater than or equal to $h(e_i)$. Map g is simplicial because of the condition $n(C'')/n(C') \geqq [K] + 1$. Since $g =_\varepsilon f$, then if $1 > \varepsilon$, $\deg f = \deg g$ and hence g is onto.

Lemma 14 together with condition $(^*)$ implies an

APPROXIMATION PROPERTY. *Let $0 < \varepsilon < 1$, an integer n, and an onto map $f : C \to C$ be given. There exists r_0 such that for any $r \geqq r_0$ there exists g in* Map $(C_{r,r}, C_{n,r})$ *such that $f =_\varepsilon g$.*

A circularly chainable continuum is said to be *embeddable* in an A-category W if it has an inverse limit representation $\{X_n, \sigma_n^m\}$ such that each X_n is an object of W and each σ_n^m is admissible in W.

THEOREM 3. *Let $X = \{X_n, \sigma_n^m\}$ be a circularly chainable continuum such that each σ_n^{n+1} has positive degree. Let P be a set of primes with the property that if p is a prime factor of the degree of some σ_n^m, then $p \in P$. Then X is embeddable in an arbitrary $A(P)$-category W.*

Proof. Let $\{\varepsilon_n\}$ be a sequence of positive numbers tending to zero. Define the following diagram, equivalent to the existence of a homeomorphism between the two inverse limits (see [1] and also [11])

$$\begin{array}{ccccc}
X_1 & \xleftarrow{\sigma_1^2} & X_2 & \leftarrow & \cdots \\
{\scriptstyle e_1}\downarrow & & {\scriptstyle e_2}\downarrow & & \\
Y_1 & \xleftarrow[\Pi_1^2]{} & Y_2 & \leftarrow & \cdots
\end{array}$$

where $e_1, e_2, \cdots$ are identities and

$$\Pi_j^n e_n \sigma_n^m =_{\varepsilon_n} \Pi_j^m e_m, \tag{1}$$

$$\sigma_j^n e_n^{-1} \Pi_n^m =_{\varepsilon_n} \sigma_j^m e_m^{-1}, \tag{2}$$

and $\{Y_n, \Pi_n^{n+1}\}$ is in W.

The construction is by induction. Let $Y_1 = C_{1,1}$. Assume that Y_j and e_j are already defined for $j \leqq n$ so that the above requirements are satisfied. According to the approximation property, there exist an integer r and a map $g \in \text{Map}\,(C_{r,r}, C_{m,r})$, such that the difference between σ_n^{n+1} and g is so small that (1) and (2) hold for any $j \leqq n$ and $m = n + 1$. Let $\Pi_n^{n+1} = g$ and $Y_{n+1} = C_{r,r}$.

The next two theorems can be proved by methods of [1] and also [11] coupled with the fact that bonding maps of the UCC are majorants.

THEOREM 4. *Let N be a set of natural numbers, and let U and W be $A(N)$-categories. Then $UCC(U)$ and $UCC(W)$ are homeomorphic.*

THEOREM 5. *If the circularly chainable continuum X can be embedded in an $A(N)$-category, then X is a continuous image of $U(N)CC$.*

A continuum Z is said to be *universal* for a class of circularly chainable continua if each member of that class can be embedded in an A-category (or an $A(N)$-category) in which Z is the universal circularly chainable continuum. Continua which are universal circularly chainable continua in some A-category or $A(N)$-category will sometimes be called universal pseudo-circles for the sake of brevity.

Theorems 3 and 4 imply

THEOREM 6. *There exists exactly one circularly chainable continuum which is universal for the class of continua which are both circularly chainable and embeddable in an A-category.*

The following theorem may also be proved by considering Čech cohomology groups.

THEOREM 7. *There exists a collection of c pseudo-circles such that no one is a continuous image of another.*

Proof. Let $P = \{P_\alpha\}$ be a collection of c sequences of primes with the property that any prime occurs at most once in any sequence and that any two sequences have at most a finite number of primes in common.

