ON EXACT SEQUENCES IN THE HOMOLOGY OF GROUPS AND ALGEBRAS

BY

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1. Introduction

The Hochschild-Serre spectral sequence for homology of groups (Lie algebras or restricted Lie algebras) gives rise to an exact sequence of terms of low degree. Let N be a normal subgroup of G with factor group Q, and let $H_n(G, M)$, $H_n(Q, M)$ denote respectively the n^{th} ordinary homology groups of G, Q with coefficients in the Q-module M. Then the sequence is

$$\text{(I)} \quad H_2(G, M) \to H_2(Q, M) \to N/[N, N] \otimes_Q M \to H_1(G, M) \to H_1(Q, M) \to 0.$$

As usual, [N, N] denotes the commutator subgroup. This sequence was studied by various authors [9], [10], [11], [7], and also applied to interesting non-homological problems. Recently Knus proved in [6] the existence of a similar exact sequence for augmented algebras under slightly restrictive conditions concerning the coefficient modules. In view of the many applications of these sequences it is desirable to have general and direct proofs. It is one of the main purposes of this paper to furnish such proofs involving merely standard homological techniques; to do so we use methods due to Barr and Rinehart [1]. The same techniques also yield various similar exact sequences, valid in higher dimensions but assuming hypotheses. These latter sequences again are known in the special case of groups (or Lie algebras) as corollaries of the Hochschild-Serre spectral sequence.

In view of the applications to the cases of groups, Lie algebras, and restricted Lie algebras we deal mainly with augmented algebras, although a large portion of the results carry over to the more general case of associative algebras. The corresponding results are listed in section 6.

Our methods and general results can be summarized as follows. Let T be an augmented algebra over the commutative ring K [8, p. 180]. The kernel of the augmentation $\varepsilon : T \to K$ is denoted by JT and called the augmentation ideal. Let $\Phi : T \to R$ be a surjective morphism of augmented algebras, with kernel S. We define a homology $\hat{H}^{\Phi}_*(T, -)$ with respect to Φ as the derived functor (in the category of R-modules) of the Φ-differentials $JT \otimes_T -$. In the special case, where $\Phi = \mathbf{1}: R \to R$, our homology $\hat{H}^{\mathbf{1}}_*(R, -)$ turns out to be essentially the ordinary (i.e. Hochschild) homology $H_*(R, -)$ (Proposi-

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tion 2.1). Our definition of a Φ-homology obviously is the "dual" of the definition of a Φ_m cohomology of Barr-Rinehart in [1]. As they do, we obtain for every R-module M a long exact sequence (Theorem 2.2)

$$\text{(II)}\quad \begin{aligned} \cdots \to \hat{H}_n^{\Phi}(T, M) \to \hat{H}_n^{1}(R, M) \to \operatorname{Tor}_{n-1}^{R}(S/JT\cdot S, M) \to \hat{H}_{n-1}^{\Phi}(T, M) \\ \to \cdots \to S/JT\cdot S \otimes_R M \to JT \otimes_T M \to JR \otimes_R M \to 0. \end{aligned}$$

The analogous statements for groups (Lie algebras, restricted Lie algebras) are obtained in Section 4, mainly by applying the results for augmented algebras to the group ring (enveloping algebra, restricted enveloping algebra).

The low dimensional part of the long exact sequence (II) and of the analogous sequence for groups (Lie algebras, restricted Lie algebras) is used in Sections 3 and 4 to deduce 5-term sequences in *ordinary* homology. We obtain in that way: in the case of augmented algebras, a generalization of the exact sequence due to Knus [6]; in the case of groups (Lie algebras, restricted Lie algebras), the exact sequence of terms of low degree in the Hochschild-Serre spectral sequence. Assuming special hypotheses, we are able to give more information about the groups at the bottom of our sequence (II).

Let $\Phi : T \to R$ be a surjective morphism of augmented algebras, with kernel S, and let M be an R-module. Suppose that R is K-projective and $H_i(T, R \otimes M) = 0$ for $1 < i < n$. Then the following sequence in ordinary homology is exact:

$$\text{(III)}\quad \begin{aligned} H_n(T, M) \to H_n(R, M) \to \operatorname{Tor}_{n-2}^{R}(S/JT\cdot S, M) \to H_{n-1}(T, M) \\ \to \cdots \to S/JT\cdot S \otimes_R M \to H_1(T, M) \to H_1(R, M) \to 0. \end{aligned}$$

In the case of groups (Lie algebras, restricted Lie algebras) the special hypotheses may be reformulated. In the setting of sequence (I), we have that *if $H_i(N, M) = 0$ for $1 < i < n$, then there is an exact sequence analogous to* (III).

