CHARACTERISTIC CLASSES

BY

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Conner and Floyd have developed a theory of characteristic classes in generalized cohomology [17]. The purpose of this paper is to present an abstract development of their theory. The theory holds for real, complex, or quaternionic bundles, and the proofs for the three cases are essentially identical. The results hold for bundles over infinite complexes provided we consider only representable cohomology theories (called r-theories). The theory is based upon one theorem, the Thom-Dold isomorphism for r-theories and infinite complexes. A simple proof of this theorem is included.

To outline the general development, consider only real vector bundles for the moment. If h is an r-theory, then any two of the following are equivalent:

(i) There exists an element $\rho \in \tilde{h}^1(RP^\infty)$ such that $h^{**}(RP^\infty)$ is an $h^{**}(\text{pt})$ power series module over ρ.
(ii) The Hopf bundle ξ over RP^∞ is h-orientable.
(iii) For each finite n, the Hopf bundle ξ_n over RP^n is h-orientable.

If h satisfies (i) above, h is said to be real orientable and ρ is said to be a real orientation for the cohomology theory h. Then the generalized Stiefel-Whitney classes exist, i.e. for each real bundle α over X, $w_i(\alpha) \in h^i(X)$. If $x \in h^i(RP^\infty)$ is zero when restricted to any RP^n, then x is zero, i.e. $h^i(PR^\infty)$ has no phantom classes. Also $h^i(BO_n)$ and $h^i(BO)$ have no phantom classes. The groups $h^{**}(BO_n)$ and $h^{**}(BO)$ are $h^{**}(\text{pt})$ power series modules over the Stiefel-Whitney classes. Every real bundle is h-orientable. Any other orientation $\bar{\rho}$ gives another set of SW classes $\bar{w}_i$, however w_i and $\bar{w}_i$ will agree when restricted to the i-skeleton X^i. The set of all orientations corresponds to the set of all series of the form $\pm\rho + a_2\rho^2 + a_3\rho^3 + \cdots$ where $a_i \in h^{-i}(\text{pt})$. Let $KO(X) = [X, BO \times Z]$ be defined for infinite complexes. Then the domain of the characteristic classes can be extended so that $w_i : KO(X) \to h^i(X)$. Since real vector bundles are orientable for the ordinary theory $H(-; Z_2)$, it follows that $H^*(PR^\infty, Z_2)$ is a Z_2 polynomial algebra, that the classical SW classes exist, and that $H^*(BO_n; Z_2)$ and $H^*(BO; Z_2)$ are polynomial algebras over the SW classes. (Here it is unnecessary to pass to the direct product H^{**}.)

Now suppose h is an r-theory such that, each complex Hopf bundle ξ_n over CP^n is h-orientable. Then any orientation $\rho \in \tilde{h}^2(CP^\infty)$ determines Chern classes $c_i : K(X) \to h^{2i}(X)$, and the theory is analogous to the real case. The

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spectrum MU determines a cohomology theory (called an s-theory) for which complex bundles are orientable. The spectrum maps

$$\begin{array}{ccccccccccc} MU & \subset & MSO & \subset & MSPL & \subset & MSTOP & \subset & MSH & \subset & K(Z) \\ & & \cap & & \cap & & \cap & & \cap & & \cap \\ & & MO & \subset & MPL & \subset & MTOP & \subset & MH & \subset & K(Z_2) \end{array}$$

determine natural transformations of cohomology theories which send orientations into orientations, and thus send Chern classes in one theory into Chern classes in another theory. For complex bundles over *finite* complexes, the theory of Chern classes holds for any of the above theories, and in fact gives a graphic way to compare $h^*(CP^n; MU)$ and $h^*(CP^n; MSO)$, for example. However, for complex bundles over infinite complexes, the theory does not hold for $h(-, MU)$ because MU is not an Ω-spectrum. The theory does hold for the associated Ω-spectra $\overline{MU} \subset \overline{MSO} \subset \cdots$.