Consider a $U(P_\alpha)CC$, for each sequence of P. Let $\{\varepsilon_n\}$ be a sequence of positive numbers with $\lim_{n\to\infty} \varepsilon_n = 0$. Let $U(P_\alpha)CC = \lim \{X_n, \sigma_n^m\}$ and let $U(P_\beta)CC = \lim \{Y_n, \Pi_n^m\}$. The existence of a continuous map from $U(P_\alpha)CC$ onto $U(P_\beta CC)$ is equivalent to the existence of the following diagram (see [1] and also [11])

$$\begin{tikzcd} X_{n_1} \arrow[d, "h_1"'] & X_{n_2} \arrow[l] \arrow[d, "h_2"'] & \cdots \arrow[l] & X_{n_l} \arrow[l] \arrow[d, "h_l"'] & \cdots \arrow[l] \\ Y_{m_1} & Y_{m_2} \arrow[l] & \cdots \arrow[l] & Y_{m_l} \arrow[l] & \cdots \arrow[l] \end{tikzcd}$$

where $\{m_k\}$ and $\{n_k\}$ are non-decreasing and unbounded sequences of positive integers and

$$\Pi_{m_l}^{m_k} h_k \sigma_{n_k}^{n_j} =_{\varepsilon_k} \Pi_{m_l}^{m_j} h_j . \tag{3}$$

Let $k > 1$ be so large that $\varepsilon_k < 1$. Then (3) implies

$$\deg (h_k \sigma_{n_k}^{n_j}) = \deg (\Pi_{m_k}^{m_j} h_j) \tag{4}$$

Let j be so large that there exists a prime p which does not occur in P_α and which does not divide the degree of h_k but which does divide the degree of $\Pi_{m_k}^{m_j}$. The existence of p, obvious from the construction of P, contradicts (4). Hence there exists no continuous map of $U(P_\alpha)CC$ onto $U(P_\beta)CC$.

THEOREM 8. *There exist continuum-many topologically distinct pseudo-circles.*

THEOREM 9. *All pseudo-circles are embeddable in E^3. $U(1)CC$ is the only universal pseudo-circle which is embeddable in the plane.*

Proof. The first fact follows immediately from known results. The second fact is a consequence of an embedding theorem by Bing [3], together with the equivalence between inverse limits and circular chains which was noted at the beginning of this section.

THEOREM 10. *Any universal pseudo-circle can be mapped continuously onto any planar pseudo-circle.*

Proof. Since one is a product of nonnegative powers of elements of any set of natural numbers, $U(1)CC$ is embeddable in any A-category. Any planar pseudo-circle, however, is a continuous image of $U(1)CC$, by Theorem 9.

An argument (with degrees of maps) similar to the proof of Theorem 7 shows

THEOREM 11. *No non-planar pseudo-circle is a continuous image of a planar pseudo-circle.*

Theorems 10 and 11 give a kind of duality between planar and non-planar pseudo-circles. Theorems 12 and 14 yield the same duality for pseudo-circles and the pseudo-arc.

THEOREM 12. *The pseudo-arc M is a continuous image of each universal pseudo-circle.*

Proof. Let M be chainable between p and q. Join M to another copy of M at the points p and q. The resulting continuum H is circularly chainable of degree one. Hence H is a continuous image of $U(1)CC$ and thus of each universal pseudo-circle. The map of H onto M obtained by folding M over into its copy homeomorphically is continuous. Consequently M is a continuous image of each universal pseudo-circle.

Theorem 6 implies that the UCC is universal for all circularly chainable continua embeddable in some A-category. This takes into consideration all circularly chainable continua except those which have a defining sequence of circular chains $\{C_i\}$ such that C_{i+1} circles zero times in C_i. Ingram [6] has proved that such continua are chainable and consequently [5] are continuous images of the pseudo-arc. These remarks and Theorem 12 prove

THEOREM 13. *UCC is universal for all circularly chainable continua and $U(1)CC$ is universal for all planar circularly chainable continua.*

Let f map the continuum X onto the continuum Y. The map f is said to be *confluent* if for each subcontinuum Q of Y and each component K of $f^{-1}(Q)$, f maps K onto Q.

LEMMA 15. *Let f map the continuum X onto the hereditarily indecomposable continuum Y. Then f is confluent.*

Proof. Let Q be a subcontinuum of Y and let K be a component of $f^{-1}(Q)$. Let $K_1, K_2, \cdots$ be a descending sequence of continua such that each K_i contains K and $K = \bigcap_{i=1}^{\infty} K_i$. Then for each i, $f(K_i)$ contains a point of Q and a point in the complement of Q. In order that $Q + f(K_i)$ not be a decomposable continuum, it is necessary that $Q \subset f(K_i)$. Consequently $Q \subset \bigcap f(K_i)$. But $\bigcap f(K_i) = f(K) \supset Q$. Hence $f(K) = Q$.