It turns out that in these three cases sequence (III) is well known as a corollary of the Hochschild-Serre spectral sequence [5], [10]. The corresponding portion of the present paper may therefore be viewed as a spectral-sequence-free proof of that sequence.

Finally similar techniques are used in Section 5 to prove the following.

Suppose R is K-projective. If $\operatorname{Tor}_i^T(R, M) = 0$ for $0 < i < n$, then the following sequence is exact:

$$\text{(IV)}\quad \begin{aligned} H_{n+1}(T, M) \to H_{n+1}(R, M) \to K \otimes_R \operatorname{Tor}_n^T(R, M) \to H_n(T, M) \\ \to H_n(R, M) \to 0. \end{aligned}$$

Again it is possible to obtain the analogous sequences in the case of groups, Lie algebras, and restricted Lie algebras as corollaries of the Hochschild-Serre spectral sequence.

We wish to thank G. S. Rinehart for many stimulating discussions. The present general form of the crucial Lemma 3.2 is his.

It is obvious that each theorem of the present paper has a straightforward "dual" in the corresponding *co*homology theory. Throughout the paper we use $- \otimes -$, without subscript, to denote the tensor product over K.

2. The long exact sequence of Barr-Rinehart in homology

Let K be any commutative ring with unit. Let R be an augmented K-algebra; i.e. a unitary K-algebra together with an algebra map $\varepsilon : R \to K$, called the *augmentation*. The kernel of ε is denoted by JR and is called the *augmentation ideal*. By ${}_R\mathbf{M}(\mathbf{M}_R)$ we denote the category of left (right) modules over R.

DEFINITION. Let $\Phi : T \to R$ be a morphism of augmented algebras. The Φ-*homology* $\hat{H}^\Phi_n(T, -)$ of T, defined in the category ${}_R\mathbf{M}(!)$, is the n^{th} left-(R, K)-relative-derived functor of the differentials $\text{Diff}^\Phi(T, -) = JT \otimes_T -$. More explicitly,

$$\hat{H}^\Phi_n(T, M) = \text{Tor}^R_n(JT \otimes_T R, M),$$

where M is in ${}_R\mathbf{M}$ and $\text{Tor}^R_n(B, -)$ denotes, for every B in $\mathbf{M}_R$, the n^{th} left-(R, K)-relative-derived functor of $B \otimes_R -$.

In [1], a Φ-cohomology was defined as the derived functor of the Φ-derivations $\text{Der}(T, M) \cong \text{Hom}_T(JT, M)$. The above therefore is the obvious "dual" of this definition.

Denoting by $H_*(R, -)$ the ordinary (relative Hochschild) homology of R we get

PROPOSITION 2.1. $\hat{H}^1_n(R, M) \cong H_{n+1}(R, M)$ *for* $n \geq 1$.

This follows immediately from the exact sequence $0 \to JR \to R \to K \to 0$ and the related Tor^R-sequence. In addition we see that the ordinary first homology group $H_1(T, M)$ may be identified with the kernel of the obvious map $JT \otimes_T M \to M$.

Dually to [1] we obtain

THEOREM 2.2. *Let* $\Phi : T \to R$ *be a surjective morphism of augmented algebras, which is K-split. Denote by S the kernel of* Φ *and let M be in* ${}_R\mathbf{M}$. *Then, for all* $n > 0$, *there is a connecting homomorphism from* $\hat{H}^1_{n+1}(R, M)$ *into* $\text{Tor}^R_n(S/JT \cdot S, M)$, *such that the following sequence is exact*:

$$\text{(V)}\quad \begin{aligned} \cdots \to \hat{H}^1_{n+1}(R, M) \to \text{Tor}^R_n(S/JT \cdot S, M) \to \hat{H}^\Phi_n(T, M) \to \hat{H}^1_n(R, M) \\ \to \cdots \to S/JT \cdot S \otimes_R M \to JT \otimes_T M \to JR \otimes_R M \to 0. \end{aligned}$$

The *proof* depends on the sequence $0 \to S/JT \cdot S \to JT \otimes_T R \to JR \to 0$, which is obviously K-split and short exact. Sequence (V) is the long exact Tor^R-sequence obtained from this short exact sequence.

3. Exact sequences in ordinary homology

As we shall see in this section we are able to replace the low dimensional Φ-homology groups in sequence (V) by ordinary homology groups. We obtain in this way, firstly, a 5-term exact sequence in *ordinary* homology, valid without any restrictive conditions, and secondly, using special hypotheses, a longer exact sequence, also in *ordinary* homology.

