SOME APPLICATIONS OF A GENERAL LEMMA ON INVARIANT MEANS

BY

ROBERT E. HUFF[1]

1. Introduction

The purpose of this note is to point out a general lemma on invariant means (Lemma 1) which is suggested by the work of Day [3] and which, together with a useful property of compact convex sets (Lemma 2), is applied to strengthen and to generalize several fixed-point theorems. Some of the techniques used here have also been used by Argabright [1] to prove Day's main result. In particular, Lemma 1 is implicit in the proofs in both [1] and [3], and the locally convex case of Lemma 2 is proven by Argabright in [1].

In this section our notation and terminology are introduced; Section 2 is devoted to the two fundamental lemmas; and in Section 3 several applications are given. Day's existence result [3] is strengthened and a corresponding uniqueness result is proven (Theorem 1, parts (1) and (2)). Theorem 2 contains a result of Cohen and Collins [2] as a special case, and yields a new proof of the Kakutani fixed-point theorem [4, p. 457] in a strengthened form.

If S is a nonempty set then $m(S)$ denotes the Banach space of all bounded functions from S to the reals, $\mathbf{R}$, with the uniform norm; 1 will be used to denote the identically one function. A *semigroup of maps from S to S is a* collection Σ of functions from S into S such that Σ is closed under compositions. If X is a subspace of $m(S)$ then an *X-semigroup* is a semigroup of maps from S to S such that $f \circ \sigma \in X$ for all f in X and all σ in Σ. If X is a subspace of $m(S)$ containing 1, then an *X-mean* is an element μ in X^* such that

(i) $\mu(f) \geqq 0$ whenever $f \in X$ and $f \geqq 0$ on S,
(ii) $\mu(1) = 1$.

Conditions (i) and (ii) are jointly equivalent to

(iii) $\| \mu \| = 1 = \mu(1)$.

If X is a subspace of $m(S)$ containing 1 and if Σ is an X-semigroup, then a *Σ-invariant X-mean* is an X-mean μ such that $\mu(f \circ \sigma) = \mu(f)$ for every f in X and every σ in Σ.

Suppose now that Σ is an abstract semigroup. Let Σ^l denote Σ acting on itself by left multiplication; i.e., $\sigma : \tau \rightarrow \sigma\tau$ for $\sigma, \tau \in \Sigma$. Then Σ is said to have a *left invariant mean* μ provided μ is a Σ^l-invariant $m(\Sigma)$-mean.

Received March 7, 1968.

[1] This paper was written while the author was a National Science Foundation Graduate Fellow.

If X is a compact Hausdorff space, $C(X)$ denotes the space of continuous functions from X to $\mathbf{R}$ with the uniform norm.

2. The fundamental lemmas

Let S and T be sets and let X be a subspace of $m(S)$ and Y a subspace of $m(T)$ such that $1 \in X$ and $1 \in Y$. Suppose that $\psi : S \rightarrow T$ is a function such that $f \circ \psi$ is in X for every f in Y. Let $\bar{\psi} : Y \rightarrow X$ be given by $\bar{\psi}(f) = f \circ \psi$ for f in Y; then $\bar{\psi}$ is a continuous linear mapping of norm 1. Thus the adjoint map $\bar{\psi}^* : X^* \rightarrow Y^*$, given by $\bar{\psi}^*(\mu) = \mu \circ \bar{\psi}$, is well defined, linear, and norm continuous of norm one. Furthermore, $\bar{\psi}^*$ is weak* continuous and maps X-means into Y-means.

LEMMA 1. *Suppose*

(1) *S is a set and X is a subspace of $m(S)$ with $1 \in X$;*

(2) *T is a set and Y is a subpace of $m(T)$ with $1 \in Y$;*

(3) *Σ_1 is an X-semigroup;*

(4) *Σ_2 is a Y-semigroup;*

(5) *Σ is an abstract semigroup for which there are semigroup homomorphisms h_1 of Σ into Σ_1 and h_2 of Σ onto Σ_2 ;*

(6) *there is a function $\psi : S \rightarrow T$ such that $f \circ \psi \in X$ for all f in Y and such that*

$$\psi \circ h_1(\sigma) = h_2(\sigma) \circ \psi \quad \textit{all} \quad \sigma \in \Sigma.$$

Let $\bar{\psi}^* : X^* \rightarrow Y^*$ be the natural second adjoint of ψ (as in the discussion above). If μ is a Σ_1-invariant X-mean, then $\bar{\psi}^*(\mu)$ is a Σ_2-invariant Y-mean.