The theories $h(-; \bar{S}p)$, $h(-; \overline{MU})$, and $h(-; \overline{MO})$ are universal in the classes of orientable theories. That is, if $h(-)$ has a complex orientation ρ, then there exists a natural transformation $T : h(-; \overline{MU}) \to h(-)$ which sends the canonical orientation for $h(-; \overline{MU})$ to ρ. The Hopf bundle ξ over CP^∞ is orientable for the cohomology theory K, and thus there is a natural transformation

$$T : h^2(-; \overline{MU}) \to K^2(-) = K(-).$$

Also there is an additive natural transformation given by the first Chern class

$$c_1 : K(-) \to h^2(-; \overline{MU}).$$

For reduced theories and connected X, the composition

$$Tc_1 : \tilde{K}(X) \to \tilde{h}^2(X; \overline{MU}) \to \tilde{K}(X)$$

is the identity. Thus K theory is a direct summand of cobordism.

Any choice of complex orientation for K theory yields Chern classes

$$c_i : K(X) \to K^{2i}(X) = K(X).$$

If we select the usual K-orientation $\xi - 1$, these classes may be expressed in terms of the exterior powers and are, of course, already known (see e.g. [27]). A corollary to these proceedings is that $K(CP^\infty = BU_1)$, $K(BU_n)$, and $K(BU)$ have no phantom classes and that they are merely power series in the Chern classes.

It is with sincere pleasure and gratitude that the author makes the following acknowledgements. The basic approach of this paper is due to Conner-Floyd [17]. The proof of the Thom-Dold isomorphism for infinite complexes was shown to the author by Ed Brown. The idea for handling the phantom classes is due to Milnor [28]. Some of the results, such as the computation of the cohomology of classifying spaces, are contained in Stong's notes [33]. The computation of $K(BU_n)$ was done originally by Atiyah-Hirzebruch [7]. Some of this paper is a presentation of conversations with Don Anderson, Dennis

Sullivan, and Brian Sanderson. A. Dold also has results on characteristic classes in generalized cohomology theories. The classical techniques used may be found in Milnor's notes [29], Hirzebruch's book [26], and Husemoller's book [27]. No claim for originality is made. The purpose of this paper is to present a unified exposition of the general theory of characteristic classes.

1. Notation

All spaces will be CW complexes. Spaces such as $X \wedge Y$ are assumed to have the weak topology. Unless otherwise specified, all cohomology theories will be multiplicative theories with unit element defined over CW complexes and not merely finite complexes. Suppose E is a ring spectrum and $S \to E$ is a map of the sphere spectrum into E (see [35, p. 270]). Suppose $1 \in \tilde{h}^0(S^0; S) = h^0(\text{pt})$ and $\mathbf{1} \in \tilde{h}^0(S^0; E) = h^0(\text{pt}; E)$ are the unit elements and the natural transformation $h^0(\text{pt}; S) \to h^0(\text{pt}; E)$ sends 1 to $\mathbf{1}$. Then $h(-; E)$ is called a multiplicative spectrum cohomology theory with unit element, or briefly, an s-theory. An s-theory satisfies the Eilenberg Steenrod axioms (except the dimension axiom) for infinite complexes, although this is proved only for finite complexes in the classical reference [35]. If F is another spectrum and $T : E \to F$ is a spectrum map with

$$\begin{array}{ccc} T_n : E_n & \longrightarrow & F_n \\ & \nwarrow \quad \nearrow & \\ & S^n & \end{array}$$

commutative, then T induces a natural transformation of s-theories, $T : h(-; E) \to h(-; F)$. In particular, $T(\mathbf{1}) = \mathbf{1}$.

If E is an Ω-spectrum, i.e., if $E_n \to \Omega E_{n+1}$ is a homotopy equivalence, then $h(-; E)$ is said to be a representable multiplicative cohomology theory with unit element, or briefly, an r-theory. Thus $h^i(X; E) = \{X, E_i\}$, free homotopy classes of maps. If X has a base point, then $\tilde{h}^i(X; E) = [X, E_i]$, based homotopy classes of maps. Every r-theory is an s-theory. An r-theory h has the following properties.