Clearly a surjective morphism $\Phi : T \to R$ induces for every M in ${}_R\mathbf{M}$ a unique map

$$\alpha_n : \hat{H}^1_n(T, M) \to \hat{H}^\Phi_n(T, M),$$

and hence by Proposition 2.1 for $n \geq 1$ a unique map

$$H_{n+1}(T, M) \to \hat{H}^\Phi_n(T, M),$$

also denoted by α_n.

LEMMA 3.1. *For every M in ${}_R\mathbf{M}$, $\alpha_1 : H_2(T, M) \to \hat{H}^\Phi_1(T, M)$ is an epimorphism.*

Proof. Consider the exact sequence $0 \to M' \to R \otimes M \to M \to 0$. This is an (R, K)-projective presentation of M in ${}_R\mathbf{M}$, and also a K-split short exact sequence in ${}_T\mathbf{M}$. Applying the functors $JT \otimes_T -$ and $(JT \otimes_T R) \otimes_R -$ we obtain the commutative diagram

$$\begin{array}{ccccccccc}
\cdots \to \hat{H}^1_1(T, M) & \to & JT \otimes_T M' & \to & JT \otimes_T (R \otimes M) & \to & JT \otimes_T M & \to & 0 \\
\downarrow \alpha_1 & & \wr\Vert & & \wr\Vert & & \wr\Vert & & \\
0 \to \hat{H}^\Phi_1(T, M) & \to & JT \otimes_T M' & \to & JT \otimes_T (R \otimes M) & \to & JT \otimes_T M & \to & 0
\end{array}$$

The Five Lemma gives the result.

LEMMA 3.2. *Suppose R is K-projective. If $H_i(T, R \otimes M) = 0$ for $1 < i < n$, then*

$$\alpha_i : H_{i+1}(T, M) \to \hat{H}^\Phi_i(T, M)$$

are isomorphisms for $1 \leq i < n - 1$ and

$$\alpha_{n-1} : H_n(T, M) \to \hat{H}^\Phi_{n-1}(T, M)$$

is an epimorphism.

Proof. Consider as above $0 \to M' \to R \otimes M \to M \to 0$. First we shall prove that $H_i(T, R \otimes M) = 0$ for $1 < i < n$ implies $H_i(T, R \otimes M') = 0$ for $1 < i < n - 1$. Since R is K-projective,

$$0 \to R \otimes M' \to R \otimes R \otimes M \to R \otimes M \to 0$$

is exact, and $H_i(T, R \otimes R \otimes M) = H_i(T, R \otimes M) \otimes R$. The assertion then follows easily from the long exact sequence in the second variable.

Now we proceed by induction. For $n = 2$ the hypothesis is empty and the conclusion follows by Lemma 3.1. Assume Lemma 3.2 is true for $n - 1$. We have to show that α_{n-2} is isomorphic and α_{n-1} is epimorphic. Applying the functors $JT \otimes_T -$ and $(JT \otimes_T R) \otimes_R -$ to the exact sequence $0 \to M' \to \mathbf{R} \otimes M \to M \to 0$, we get the following commutative diagram ($k \geq 2$):

$$\begin{array}{ccccccccc}
\cdots \to & \hat{H}^1_k(T, R \otimes M) & \to & \hat{H}^1_k(T, M) & \to & \hat{H}^1_{k-1}(T, M') & \to & \hat{H}^1_{k-1}(T, R \otimes M) \\
& \downarrow & & \downarrow \alpha_k & & \downarrow \beta_{k-1} & & \downarrow \\
\cdots \to & 0 & \to & \hat{H}^\Phi_k(T, M) & \to & \hat{H}^\Phi_{k-1}(T, M') & \to & 0
\end{array}$$

Using the remark at the beginning of the proof we have $H_i(T, R \otimes M') = 0$ for $1 < i < n - 1$. Applying the induction hypothesis we know that β_{n-3} is isomorphic and β_{n-2} is epimorphic. Since

$$\hat{H}^1_{n-2}(T, R \otimes M) \cong H_{n-1}(T, R \otimes M) = 0,$$

we have, by the Five Lemma applied to the above diagram, α_{n-2} isomorphic and α_{n-1} epimorphic.

Remark. For $n = 3$, Lemma 3.2 is also true without the assumption that R is K-projective.