Conditions (3), (4), and (5) together can be interpreted as saying that Σ acts both as an X-semigroup and as Y-semigroup; condition (6) gives the connection between the two actions. For the proof, let μ be a Σ_1-invariant X-mean, let f be in Y, and let $h_2(\sigma)$ be in Σ_2 where $\sigma \in \Sigma$. Then

$$[\bar{\psi}^*(\mu)](f \circ h_2(\sigma)) = \mu(f \circ h_2(\sigma) \circ \psi) = \mu(f \circ \psi \circ h_1(\sigma)) = \mu(f \circ \psi) = [\bar{\psi}^*(\mu)](f).$$

As an example of application of the lemma, suppose Σ is a group of continuous maps of a compact Hausdorff space Z into itself, and suppose Σ has a compact Hausdorff group topology such that for some y_0 in Z the map $\psi : \sigma \rightarrow \sigma(y_0)$ of Σ to Z is continuous. Then there is a Σ-invariant regular Borel probability measure on Z. For the proof, take $S = \Sigma$, $X = C(\Sigma)$, $T = Z$, $Y = C(Z)$, $\Sigma_1 = \Sigma^l$, $\Sigma_2 = \Sigma$ (with natural homomorphisms), and take ψ as given. Left Haar measure μ for Σ is a Σ_1-invariant X-mean, and so by the theorem $\bar{\psi}^*(\mu)$ is a Σ-invariant $C(Z)$-mean; by the Riesz representation theorem there is a Σ-invariant regular Borel probability measure on Z representing $\bar{\psi}^*(\mu)$.

Suppose E is a real linear space, K is a convex subset of E and T is some compact topology for K. Let $A(K)$ denote the set of all T-continuous affine maps of K to $\mathbf{R}$. Then $A(K)$ is a closed subspace of $C(K)$. Let

$Q : K \to A(K)^*$ be the natural map given by $Q(x)(f) = f(x)$ for x in K and f in $A(K)$. Since the members of $A(K)$ are T-continuous, Q is weak* continuous; since the members of $A(K)$ are affine, Q is affine; and it is clear that Q maps K into the $A(K)$-means. In fact, $Q(K)$ is precisely the set of all $A(K)$-means: Let μ be an $A(K)$-mean and suppose μ is not in $Q(K)$. Since $Q(K)$ is a weak* compact convex set, by the separation theorem [3, p. 417], there is an f in $A(K)$ such that

$$\mu(f) < \inf\{f(k) : k \in K\}.$$

If we let $\alpha = \inf\{f(k) : k \in K\}$, then $\alpha \cdot 1 \leqq f$ so that $\alpha = \mu(\alpha \cdot 1) \leqq \mu(f)$, a contradiction. Thus Q is a weak* continuous affine map of K onto the set of all $A(K)$-means; if $A(K)$ separates points of K, then Q is an affine homeomorphism.

For the remainder of this paper we will assume that E is a fixed real linear space, that K is a convex subset of E, and that T is a compact topology for K such that the Banach space $A(K)$ of all T-continuous affine maps of K into $\mathbf{R}$ *separates points of K.* Topological properties of K will always be with respect to the topology T. Note that the preceding remarks imply that K is affinely homeomorphic to a compact convex subset of a locally convex linear topological Hausdorff space; conversely, any compact convex set in a locally convex linear topological Hausdorff space satisfies the above assumptions on K. The next lemma is an immediate consequence of the proceding remarks.