(i) If $X = \bigcup_{a \in \alpha} X_a$ where the X_a are pairwise disjoint open sets, then $h^k(X) \approx \prod_{a \in \alpha} h^k(X_a)$.

(ii) If $X = \bigcup_{n \geq 0} X_n$, $X_n \subset X_{n+1}$, $a_n \in h^k(X_n)$, and $i_n^*(a_{n+1}) = a_n$ where $i_n : X_n \to X_{n+1}$ is the inclusion, then $\exists$ a class $a \in h^k(X) \ni j_n^*(a) = a_n$ where $j_n : X_n \to X$ is the inclusion.

(Each of (i) and (ii) holds for s-theories provided X is a finite complex.)

Suppose E is a spectrum. Then the associated Ω-spectrum $\bar{E}$ is defined by $\bar{E}_n = \lim_{k \to \infty} \Omega^k E_{n+k}$. The induced natural transformation $h(-; E) \to h(-; \bar{E})$ sends the s-theory to an r-theory and is a natural equivalence when restricted to finite complexes.

If h is an s-theory, then h^n will denote $h^n(\text{pt}) = \tilde{h}^n(S^0)$ and h^* will denote the coefficient ring $\oplus h^n$ where the sum is over all integers. If X has a base point, then $\tilde{h}^n(X) = h^n(X, x_0)$ and h^n will be considered as subgroups of $h^n(X)$. Under this convention we have the "equality" $h^n(X) = \tilde{h}^n(X) \oplus h^n$. If X is connected and without base point, then $\tilde{h}^n(X)$ and h^n are still well defined subgroups of $h^n(X)$ because $x_0 \to X$ is unique up to homotopy.

Let $\mathbf{1}^k \in \tilde{h}^k(S^k)$ correspond to $\mathbf{1}^0 = \mathbf{1} \in \tilde{h}^0(S^0)$ under the suspension isomorphism. Then $\tilde{h}^n(X) \approx \tilde{h}^{n+k}(X \wedge S^k)$ corresponds to multiplication by $\mathbf{1}^k$, and so $\mathbf{1}^k \wedge \mathbf{1}^l = \mathbf{1}^{k+l}$. If h is an s-theory, α^k is a real vector bundle over a connected X, and $T(\alpha)$ is the Thom space of α, then an h-orientation is a class $U \in \tilde{h}^k(T(\alpha))$ which, when restricted to a "fibre" S^k gives $\pm\mathbf{1}^k$. Some classical s-theories come from the spectra

$$\begin{array}{cccccccccccccc} MSp & \subset & MSU & \subset & MU & \subset & MSO & \subset & MSPL & \subset & MSTOP & \subset & MSH & \subset & K(Z) \\ & & & & & & \cap & & \cap & & \cap & & \cap & & \cap \\ & & & & & & MO & \subset & MPL & \subset & MTOP & \subset & MH & \subset & K(Z_2) \end{array}$$

and their associated Ω-spectra. Any simplectic bundle α^n has a *canonical* orientation $U \in \tilde{h}^{4n}(T(\alpha); MSp)$. Let $E(\alpha) \to E(\gamma_n)$ be the map of total spaces covering the classifying map $X \to BSp_n$. The class U is determined by the induced map $T(\alpha) \to MSp_n$. The natural transformations

$$h(-; MSp) \to h(-; MSU) \to \cdots \to K(Z_2)$$

and

$$h(-; MSp) \to h(-; \overline{MSp}) \to h(-; \overline{MSU}) \to \cdots \to K(Z_2)$$

send orientation classes into orientation classes. Similarly, complex bundles and real bundles have *canonical* orientations in the theories determined by MU and MO respectively. Since the spectrum map $MU \to K(Z)$ sends an orientation class to an orientation class, this gives a (strange) way to prove that complex bundles have an orientation class in ordinary cohomology with Z coefficients.