Lemma 3.1 and sequence (V) lead to

THEOREM 3.3. *Let $\Phi : T \to R$ be a surjective homomorphism of augmented algebras, with kernel S. Suppose Φ is K-split. Then, for every M in ${}_R\mathbf{M}$, the following sequence is exact:*

$$\text{(VI)} \quad \begin{array}{l} H_2(T, M) \to H_2(R, M) \to S/JT \cdot S \otimes_R M \\ \qquad\qquad\qquad \to H_1(T, M) \to H_1(R, M) \to 0. \end{array}$$

Remark. Denoting by P the pull-back of the two maps

$$H_2(T, M) \to \hat{H}^\Phi_1(T, M) \quad \text{and} \quad \mathrm{Tor}^R_1 (S/JT \cdot S, M) \to \hat{H}^\Phi_1(T, M),$$

we are able to extend sequence (VI) by two further terms:

$$\begin{array}{ccccccccc}
& & P & \longrightarrow & & H_2(T, M) & & & \\
& \nearrow & \downarrow & & & \downarrow & \searrow & & \\
H_3(R, M) & \to & \mathrm{Tor}^R_1 (S/JT \cdot S, M) & \to & & \hat{H}^\Phi_1(T, M) & \to & H_2(R, M) & \to \cdots
\end{array}$$

It is easy to see that $H_3(R, M) \to P \to H_2(T, M) \to H_2(R, M) \to \cdots$ is exact.

Lemma 3.2 and sequence (V) lead to

THEOREM 3.4. *Let $\Phi : T \to R$ be a surjective homomorphism of augmented algebras, with kernel S. Suppose R is K-projective, and $H_i(T, R \otimes M) = 0$ for $1 < i < n$. Then the following sequence is exact:*

$$\text{(VII)} \quad \begin{array}{l} H_n(T, M) \to H_n(R, M) \to \mathrm{Tor}^R_{n-2}(S/JT \cdot S, M) \to H_{n-1}(T, M) \\ \qquad\qquad \to \cdots \to S/JT \cdot S \otimes_R M \to H_1(T, M) \to H_1(R, M) \to 0. \end{array}$$

4. The case of groups, Lie algebras, restricted Lie algebras

(a) *Groups.* Let $\varphi : G \to Q$ be a group homomorphism. φ induces a map of the group rings over the integers $\Phi : ZG \to ZQ$, which is a map of augmented algebras over Z. If $\varphi : G \to Q$ is surjective, so is $\Phi : ZG \to ZQ$.

PROPOSITION 4.1. *Let $\varphi : G \to Q$ be a surjective group homomorphism with kernel N. Denote by S the kernel of the corresponding map $\Phi : ZG \to ZQ$. Then $S/JZG\cdot S \cong N/[N, N]$, where $[N, N]$ denotes the subgroup of N generated by all elements of the form $nmn^{-1}m^{-1}$ with $n, m \in N$.*

Proof. It is to be shown that the kernel of the map $JZG \otimes_{ZG} ZQ \to JZQ$ is isomorphic to $N/[N, N]$. This is done by considering the exact sequence $0 \to JZG \to ZG \to Z \to 0$ and its tensor product over ZG with ZQ:

$$0 \to \mathrm{Tor}_1^{ZG}(Z, ZQ) \to JZG \otimes_{ZG} ZQ \to ZQ \to Z \to 0.$$

By [2, Proposition 7.4, p. 196] we have

$$\mathrm{Tor}_1^{ZG}(Z, ZQ) \cong \mathrm{Tor}_1^{ZN}(Z, Z) = H_1(N, Z) \cong N/[N, N].$$

The Q-module-structure in $N/[N, N]$ is induced by the conjugation G.

Let $\hat{H}_n^{\varphi}(G, M)$ denote the group $\hat{H}_n^{\Phi}(ZG, M)$ and $H_n(G, M)$ the n^{th} ordinary (Eilenberg-MacLane) homology group. Proposition 2.1 says that $\hat{H}_n^1(Q, M)$ is isomorphic to $H_{n+1}(Q, M)$ for $n \geq 1$. We shall now use the notation $\bar{N}$ for $N/[N, N]$, $-\otimes_Q -$ for $-\otimes_{ZQ} -$, and $\mathrm{Tor}_n^Q(-, -)$ for $\mathrm{Tor}_n^{ZQ}(-, -)$. Recall that $\mathrm{Tor}^{ZQ}(B, -)$ denotes the n^{th} (ZQ, Z)-*relative* derived functor of $B \otimes_Q -$ for every right Q-module B. Then Theorem 2.2, with the remark after Proposition 2.1, leads to