LEMMA 2. *Let Σ be a semigroup of continuous affine maps of K into K. Then there is a Σ-invariant $A(K)$-mean if and only if there is a Σ-fixed point in K.*

3. Applications

Let Σ be a semigroup of continuous affine maps of K into K and suppose Σ has a left invariant mean. In Lemma 1, take $S = \Sigma$, $X = m(\Sigma)$, $T = K$, $Y = A(K)$, $\Sigma_1 = \Sigma^l$, $\Sigma_2 = \Sigma$, and ψ given by $\psi(\sigma) = \sigma(y_0)$ for some fixed y_0 in K. Lemma 1 then asserts the existence of a Σ-invariant $A(K)$-mean; by Lemma 2, there is a Σ-fixed point in K. Day [3] obtained this result which we shall strengthen in Theorem 1 below. Day [3] gives an example to show that it is not sufficient to assume that Σ has a right invariant mean; i.e., "left amenability" implies existence of fixed-points in K, "right amenability" does not. This is because of the nature in which functions compose, and no further considerations of the implications of the existence of right invariant means were made in [3] or in [1]. In Theorem 1 we shall prove that the existence of a right invariant mean for the semigroup is connected with uniqueness of the fixed point.

Note that if Σ is a collection of affine maps on K to K then $\Sigma \subset K^K \subset E^K$ (product sets) and K^K is a convex subset of the real linear space E^K with pointwise operations. We can give K^K the product topology and it is then compact by the Tychonoff theorem. Thus it makes sense to speak of the closed convex hull of Σ in K^K. Note that K^K has sufficiently many real continuous affine functions to separate points of K^K.

The Σ-*orbit* of a point x in K is the *closed convex hull of the set* $\{\sigma(x) : \sigma \in \Sigma\}$.

THEOREM 1. *Let Σ be a semigroup of continuous affine maps on K to K. Let Φ denote the closed convex hull of Σ in K^K.*

(1) *If Σ has a left invariant mean, then there is some $\bar\phi$ in Φ such that*

$$\phi(\bar\phi(x)) = \bar\phi(x) \quad \textit{all } \phi \in \Phi,\ x \in K;$$

thus for every x in K, $\bar\phi(x)$ is a Φ-fixed point in the Σ-orbit of x.

(2) *If Σ has a right invariant mean and if Σ is equicontinuous, then there is some $\bar{\bar\phi}$ in Φ such that*

$$\bar{\bar\phi}(\phi(x)) = \bar{\bar\phi}(x) \quad \textit{all } \phi \in \Phi,\ x \in K;$$

thus for every x in K, $\bar{\bar\phi}(x)$ is the only possible Σ-fixed point in the Σ-orbit of x in K.

Proof. (1) Φ is a compact convex subset of K^K. Let Σ act on Φ by left composition; i.e.,

$$\sigma^L : \phi \to \sigma \circ \phi \quad \sigma \text{ in } \Sigma,\ \phi \text{ in } \Phi.$$

Then each σ^L acts as a continuous affine map of Φ into Φ. Since Σ, as an abstract semigroup, has a left invariant mean, by the preceding remarks there is some $\bar\phi$ in Φ such that $\sigma \circ \bar\phi = \bar\phi$ for all σ in Σ. From the definition of Φ, it follows that $\phi \circ \bar\phi = \bar\phi$ for all ϕ in Φ.

(2) Next, let Σ act on Φ by right composition; i.e.,

$$\sigma^R : \phi \to \phi \circ \sigma \quad \sigma \text{ in } \Sigma,\ \phi \text{ in } \Phi.$$

Then each σ^R acts as a continuous affine map on Φ to Φ. Since Σ, as an abstract semigroup has a right invariant mean, $\{\sigma^R : \sigma \in \Sigma\}$ has a left invariant mean. Thus there is some $\bar{\bar\phi}$ in Φ such that $\bar{\bar\phi} \circ \sigma = \bar{\bar\phi}$ for all σ in Σ. Since Σ is equicontinuous, $\bar{\bar\phi}$ is continuous [6, p. 232]. It follows that $\bar{\bar\phi} \circ \phi = \bar{\bar\phi}$ for all ϕ in Φ. If y is a Σ-fixed point in the Σ-orbit of x, then y is a Φ-fixed point and $y = \phi_0(x)$ for some ϕ_0 in Φ; thus

$$y = \bar{\bar\phi}(y) = \bar{\bar\phi}(\phi_0(x)) = \bar{\bar\phi}(x).$$

It is well known that every commutative semigroup has a (two-sided) invariant mean (see [3]) so that Theorem 1 gives the following strengthening of the Markov-Kakutani fixed-point theorem [4, p. 456].