This paper is developed along three lines—real, complex, and quaterionic. WP^n will denote RP^n, CP^n, or HP^n; the real, complex, or quaterionic projective spaces. Thus $WP^0 =$ pt, and $WP^1 = S^1, S^2$, or S^4. Let ξ_n denote (ambiguously) the real, complex, or quaterionic Hopf bundle over WP^n with group W_1. In this last instance, W_1 denotes O_1, U_1, or Sp_1. Let ξ denote (ambiguously) the Hopf bundle over WP^∞. Now fix once and for all copies of WP^n with $WP^0 \subset WP^1 \subset WP^2 \subset \cdots = WP^\infty$. Also identify the total space $E(\xi_n)$ with (WP^{n+1} − pt.) and $T(\xi_n)$ with WP^{n+1}. Under this convention, $\xi_n \mid WP^{n-1}$ is not equal to ξ_{n-1}, although it is equivalent to ξ_{n-1}.

2. The Thom-Dold isomorphism

THEOREM 2.1. *Suppose h is an r-cohomology theory, E, B, and F are CW complexes, and $\pi : E \to B$ is a continuous function mapping E onto B. Suppose $A \subset B$, $E' \subset E$, $F' \subset F$, and $\pi^{-1}(A) \subset E$ are subcomplexes, and the following conditions hold.*

(1) *If* $K \subset L \subset B$ *are subcomplexes with* K *a deformation retract of* L, *then the pair* $(\pi^{-1}(K), \pi^{-1}(K) \cap E')$ *is a deformation retract of* $(\pi^{-1}(L), \pi^{-1}(L) \cap E')$.

(2) *For each* $b \in B$, $\exists$ *a homotopy equivalence of pairs*

$$j_b : (F, F') \to (\pi^{-1}(b), \pi^{-1}(b) \cap E').$$

(3) $\exists$ *classes* $c_i \in h^{n(i)}(E, E')$ *for* $i = 1, 2, \cdots k$ *such that for each* $b \in B$,

$$j_b^*(c_1), j_b^*(c_2), \cdots, j_b^*(c_k)$$

is a free h^**-basis for* $h^*(F, F')$.

Then

$$f : h^*(B, A) \oplus h^*(B, A) \oplus \cdots \oplus h^*(B, A) \to h^*(E, E' \cup \pi^{-1}(A))$$

defined by

$$f(x_1, x_2, \cdots, x_k) = \pi^*(x_1)c_1 + \cdots + \pi^*(x_k)c_k$$

is an (additive) isomorphism.

(*Note* 1. The map f is an h^*-module homomorphism but it is not a ring homomorphism. If $A = \emptyset$, then $c_1, c_2, \cdots, c_k$ is a free $h^*(B)$-basis for $h^*(E, E')$.

Note 2. If B is a finite complex, then the hypothesis may be weakened by allowing $c_i \in h^*(E, E')$, and the proof below holds. For B infinite, the author does not know whether or not the theorem is true, even if B is connected.

Note 3. If B is finite, then the theorem holds for any s-theory h.)

Sketch of proof. Consider first the case $A = \emptyset$. The conclusion may be restated as follows: Any $e \in h^n(E, E')$ may be written uniquely as

$$e = \pi^*(x_1)c_1 + \cdots + \pi^*(x_k)c_k \quad \text{where } x_i \in h^{n-n(i)}(B).$$

The proof follows from the following observations.

(I) Suppose B is the union of disjoint open sets, $B = \bigcup_{a \in \alpha} B_a$. If for each $a \in \alpha$, the conclusion holds for the bundle restricted to B_a and the classes $c_1, c_2, \cdots, c_k$ restricted to this bundle, then the conclusion holds for the bundle over B. This follows from $h^n(B) \approx \prod h^n(B_a)$.

(II) If $B = B_1 \cup B_2$ and the conclusion holds for the bundle restricted to B_1, B_2, and $B_1 \cap B_2$, then the conclusion holds for the bundle over B. This follows from two Mayer-Vietoris sequences and the five lemma.