THEOREM 4.2. *Let $\varphi : G \to Q$ be a surjective group homomorphism with kernel N. Then for every Q-module M we have the following exact sequence:*

$$\text{(V')} \quad \begin{aligned} \cdots &\to H_{n+2}(Q, M) \to \mathrm{Tor}_n^Q(\bar{N}, M) \to \hat{H}_n^{\varphi}(G, M) \to H_{n+1}(Q, M) \\ &\to \cdots \to \bar{N} \otimes_Q M \to H_1(G, M) \to H_1(Q, M) \to 0. \end{aligned}$$

THEOREM 4.3. *Let $\varphi : G \to Q$ be a surjective group homomorphism with kernel N. Then for every Q-module the following sequence is exact:*

$$\text{(VI')} \quad H_2(G, M) \to H_2(Q, M) \to \bar{N} \otimes_Q M \to H_1(G, M) \to H_1(Q, M) \to 0.$$

If further $H_i(N, M) = 0$ for $1 < i < n$, then (VI′) *may be extended to*

$$\text{(VII')} \quad \begin{aligned} &H_n(G, M) \to H_n(Q, M) \to \mathrm{Tor}_{n-2}^Q(\bar{N}, M) \to H_{n-1}(G, M) \\ &\to \cdots \to \bar{N} \otimes_Q M \to H_1(G, M) \to H_1(Q, M) \to 0. \end{aligned}$$

Proof. We merely have to show that $H_i(N, M) \cong H_i(G, ZQ \otimes M)$. Since N acts trivially in M, this is done by [2, Proposition 7.4, p. 196].

(b) *Lie algebras.* Let $\varphi : G \to Q$ be a homomorphism of Lie algebras over K. φ induces a map of the enveloping algebras $\Phi : UG \to UQ$, which is a morphism of augmented algebras over K. If $\varphi : G \to Q$ is surjective, so is $\Phi : UG \to UQ$.

PROPOSITION 4.4. *Let $\varphi : G \to Q$ be a surjective homomorphism of Lie algebras over K, with kernel N. Suppose N and Q are K-free. Denote by S the kernel of the corresponding map $\Phi : UG \to UQ$. Then $S/JUG \cdot S \cong N/[N, N]$, where $[N, N]$ denotes the Lie ideal generated by all Lie products $[n, m]$ with $n, m \in N$.*

Since the Lie algebras are K-free, the *proof* is similar to that of Proposition 4.1. The Q-module structure in $N/[N, N]$ is easily seen to be induced by the Lie product in G.

Let $\hat{H}^{\varphi}_n(G, M)$ denote the group $\hat{H}^{\Phi}_n(UG, M)$ and $H_n(G, M)$ the n^{th} ordinary homology group. Proposition 2.1 says that $\hat{H}^1_n(Q, M)$ is isomorphic to $H_{n+1}(Q, M)$ for $n \geq 1$. We shall use the notation $\bar{N}$ for $N/[N, N]$, $- \otimes_Q -$ for $- \otimes_{UQ} -$, and $\mathrm{Tor}^Q_n(-, -)$ for $\mathrm{Tor}^{UQ}_n(-, -)$. Recall that $\mathrm{Tor}^{UQ}_n(B, -)$ denotes the n^{th} (UQ, K)*-relative* derived functor of $B \otimes_Q -$ for every right Q-module B. Then Theorem 2.2, together with the remark after Proposition 2.1, leads to results analogous to Theorems 4.2 and 4.3, and exact sequences (V′), (VI′), (VII′). The only thing to check is $H_i(N, M) \cong H_i(G, UQ \otimes M)$. Since the Lie algebras are K-free, this is done by [2, Proposition 4.2, p. 275].

(c) *Restricted Lie algebras.* Let $\varphi : G \to Q$ be a homomorphism of restricted Lie algebras over K. Denote the p-map by $m \to m^{[p]}$. The map φ induces a map of the restricted enveloping algebras $\Phi : VG \to VQ$, which is a morphism of augmented algebras over K. If $\varphi : G \to Q$ is surjective, so is $\Phi : VG \to VQ$.

PROPOSITION 4.5. *Let $\varphi : G \to Q$ be a surjective homomorphism of restricted Lie algebras over K, with kernel N. Suppose N and Q are K-free. Denote by S the kernel of the corresponding map $\Phi : VG \to VQ$. Then $S/JVG \cdot S \cong N/[N, N]'$, where $[N, N]'$ denotes the ideal generated by all $[n, n'] + m^{[p]}$ with $n, n', m \in N$.*

Proof. First we must show that in the given circumstances VG is VN-free. This is well known, and proved analogously to the corresponding statement for the enveloping algebras. The rest is again done by [2, p. 196], similarly to the proof of Proposition 4.1; it is well known that $H_1(N, K) \cong N/[N, N]'$. The Q-module structure in $N/[N, N]'$ is easily seen to be induced by the Lie product in G.

