LOCAL RINGS WITH THE OUTER PRODUCT PROPERTY

BY
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A commutative ring R is said to have the outer product property if, given any ordered n-tuple of elements in R, there exists an $(n - 1) \times n$ matrix over R whose $(n - 1) \times (n - 1)$ subdeterminants are exactly the given n elements. The object of this paper is to prove that a local ring (R, M) has the outer product property if and only if the V-dimension of R, i.e. the minimal number of generators of M, is ≤ 2.

In [2] and [3], D. Lissner defined the outer product property, observed that a classical result of Hermite asserts the ring Z has this property, and proved that a polynomial ring $A[x]$ (with A a principal ideal domain) has the outer product property (thus generalizing a well-known result of Seshadri [4]) and also that Dedekind domains have this property. The author, in [5], proved that $D[x]$ (with D a Dedekind domain) and regular local rings of dimension ≤ 2 have a rather stronger property, which implies the outer product property; the same paper gives a necessary condition for a ring to have the outer product property (see Prop. 1.4 in the present paper), and the result we are about to prove asserts that for local rings this condition is also sufficient.

1. Definitions and preliminaries

Let R be a commutative ring with identity. With R^n denoting the R-module of ordered n-tuples of elements in R, we say two elements α, β in R^n are *R-equivalent* if there exists an invertible R-homomorphism $R^n \to R^n$ which maps α into β. We say

$$\alpha = \| a_1, \cdots, a_n \|$$

is the *outer product* of the $(n - 1) \times n$ matrix B over R (or of the rows of this matrix) if $(-1)^{i-1}\alpha_i$ equals the determinant of the matrix obtained from B by deleting its i-th column. If such a matrix B exists, we say *α is an outer product over R*. R has the *outer product property* if, for all $n \geq 2$, every element of R^n is an outer product over R. If $\alpha = \| a_1, \cdots, a_n \|$, we denote the ideal $\sum Ra_i$ by $I(\alpha)$.

PROPOSITION 1.1. *If α is an outer product over R, so is $r\alpha$ $(r \in R)$.*

PROPOSITION 1.2. *If α, β in R^n are R-equivalent, then α is an outer product over R iff β is.*

Proof. If α is the outer product of the matrix B, and T is an invertible $n \times n$ matrix over R, then αT is the outer product of the matrix $B^t T^{-1}$.

Received February 26, 1968.

PROPOSITION 1.3. *If R has the outer product property, so does any homomorphic image of R.*

PROPOSITION 1.4. *If R has the outer product property, and P is any prime ideal of R, then R_P has V-dimension ≤ 2.*

Proof. This is Theorem 2 in [5].

2. The outer product property

PROPOSITION 2.1. *Let R be a local ring, α and β in R^n; then α and β are R-equivalent if and only if $I(\alpha) = I(\beta)$.*

Proof. "Only if" is clear; "if" is a consequence of Nakayama's Lemma.

PROPOSITION 2.2. *Let (R, M) be a local ring, $\{\alpha_i\}$ a sequence of elements in R^n converging in the M-adic topology to the element α in R^n, and suppose $I(\alpha)$ is M-primary or R. Then, for all i large enough, α_i and α are R-equivalent.*

Proof. $I(\alpha) \supseteq M^s$ for some s; for i large enough,

$$\alpha_i - \alpha \in M^{s+1}R^n \subseteq MI(\alpha)R^n.$$

By Nakayama's lemma, $I(\alpha_i) = I(\alpha)$; hence α_i and α are R-equivalent.

THEOREM. *A necessary and sufficient condition for a local ring (R, M) to have the outer product property is that it have V-dimension ≤ 2.*

Proof. Necessity follows from Prop. 1.4; we now prove sufficiency. Let (R, M) be a local ring of V-dimension ≤ 2.

By [5, Theorem 1.1], regular local rings of dimension ≤ 2 have the outer product property; hence, if R is complete, Prop. 1.3 and the following lemma imply that R has the outer product property.