THEOREM 14. *No pseudo-circle is a continuous image of the pseudo-arc M.*

Proof. Suppose that f were a map of M onto the pseudo-circle H. Since H is hereditarily indecomposable, the above lemma implies that f is confluent. Lelek [8] has observed that such a confluent mapping induces a monomorphism,

$$f^* : H^1(H) \to H^1(M)$$

of Čech cohomology groups with integer coefficients. Since $H^1(M)$ is trivial and $H^1(H)$ is non-trivial, such a monomorphism cannot exist. This contradiction proves the theorem.

Characterization of continuous images of pseudo-circles

Fearnley [5] has characterized the continuous images of the pseudo-arc in terms of p-chains. In this section the author characterizes the continuous images of pseudo-circles in terms of q-chains and proves that p-chainable continua are q-chainable.

DEFINITION. A q-chain $Q = (q_0, \cdots, q_n)$ is a finite sequence of sets such that $q_i \cap q_j \neq \emptyset$ whenever $|i - j| \leqq 1 \pmod{n+1}$. Each q_i is called a link of Q.

DEFINITION. If $P = (p_0, \cdots, p_n)$ and $Q = (q_0, \cdots, q_m)$ are q-chains and each link p_i of P is a subset of a link q_{x_i} of Q, then the sequence of ordered pairs $(0, x_0)$, $(1, x_1)$, $\cdots$, (n, x_n) is said to be a pattern of P in Q if $|x_i - x_j| \leqq 1 \pmod{m+1}$ whenever $|i - j| \leqq 1 \pmod{n+1}$, $0 \leqq i, j \leqq n$.

DEFINITION. When P and Q are q-chains, P is said to refine Q if there is a pattern of P in Q.

In considering the relationships between q-chains, it is obvious that there might be several patterns of P in Q. When a pattern of P in Q is mentioned in an argument, it is assumed that a chosen pattern is fixed throughout the proof.

DEFINITION. Let P refine Q. The degree of P in Q is the number of times in the pattern of P in Q that $x_j = m$ and $x_{j+1} = 0$, $j = 0, 1, \cdots, m-1$, diminished by the number of times that $x_j = 0$ and $x_{j+1} = m$, $j = 0, 1, \cdots, m-1$.

DEFINITION. The continuum M is q-chainable if there exists a sequence of q-chains $Q_1, Q_2, Q_3, \cdots$ such that for each natural number i,

(1) Q_i covers M,

(2) Q_{i+1} refines Q_i,

(3) each link of Q_i has diameter less than $1/i$, and

(4) the closure of each link of Q_{i+1} is a subset of the link of Q_i to which it corresponds under the pattern of Q_{i+1} in Q_i.

The sequence $\{Q_i\}$ is said to be associated with M.

DEFINITION. Let N be a set of natural numbers. The continuum M is q-chainable of degree N if there is a sequence of q-chains $Q_1, Q_2, \cdots$ associated with M such that for each natural number $j > 1$ the degree of Q_j in Q_{j-1} is zero or a product of nonnegative powers of elements of N. In case the only element of N is the positive integer n, M is said to be q-chainable of degree n.

THEOREM 15. *The continuum M is a continuous image of UCC if and only if M is q-chainable.*

Proof. Suppose that M is the image of UCC under the map f. Let $D_1, D_2, \cdots$ be a sequence of circular chains in UCC, considered as space, such that for each natural number i, (1) D_{i+1} refines D_i, (2) the diameter of each link of D_i is less than $1/i$, (3) the closure of each link of D_{i+1} is a subset of some link of D_i, and (4) D_i covers M.

Consider the sequence of q-chains $f(D_1), f(D_2), \cdots$. Since f is uniformly continuous, one may assume that the diameter of each link of $f(D_i)$ is less than $1/i$. Each $f(D_i)$ covers M because each D_i covers UCC and f is onto. Choose a pattern of D_i in D_{i-1} which assigns to each link of D_i a link of D_{i-1} which contains its closure. Such a pattern may be chosen because each D_i is a circular chain. It is obvious that each $f(D_{i+1})$ refines $f(D_i)$ and that the closure of each link of $f(D_{i+1})$ is a subset of the link of $f(D_i)$ to which it corresponds. Hence $f(D_1), f(D_2), \cdots$ is a sequence of q-chains associated with M, and M is q-chainable.

Now suppose M is a q-chainable continuum, and let $Q_1, Q_2, \cdots$ be a sequence of q-chains associated with M. Since each circularly chainable continuum is a continuous image of UCC, it suffices to define a circularly chainable continuum K of which M is a continuous image.