Let $\hat{H}^{\varphi}_n(G, M)$ denote the group $\hat{H}^{\Phi}_n(VG, M)$, and $H_n(G, M)$ the n^{th} ordinary homology group (see [4]). Proposition 2.1 says that $\hat{H}^1_n(Q, M)$ is isomorphic to $H_{n+1}(Q, M)$ for $n \geq 1$. We shall use the notation $\bar{N}$ for $N/[N, N]'$, $- \otimes_Q -$ for $- \otimes_{VQ} -$, and $\mathrm{Tor}^Q_n(-, -)$ for $\mathrm{Tor}^{VQ}_n(-, -)$. Also recall that $\mathrm{Tor}^{VQ}_n(B, -)$ denotes for every right Q-module B the n^{th} (VQ, K)*-relative* derived functor of $B \otimes_Q -$. Then Theorem 2.2 leads to results analogous to Theorems 4.2, 4.3 and exact sequences (V′), (VI′), (VII′). Again the only thing to check is $H_i(N, M) \cong H_i(G, VQ \otimes M)$. This is done by [2, Proposition 7.2, p. 196] using the fact that in the given circumstances VG is free as a VN-module.

Remark. In the cases (a), (b), and (c) considered in this section there exists a Hochschild-Serre spectral sequence

$$H_p(Q, H_q(N, M)) \Rightarrow H_n(G, M).$$

It is easily checked that sequence (VI′) is the exact sequence of terms of low degree in the above spectral sequence. Assuming $H_i(N, M) = 0$ for $1 < i < n$ a longer exact sequence may be deduced [5], [10]:

$$\text{(VIII)} \quad \begin{aligned} &H_n(G, M) \to H_n(Q, M) \to H_{n-2}(Q, H_1(N, M)) \to H_{n-1}(G, M) \\ &\qquad \to \cdots \to \bar{N} \otimes_Q M \to H_1(G, M) \to H_1(Q, M) \to 0. \end{aligned}$$

To show that the sequences (VII′) and (VIII) are identical, it remains to prove, that for $0 \leq k < n - 1$,

$$H_k(Q, H_1(N, M)) \cong \mathrm{Tor}_k^Q(\bar{N}, M).$$

We shall prove this in the group case only; the proofs for Lie algebras and restricted Lie algebras are similar. For $n = 2$ we obviously have

$$Z \otimes_Q H_1(N, M) \cong Z \otimes_Q(\bar{N} \otimes M) = \bar{N} \otimes_Q M.$$

Let $n > 2$. Then $H_2(N, M) = 0$. This implies $\mathrm{Tor}_1^K(\bar{N}, M) = 0$ by an argument similar to that used in the proof of Lemma 3.1. Consider the (ZQ, Z)-relative-projective presentation of M

$$0 \to M' \to ZQ \otimes M \to M \to 0.$$

Tensoring with $\bar{N}$ over Z leads to

$$0 \to \bar{N} \otimes M' \to \bar{N} \otimes ZQ \otimes M \to \bar{N} \otimes M \to 0.$$

In each of these modules we consider the Q-action via the diagonal map of ZQ. The sequence remains exact; $\bar{N} \otimes ZQ \otimes M$ with the new Q-module-structure is still relative-projective. We therefore obtain exact sequences

$$\begin{aligned} &0 \to \mathrm{Tor}_{n-2}^Q(Z, \bar{N} \otimes M) \to \mathrm{Tor}_{n-3}^Q(Z, \bar{N} \otimes M') \\ &\qquad\qquad \to \mathrm{Tor}_{n-3}^Q(Z, \bar{N} \otimes ZQ \otimes M) \to \cdots, \end{aligned}$$

$$0 \to \mathrm{Tor}_{n-2}^Q(\bar{N}, M) \to \mathrm{Tor}_{n-3}^Q(\bar{N}, M') \to \mathrm{Tor}_{n-3}^Q(\bar{N}, ZQ \otimes M) \to \cdots.$$

For $n = 3$ the claimed isomorphism follows immediately. For $n > 3$ the right most terms are trivial. We therefore have to prove that

$$\mathrm{Tor}_{n-3}^Q(K, \bar{N} \otimes M') \cong \mathrm{Tor}_{n-3}^Q(\bar{N}, M').$$

This is done by induction, since $0 = H_i(N, M) \cong H_i(G, ZQ \otimes M)$ for $1 < i < n$ implies

$$H_i(G, ZQ \otimes M') \cong H_i(N, M') = 0$$

for $1 < i < n - 1$ by an argument used in the proof of Lemma 3.2.