(III) The theorem holds for $B = D^t$ and $B = S^t$. In the case of $B = D^t$, it follows from the hypothesis that (E, E') has the homotopy type of (F, F'). The case of $B = S^t$ follows from induction on t, part (II), and the fact that $S^t = D_1^t \cup D_2^t$ where $D_1^t \cap D_2^t = S^{t-1}$.

(IV) The conclusion holds for the bundle restricted to any skeleton B^t. It holds for B^0 by (I). Suppose inductively that it holds for $B_1 = B^{t-1}$. Let B_2 be the disjoint union of t-cells, $B_2 = \bigcup_{a \in \alpha} D_a^t$. Then $B^t = B_1 \cup B_2$ and $B_1 \cap B_2 = \bigcup_{a \in \alpha} S_a^{t-1}$. (Actually B_1 is the t-skeleton expanded slightly and B_2

is the t-cells contracted slightly.) The conclusion now follows from (I), (II), (III).

(V) The conclusion holds (when $A = \emptyset$). Let

$$\bar{B} = B^0 \times [0, 1] \cup B^1 \times [1, 2] \cup B^2 \times [2, 3] \cup \cdots$$

where $B^i \times [i+1] \subset B^i \times [i, i+1]$ is identified with

$$B^{i+1} \times [i+1] \subset B^{i+1} \times [i+1, i+2].$$

Then $\bar{B}$ has the same homotopy type as B. Let

$$B_1 = \bigcup B^i \times [i, i+\tfrac{1}{2}] \quad \text{and} \quad B_2 = \bigcup B^i \times [i+\tfrac{1}{2}, i+1].$$

Then the conclusion holds for B_1, B_2, $B_1 \cap B_2$, and thus the conclusion holds.

(VI) The conclusion holds in general (A void or not). This follows from the exact sequences over the pair (B, A), the fact that the theorem holds over B and over A, and the five lemma.

3. Phantom classes

A class $x \in h^i(X)$ or $x \in h^*(X)$ is called a phantom class if it is non-zero but its restriction to each finite skeleton X^i is zero. The skeleton X^i may be infinite. If $f : Y \to X$ is a homotopy equivalence then x is a phantom class iff $f^*(x)$ is a phantom class. In particular, the notion of phantom class does not depend upon the CW structure of X. Note that if X is pointed, $h^i(X)$ has no phantom classes iff $\tilde{h}^i(X)$ has none.

THEOREM 3.1. *Let h be an r-theory, i an integer, X a CW complex, and $X_1 \subset X_2 \subset \cdots = X$ be a sequence of subcomplexes with $h^{i-1}(X_{n+1}) \to h^{i-1}(X_n)$ onto for all $n > 0$. Then*

(1) *if $x \in h^i(X)$ is zero when restricted to each X_n, then x is zero;*

(2) *if each X_n is finite-dimensional and each finite skeleton X^i is contained in some X_n, then $h^i(X)$ has no phantom classes;*

(3) *if t is a positive integer and $X_n = X^{tn}$, then X has no phantom classes;*

(4) *if*

$$X = A_1 \times A_2 \times \cdots \times A_k,$$

t is a positive integer, and $X_n = A_1^{tn} \times A_2^{tn} \times \cdots \times A_k^{tn}$, then X has no phantom classes.

Proof. Parts (3) and (4) are special cases of (2), and (2) follows immediately from (1). To prove (1) it suffices to consider the reduced case $x \in \tilde{h}^i(X)$. Let E_i be the classifying space for $\tilde{h}^i$ and let $f : X \to E_i$ represent x. Suppose inductively that $f(X_{n-1}) =$ base pt. Since $\tilde{h}^{i-1}(X_n) \to \tilde{h}^{i-1}(X_{n-1})$ is onto, the following is exact.

$$0 \to \tilde{h}^i(X_n/X_{n-1}) \to \tilde{h}^i(X_n) \to \tilde{h}_i(X_{n-1}),$$

or

$$0 \to [(X_n/X_{n-1}), E_i] \to [X_n, E_i] \to [X_{n-1}, E_i],$$
$$[g_n] \longrightarrow [f_n]$$