LEMMA 2.3. *If (R, M) is a complete local ring, then R is the homomorphic image of a regular local ring of dimension equal to the V-dimension of R.*

Proof. It follows from the Cohen structure theorem [1, Theorem 9 and Theorem 12] that R is the homomorphic image of a ring S which is the ring of formal power series in n indeterminates, either over R/M in equal characteristic case, or over a complete local domain in which the maximal ideal is principal, and generated by a prime integer p, if R/M has unequal characteristic p. Thus, S is a regular local ring of dimension n or $n + 1$; suppose $R = S/I$. If S has dimension $n + 1$, then since

$$V\text{-dim } R = V\text{-dim } S - 1,$$

I contains an element a in $N - N^2$, and R is a homomorphic image of the regular local ring S/aS of dimension n.

We have established the truth of our theorem when R is regular or complete. R has dimension ≤ 2; if $\dim R = 2$, or $\dim R = V\text{-dim } R = 1$, then

R is regular, while if $\dim R = 0$, R is complete. Only one more case need be considered; we now assume

$$\dim R = 1, \qquad V\text{-}\dim R = 2.$$

We must show every α in R^n is an outer product; we consider two cases.

Case 1. *There is no minimal prime containing all coordinates of* α. Note that here $I(\alpha)$ is M-primary. Since the completion of R has the outer product property, there exists a sequence $\{B_i\}$ of $(n - 1) \times n$ matrices over R, converging element-wise to a matrix B over R^* whose outer product is α. Denoting by α_i the outer product of B_i, we see that α_i converges to α, and each α_i is an outer product over R. By Prop. 2.2, α_i is R-equivalent to α for i large enough, and so α is an outer product.

Case 2. *Some minimal prime P contains all coordinates of* α. We begin by showing every minimal prime P is principal. It suffices to show $PR^* = P^*$ is principal in the completion R^* of R. M^* is not one of the prime divisors of P^*, because

$$P^*:M^* = (P:M)^* = P^*$$

and since $\dim R^* = 1$, P^* is of unmixed dimension 1. By Lemma 2.3, there is a regular local ring (S, N) of dimension 2 and a homomorphism Φ from S onto R^*, since $V\text{-}\dim R^* = 2$. Since $\Phi^{-1}(P^*)$ has unmixed dimension 1, and hence unmixed rank 1, in S, and S is a unique factorization domain, it follows that $\phi^{-1}(P^*)$, hence P^*, and hence P, are principal. Let $P_i = p_i R$, $\alpha = \| a_1, \cdots, a_n \|$, all a_i in P_1. We must show α is an outer product. This is clear if $\alpha = 0$; if $\alpha \neq 0$, let $p_1^{n(1)}$ be the highest power of p_1 dividing all non-zero entries in α, and let

$$a_i = p_1^n b_i, \qquad \alpha_1 = \| b_1, \cdots, b_n \|,$$

so

$$\alpha = p_1^{n(1)} \alpha_1 .$$

Not all coordinates of α_1 lie in P_1; if all coordinates lie in, say, P_2, the same reasoning gives $\alpha_1 = p_2^{n(2)} \alpha_2$, $\alpha = p_1^{n(1)} p_2^{n(2)} \alpha_2$, with neither P_1 nor P_2 containing all coordinates of α_2. Continuing in this way, we obtain

$$\alpha = p_1^{n(1)} \cdots p_s^{n(s)} \alpha_s \qquad (1 \leq s \leq r)$$

with no minimal prime containing all coordinates of α_s. By Case 1, α_s is an outer product. Hence, α is an outer product by Prop. 1.1, and our proof is complete.

Remark. This proof also shows that if R is any commutative ring with identity, the condition "for all prime ideals P in R, $V\text{-}\dim R_p \leq 2$" of Prop. 1.4 is satisfied if it holds for all maximal ideals P. Namely, if M is maximal, M contains P properly, and $V\text{-}\dim R_M \leq 2$; then either R_M is a regular local ring of dimension 1, whence

$$PR_M = PR_p = 0$$

or R_M is a regular local ring of dimension 2, whence PR_M and hence PR_p is principal, or dim $R_M = 1$ and V-dim $R_M = 2$, in which case by the argument in Case 2 of the above proof, PR_M and hence PR_p is principal. Thus, V-dim $R_p = 0$ or 1.

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