5. Exact sequences in higher dimensions

Under suitable conditions the Hochchild-Serre spectral sequence also gives rise to 5-term exact sequences in higher dimensions. Methods which are closely related to the techniques used so far in this paper also lead to spectral-sequence-free proofs of these sequences. In addition, an analogous sequence in the case of augmented algebras is proved.

THEOREM 5.1. *Let $\Phi : T \to R$ be a surjective morphism of augmented algebras over K. Suppose R is K-projective, and let M be in ${}_R\mathbf{M}$. If $\mathrm{Tor}_i^T(R, M) = 0$ for $0 < i < n$, then $H_k(T, M) \cong H_k(R, M)$ for $0 \leq k < n$ and the following sequence is exact:*

$$\text{(IX)}\qquad \begin{aligned} H_{n+1}(T, M) \to H_{n+1}(R, M) \to K \otimes_R \mathrm{Tor}_n^T(R, M) \to H_n(T, M) \\ \to H_n(R, M) \to 0. \end{aligned}$$

Proof. Let $\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$ be a (T, K)-relative projective resolution of M. Define Q_i for $i \geq 1$ to be the kernel of $P_i \to P_{i-1}$, and Q_0 to be the kernel of $P_0 \to M$. The fact that $\mathrm{Tor}_i^T(R, M) = 0$ for $0 < i < n$ gives rise to exact sequences

$$0 \to R \otimes_T Q_0 \to R \otimes_T P_0 \to R \otimes_R M \to 0,$$

$$0 \to R \otimes_T Q_i \to R \otimes_T P_i \to R \otimes_T Q_{i-1} \to 0 \qquad \text{for } 0 < i < n - 1,$$

$$0 \to \mathrm{Tor}_n^T(R, M) \to R \otimes_T Q_{n-1} \xrightarrow{\gamma} R \otimes_T P_{n-1} \to R \otimes_T Q_{n-2} \to 0.$$

Trivially $H_0(T, M) \cong K \otimes_T M \cong K \otimes_R M = H_0(R, M)$. Let $0 < k < n$, then

$$H_k(T, M) = \mathrm{Tor}_k^T(K, M) \cong \cdots \cong \mathrm{Tor}_1^T(K, Q_{k-2}).[2]$$

Since R is K-projective the absolute $\mathrm{Tor}_*^R(K, -)$ is equal to the (R, K)-relative $\mathrm{Tor}_*^R(K, -)$. Since $R \otimes_T P_j$ are (R, K)-relative projective, we obtain

$$\mathrm{Tor}_1^T(K, Q_{k-2}) \cong \mathrm{Tor}_1^R(K, R \otimes_T Q_{k-2}) \cong \cdots \cong \mathrm{Tor}_n^R(K, M) = H_k(R, M).$$

Denote the image of γ by B, and apply the functor $K \otimes_R -$ to the resulting short exact sequence

$$0 \to \mathrm{Tor}_n^T(R, M) \to R \otimes_T Q_{n-1} \to B \to 0.$$

We obtain

$$\begin{aligned} \cdots \to \mathrm{Tor}_1^R(K, R \otimes_T Q_{n-1}) \to \mathrm{Tor}_1^R(K, B) \to K \otimes_R \mathrm{Tor}_n^T(R, M) \\ \to K \otimes_T Q_{n-1} \to K \otimes_R B \to 0. \end{aligned}$$

Because $R \otimes_T P_k$ is (R, K)-relative projective, we have

$$\begin{aligned} \mathrm{Tor}_1^R(K, B) &\cong \mathrm{Tor}_2^R(K, R \otimes_T Q_{n-2}) \\ &\cong \cdots \cong \mathrm{Tor}_{n+1}^R(K, R \otimes_R M) = H_{n+1}(R, M). \end{aligned}$$

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[2] Set $Q_{-1} = M$.

Moreover the kernels of the maps

$$K \otimes_T Q_{n-1} \to K \otimes_T P_{n-1} \quad \text{and} \quad K \otimes_R B \to K \otimes_T P_{n-1}$$

may be identified with $H_n(T, M)$ and $H_n(R, M)$ respectively. Finally, it is easy to prove (using the method of Lemma 3.1) that $H_{n+1}(T, M) \cong \mathrm{Tor}_1^T(K, Q_{n-1})$ is mapped epimorphically onto $\mathrm{Tor}_1^R(K, R \otimes_T Q_{n-1})$. This completes the proof of Theorem 5.1.

In the cases considered in Section 4, i.e. groups, K-free Lie algebras, and K-free restricted Lie algebras, sequence (IX) again may be obtained using the corresponding Hochschild-Serre spectral sequence.