Let $f_n = f \mid X_n$ and let $g_n : (X_n/X_{n-1}) \to E_i$ be the map such that the composition

$$X_n \to (X_n/X_{n-1}) \xrightarrow{g_n} E_i$$

is f_n. Since $[f_n]$ is the zero class, $[g_n]$ is the zero class. Thus g_n is null homotopic and thus f_n is null homotopic mod X_{n-1}. By the homotopy extension property, f is homotopic mod X_{n-1} to f_1 with $f_1(X_n) =$ base pt. This allows the construction of a null homotopy of f and completes the proof.

4. The cohomology of projective space

THEOREM 4.1. *Suppose h is an s-theory, $n \geq 0$, and*

$$\rho_{n+1} \in \tilde{h}^1(RP^{n+1}), \quad \tilde{h}^2(CP^{n+1}), \quad \text{or } \tilde{h}^4(HP^{n+1}).$$

For any t with $0 < t \leq n + 1$, let $i_t : WP^t \to WP^{n+1}$ be the inclusion and $\rho_t = i_t^*(\rho_{n+1})$. Then (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e), *and if $h^0 \approx Z$ or Z_2,* (e) $\Rightarrow$ (a).

(a) *The class ρ_{n+1} is an h-orientation for ξ_n.*

(b) *The class $\rho_1 \in \tilde{h}^1(RP^1 = S^1)$, $\tilde{h}^2(CP^1 = S^2)$, or $\tilde{h}^4(HP^1 = S^4)$ is an orientation for ξ_0.*

(c) *For any t with $0 < t \leq n + 1$, $\rho_t \in \tilde{h}^1(RP^t)$, $\tilde{h}^2(CP^t)$, or $\tilde{h}^4(HP^t)$ is an h-orientation for ξ_{t-1}.*

(d) *For any t with $0 < t \leq n + 1$, $\tilde{h}^*(WP^t)$ is a free h^*-module with free basis $\rho_t, \rho_t^2, \cdots, \rho_t^t$. The kernel of*

$$j^* : \tilde{h}^*(WP^t) \to \tilde{h}^*(WP^{t-1})$$

is the sub-(h^*-module) generated by ρ_t^t. Also $h^*(WP^t)$ is a free h^*-module with free basis $\mathbf{1}, \rho_t, \rho_t^2, \cdots, \rho_t^t$.

(e) *There exists a unit $u \in h^0$ such that $u\rho_{n+1}$ is an h-orientation of ξ_n.*

Proof. The proof is essentially the same for the three cases $W = R, C$, or H, and therefore is presented only for the case $W = R$. First note that for $n = 0$, (b) and (c) are merely restatements of (a).

Show (a) $\Leftrightarrow$ (b). Assume $n > 0$ (no induction is necessary). There exists a homeomorphism $u : RP^{n+1} \to RP^{n+1} = T(\xi_n)$ which is isotopic to the identity and which maps $S^1 = RP^1$ onto $T(\xi_n \mid \text{pt.})$. This is because $\exists$ an orthogonal homeomorphism $R^{n+2} \to R^{n+2}$ which interchanges the x_2 and x_{n+2} axes. If $j = u \mid S^1 = RP^1$, then by definition, ρ_{n+1} is an h-orientation for ξ_n iff $j^*(\xi_{n+1}) = \pm\mathbf{1}^1$. Since j is homotopic to the inclusion $i_1 : RP^1 \to PR^{n+1}$, this proves (a) $\Leftrightarrow$ (b).

To show (c) $\Rightarrow$ (a), take $t = n + 1$. The proof that (a) $\Rightarrow$ (c) is obvious using (b). Thus (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c).