In order to construct K, define a new sequence $P_1, P_2, \cdots$ of q-chains associated with M and a sequence of circular chains $D_1, D_2, \cdots$, such that for each natural number n, (1) D_n is a circular chain of open balls in E^3, (2) D_{n+1} has the same pattern in D_n as P_{n+1} has in P_n, (3) the diameter of each link of D_n is less than $1/n$, and (4) the closure of each link of D_{n+1} is contained in the link of D_n to which it corresponds under the pattern of D_{n+1} in D_n. Define K to be the common part of the D_i's.

The sequences $\{P_i\}$ and $\{D_i\}$ may be constructed by an induction argument. Namely, let $P_1 = Q_1$ and let D_1 satisfy (1) and (3) and have the same number of links as Q_1. Let $d_0, d_1, \cdots, d_m$ denote the links of D_1.

Let $(0, x_0), (1, x_1), \cdots, (n, x_n)$ be the pattern of Q_2 in Q_1. Let A_0 be an open ball of diameter less than $1/n$ such that the closure of A_0 is a subset of d_{x_0}. Call A_0 the first link of D_2. Let C be a chain of open balls of diameter less than $1/n$ such that C is a closed refinement of $\{d_{x_0}, d_{x_1}\}$ and the last link A_l of C is the first link of C whose closure is contained in d_{x_1}. Let $A_0, A_1, \cdots, A_l$ be the first $l + 1$ links of D_2. Define a new q-chain Q_2' by $q_1' = q_0$ for $0 \leqq i < l$, $q_{i+l}' = q_i$, for $i \geqq l$ and a pattern of Q_2' in Q_1 by (i, x_0) for $0 \leqq i < l$ and $(i + l, x_i)$ for $i \geqq l$.

It is obvious how to continue the construction of the links of D_2 so that D_2 satisfies (1)–(4). At the end of the construction, denote the q-chain replacing Q_2 by $Q_{2,2}$. It is now necessary to modify each Q_n, $n > 2$, so that the modification, $Q_{n,2}$, will be a refinement of $Q_{n-1,2}$. It is clear how to construct the remaining circular chains $D_3, D_4, \cdots$ so that each D_n satisfies (1)–(4) for the sequences of q-chains $\{P_n\} = \{Q_{n,n}\}$, the diagonal sequence of the matrix.

Using the circular chains $D_1, D_2, \cdots$ define a map f of K onto M as follows. For any x in K, let $J_n(x)$ denote the union of the links of P_n such that x belongs to the links of D_n having the same indices. For each n, $J_n(x)$ is the union of at most two links of P_n, and hence each $J_n(x)$ has diameter less than $2/n$. Because of the restrictions on the circular chains, $J_{n+1}(x) \subset J_n(x)$. The sequence of closed sets $\bar{J}_1(x), \bar{J}_2(x), \cdots$ therefore forms a decreasing sequence of sets with diameters tending to zero. Define $f(x)$ to be the unique common point of the closure of the $J_n(x)$'s.

It remains to prove that f is continuous and onto. Let g be an open set in M and let k be a natural number such that g contains three consecutive links l_1, l_2, l_3 of R_k. Let d_1, d_2, d_3 denote the corresponding links of D_k. Since $f(d_2) \subset J_k(x)$, for any x in d_2, the open set $d_2 \cap K$ is mapped into the open set g. Consequently f is continuous and f is onto since $f(K)$ is dense in the compact space M.

Consideration of the degree of q-chainability of the continuum M and the proof of Theorem 15 yield the more precise

THEOREM 16. *Let N be a set of natural numbers. In order that the continuum M be a continuous image of $U(N)CC$ it is necessary and sufficient that M be q-chainable of degree N.*

COROLLARY. *The continuum M is a continuous image of a planar circularly chainable continuum if and only if M is q-chainable of degree one.*

COROLLARY. *In order that the continuum M be a continuous image of a pseudo-circle, it is necessary and sufficient that M be q-chainable.*

One usually thinks of different kinds of chains having links which are open sets. The following theorem, the proof of which is omitted, shows that one gains no generality in admitting arbitrary sets as links of q-chains.

THEOREM 17. *The continuum M is a continuous image of a pseudo-circle if and only if M is q-chainable with q-chains whose links are open sets.*

Finally we show

THEOREM 18. *All p-chainable continua, and hence the pseudo-arc, are q-chainable.*

Proof. See [5] for the definitions of undefined terms. Let $\{P_i\}$ be a sequence of p-chains associated with the continuum M. Let Q_n be the p-chain

P_n plus its conjugate. That $\{Q_n\}$ is a sequence of q-chains associated with M is obvious.

COROLLARY. *All p-chainable continua are continuous images of a pseudo-circle.*

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