THEOREM 5.2. *If $H_i(N, M) = 0$ for $0 < i < n$, then $H_k(G, M) \cong N_k(Q, M)$ for $0 \leq k < n$ and the following sequence is exact:*

$$\text{(IX}') \quad \begin{aligned} &H_{n+1}(G, M) \to H_{n+1}(Q, M) \to K \otimes_Q H_n(N, M) \to H_n(G, M) \\ &\qquad \to H_n(Q, M) \to 0. \end{aligned}$$

Proof. The only thing to check is that

$$H_k(N, M) = \mathrm{Tor}_k^N(K, M) \cong \mathrm{Tor}_k^G(R, M),$$

where R denotes ZQ, UQ, VQ respectively, and K denotes the ground ring Z, K, K respectively. For proofs, see [2, Proposition 7.5, p. 196] and [2, Corollary 4.4, p. 275].

6. The case of associative algebras

In this section we denote by $\Phi : T \to R$ a morphism of unitary associative K-algebras. T^* and R^* are the opposite algebras. Define JT to be the kernel of the map $\varepsilon : T \otimes T^* \to T$ given by $\varepsilon(s \otimes t^*) = s \cdot t$ for $s, t \in T$. Let M be an R-bimodule. We define a Φ-homology by

$$\hat{H}_n^\Phi(T, M) = \mathrm{Tor}_n^R(R \otimes_T JT \otimes_T R, M),$$

where $\mathrm{Tor}_n^R(B, -)$ denotes, for every R-bimodule B, the n^{th} left-$(R \otimes R^*, K)$-relative derived functor of $B \otimes_{R \otimes R^*} -$. Again $\hat{H}_*^1(R, M)$ for $1 : R \to R$ is essentially the ordinary (i.e. relative Hochschild) homology $H_*(R, M)$. From now on let $\Phi : T \to R$ denote a surjective morphism with kernel S. By S^2 we understand the ideal generated by $s \cdot t$ with $s, t \in S$. S/S^2 carries a natural R-bimodule-structure induced by the multiplication in T. It was shown in [1] that the sequence

$$0 \to S/S^2 \to R \otimes_T JT \otimes_T R \to JR \to 0$$

is exact and K-split. This leads immediately to

THEOREM 6.1. *Let $\Phi : T \to R$ be a surjective morphism of unitary K-algebras, with kernel S. Let M be an R-bimodule, and suppose Φ is K-split. Then there is an exact sequence:*

$$\text{(V}'') \quad \begin{aligned} &\cdots \to \hat{H}_{n+1}^1(R, M) \to \mathrm{Tor}_n^R(S/S^2, M) \to \hat{H}_n^\Phi(T, M) \to \hat{H}_n^1(R, M) \\ &\qquad \to \cdots \to S/S^2 \otimes_{R \otimes R^*} M \to JT \otimes_{T \otimes T^*} M \to JR \otimes_{R \otimes R^*} M \to 0. \end{aligned}$$

It is easy to see that the statements corresponding to Lemmas 3.1 and 3.2 are also valid in the case of associative algebras. Therefore

THEOREM 6.2. *Let $\Phi : T \to R$ be as in Theorem 6.1. Then the following sequence is exact:*

$$\text{(VI}''\text{)} \quad H_2(T, M) \to H_2(R, M) \to S/S^2 \otimes_{R \otimes R^*} M \to H_1(T, M) \to H_1(R, M) \to 0.$$

If further R is K-projective and if $H_i(T, R \otimes R^* \otimes M) = 0$ for $1 < i < n$, then (VI$''$) may be extended to

$$\text{(VII}''\text{)} \quad H_n(T, M) \to H_n(R, M) \to \operatorname{Tor}^R_{n-2}(S/S^2, M) \to H_{n-1}(T, M) \to \cdots \to S/S^2 \otimes_{R \otimes R^*} M \to H_1(T, M) \to H_1(R, M) \to 0.$$

Finally the procedure used in Section 5 also carries over to associative algebras. We obtain

THEOREM 6.3. *Suppose R is K projective. If $\operatorname{Tor}^T_i(R \otimes R^*, M) = 0$ for $0 < i < n$, then $H_k(T, M) \cong H_k(R, M)$ for $0 \leq k < n$ and the following sequence is exact:*

$$\text{(IX}''\text{)} \quad H_{n+1}(T, M) \to H_{n+1}(R, M) \to R \otimes_{R \otimes R^*} \operatorname{Tor}^T_n(R \otimes R^*, M) \to H_n(T, M) \to H_n(R, M) \to 0.$$

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