COROLLARY. *Let Σ be a commutative semigroup of continuous affine maps on K to K and let Φ denote the closed convex hull of Σ in K^K. There is a function $\bar\phi$ in Φ such that for every x in K, $\bar\phi(x)$ is a fixed point in the Σ-orbit of x; if Σ is equicontinuous, $\bar\phi(x)$ is the only fixed point in the Σ-orbit of x.*

As Day [3] has pointed out, the full assumption of a Σ^l-invariant $m(\Sigma)$-mean is not necessary for his fixed-point theorem; often a Σ^l-invariant subspace of $m(\Sigma)$ is all that need be considered. If Σ has a topology τ let $C(\Sigma; \tau)$ denote the subspace of $m(\Sigma)$ consisting of the τ-continuous members of $m(\Sigma)$. If τ_1 and τ_2 are two topologies, if $C(\Sigma; \tau_1)$ and $C(\Sigma; \tau_2)$ are both Σ^l-invariant (i.e., $f \circ \sigma^l$ is in $C(\Sigma; \tau_i)$ whenever f is in $C(\Sigma; \tau_i)$ and σ^l is in Σ^l, $i = 1, 2$), and if $\tau_1 \subset \tau_2$, then there is a Σ^l-invariant $C(\Sigma; \tau_1)$-mean whenever there is a Σ^l-invariant $C(\Sigma; \tau_2)$-mean. This follows easily from Lemma 1.

Now suppose Σ is a semigroup of continuous affine maps on K to K and let y_0 be a point in K. If $\tau(y_0)$ denotes the topology on Σ of pointwise convergence at the point y_0, then $C(\Sigma; \tau(y_0))$ is Σ^l-invariant. If there exists a Σ^l-invariant $C(\Sigma; \tau(y_0))$-mean, then there is a Σ-fixed point in the Σ-orbit of y_0. The proof is just as in the first paragraph of this section where we take $X = C(\Sigma; \tau(y_0))$ instead of $m(\Sigma)$.

With these considerations, we can improve Theorem 1.

THEOREM 1′. *Let Σ be a semigroup of continuous affine maps on K to K and give Σ the topology τ_p of pointwise convergence on K (i.e., consider Σ as a subspace of K^K with the product topology). Let Φ denote the closed convex hull of Σ in K^K. $C(\Sigma; \tau_p)$ is Σ^l-invariant and Σ^r-invariant.*

(1) *If there exists a Σ^l-invariant $C(\Sigma; \tau_p)$-mean, then there is some $\bar{\phi}$ in Φ such that*

$$\phi(\bar{\phi}(x)) = \bar{\phi}(x) \quad \text{all } \phi \text{ in } \Phi,\ x \text{ in } K;$$

thus for every x in K, $\bar{\phi}(x)$ is a Φ-fixed point in the Σ-orbit of x.

(2) *If there exists a Σ^r-invariant $C(\Sigma; \tau_p)$-mean, and if Σ is equicontinuous, then there is some $\bar{\bar{\phi}}$ in Φ such that*

$$\bar{\bar{\phi}}(\phi(x)) = \bar{\bar{\phi}}(x) \quad \text{all } \phi \text{ in } \Phi,\ x \text{ in } K;$$

thus for every x in K, $\bar{\bar{\phi}}(x)$ is the only possible Σ-fixed point in the Σ-orbit of x.

The proof is simply a matter of carrying over the proof of Theorem 1 and applying the remarks preceding the Theorem; we omit the details.

As a particular case of the remarks preceding Theorem 1′, if Σ is a compact Hausdorff group with respect to some topology stronger than the topology of pointwise convergence at some point y_0 of K, then the existence of the Haar integral for Σ implies the existence of a Σ-fixed point in the Σ-orbit of y_0 in K. This generalizes (4.4) of Klee [7]. We apply this observation to obtain a result which generalizes the second theorem of Cohen and Collins [2], and whose corollary is a strengthening of the Kakutani fixed-point theorem [4, p. 457].

THEOREM 2. *Suppose that, in addition to the properties assumed in Section 2, K is also a semigroup under a binary operation $(\cdot)$ which is continuous and affine in each variable separately. Let G be a closed subgroup of K, and let T be the closed convex hull of G in K. Then there is a unique element $\bar{t}$ in T such that*

$$t \cdot \bar{t} = \bar{t} \cdot t = \bar{t} \quad \text{all } t \in T;$$

i.e., T has a zero element.

Proof. Since G is a group with a compact Hausdorff topology such that multiplication $(\cdot)$ is continuous in each variable separately, by Ellis' theorem (the main result of [5]), G must in fact be a compact Hausdorff topological group under this topology.

Now let G act on T by left multiplication; i.e., $g : t \rightarrow g \cdot t$ for t in T, g in G. From above, there is some $\bar{t}$ in T such that $g \cdot \bar{t} = \bar{t}$ for all g in G; by the definition of T, this implies $t \cdot \bar{t} = \bar{t}$ for all t in T.

Similarly, there is some $\bar{\bar{t}}$ in T such that $\bar{\bar{t}} \cdot t = \bar{\bar{t}}$ for all t in T; thus $\bar{t} = \bar{\bar{t}} \cdot \bar{t} = \bar{\bar{t}}$ and the theorem holds.

COROLLARY. *Let Σ be an equicontinuous group of continuous affine maps on K to K, and let Φ denote the closed convex hull of Σ in K^K. Then Φ consists of continuous affine functions and there is a unique element $\bar{\phi}$ in Φ such that*

$$\phi \circ \bar{\phi} = \bar{\phi} \circ \phi = \bar{\phi} \quad \text{all } \phi \text{ in } \Phi;$$

in particular, for every x in K, $\bar{\phi}(x)$ is a Φ-fixed point and is the only Σ-fixed point in the Σ-orbit of x.

Proof. It is clear that the convex hull of Σ in K^K is equicontinuous on K. Then Φ is equicontinuous and the map $(\phi, x) \rightarrow \phi(x)$ on $\Phi \times K$ to K is jointly continuous by [6, p. 232]. Φ is closed under composition and satisfies the hypotheses on K in the Theorem. Let G denote the closure of Σ in Φ; then the closed convex hull of G in Φ is all of Φ. By the theorem, it is sufficient to prove that G is a group. It is clear that G is closed under composition; it remains only to prove that if g is in G then g is invertible and g^{-1} is in G. Suppose $\{\sigma_\alpha\}$ is a net in Σ with $\sigma_\alpha \rightarrow g \in G$ pointwise. Since G is compact, we may assume that there is some h in G with $\sigma_\alpha^{-1} \rightarrow h$ pointwise. If x is in K, then $(\sigma_\alpha, \sigma_\alpha^{-1}(x)) \rightarrow (g, h(x))$ in $\Phi \times K$ and so $x = \sigma_\alpha(\sigma_\alpha^{-1}(x)) \rightarrow g(h(x))$; i.e., $x = g(h(x))$. Similarly $x = h(g(x))$ so that $h = g^{-1}$.

REFERENCES

1. L. N. ARGABRIGHT, *Invariant means and fixed points; a sequel to Mitchell's paper*, Trans. Amer. Math. Soc., vol. 130 (1968), pp. 127–130.
2. H. COHEN AND H. S. COLLINS, *Affine semigroups*, Trans. Amer. Math. Soc., vol. 93 (1959), pp. 97–113.
3. M. M. DAY, *Fixed-point theorems for compact convex sets*, Illinois J. Math., vol. 5 (1961), pp. 585–590.
4. N. DUNFORD AND J. T. SCHWARTZ, *Linear operators, Part I*, Interscience, New York, 1963.
5. R. ELLIS, *Locally compact transformation groups*, Duke Math. J., vol. 24 (1957), pp. 119–126.
6. J. L. KELLEY, *General topology*, van Nostrand, Princeton, 1955.
7. V. L. KLEE, JR., *Invariant extension of linear functionals*, Pacific J. Math., vol. 4 (1954), pp. 37–46.

UNIVERSITY OF NORTH CAROLINA
CHAPEL HILL, NORTH CAROLINA