Show (c) $\Rightarrow$ (d). Suppose $n = 0$ and $\rho_1 \in \tilde{h}^1(RP^1 = S^1)$ is an orientation for ξ_0, i.e. $\rho_1 = \pm\mathbf{1}^1$. Since $\tilde{h}^*(S^0)$ is a free h^*-module with basis $\pm\mathbf{1}$, and the suspension isomorphism $\tilde{h}^*(S^0) \approx \tilde{h}^*(S^1)$ is an h^*-module homomorphism, it follows that $\tilde{h}^1(S^1)$ is a free h^*-module with basis $\rho_1 = \pm\mathbf{1}^1$.

Now suppose $n > 0$, ρ_{n+1} is an orientation for ξ_n, and $\rho_n, \rho_n^2, \cdots, \rho_n^n$ is a free basis of $\tilde{h}^n(RP^n)$. Show $\rho_{n+1}, \rho_{n+1}^2, \cdots, \rho_{n+1}^{n+1}$ is a free basis of $\tilde{h}^{n+1}(RP^{n+1})$. Consider the following commutative diagram where i and j are inclusions, π is the bundle projection, and T is the map induced by the Thom-Dold isomorphism.

$$\begin{array}{ccccc}
 & & h^* \oplus \tilde{h}^*(RP^n) & \xleftarrow{i^*} & \tilde{h}^*(RP^{n+1}) \\
 & & = \big\downarrow \quad \approx \big\downarrow \pi^* & & \approx \big\downarrow g \\
h^*(E(\xi_n)) & = & h^* \oplus \tilde{h}^*(E(\xi_n)) & \xleftarrow{j^*} & h^*[E(\xi_n), S(\xi_n)] \\
 & & T\searrow & & \swarrow \otimes g(\rho_{n+1}) \\
 & & & h^*[E(\xi_n), S(\xi_n)]. &
\end{array}$$

By the inductive hypothesis, $h^*(E(\xi_n))$ has a free basis

$$\mathbf{1}, j^*[g(\rho_{n+1})], j^*[g(\rho_{n+1}^2)], \cdots, j^*[g(\rho_{n+1}^n)].$$

Also $T(\mathbf{1}) = g(\rho_{n+1})$ and by the bottom triangle,

$$T(j^*[g(\rho_{n+1}^t)]) = g(\rho_{n+1}^t) \otimes g(\rho_{n+1}) = g(\rho_{n+1}^{t+1}).$$

Thus $h^*[E(\xi_n), S(\xi_n)]$ has a basis $g(\rho_{n+1}), \cdots, g(\rho_{n+1}^{n+1})$ and the first statement of (d) follows because g is an isomorphism. The statement that kernel

$$j^* : \tilde{h}^*(WP^t) \to \tilde{h}^*(WP^{t-1})$$

is generated by ρ_t^t follows from Theorem 4.2 below. Thus (c) $\Rightarrow$ (d).

Now suppose (d) is true and show (e). The class ρ_1 is a generator for the free h^*-module $\tilde{h}^*(RP^1 = S^1)$. Thus $\exists$ a unique class $u \in h^* \ni u\rho_1 = \mathbf{1}^1 \in \tilde{h}^1(S^1)$, and in fact, $u \in h^0$. Thus $u\rho_1$ is an orientation for ξ_0 and since (b) $\Rightarrow$ (a), $u\rho_{n+1}$ is an orientation for ξ_n. To see that u is a unit of h^0, note that if $v \in h^0 = \tilde{h}^0(S^0)$ is the unique class which suspends to ρ_1, then $uv = \mathbf{1}$. Thus (d) $\Rightarrow$ (e).

If $h^0 = Z$ or Z_2, then $\pm\mathbf{1}$ are the only units of h^0, and thus (e) $\Rightarrow$ (a). This completes Theorem 4.1.

The above theorem yields the classical computations of $H^*(RP^n; Z_2)$, $H^*(CP^n; Z)$, and $H^*(HP^n; Z)$. It also applies to $K^*(CP^n)$, $KO^*(HP^n)$, $h^*(CP^n; MU)$, $h^*(CP^n; MSPL)$, etc.

The following lemma is Lemma (7.1) of [17], except the restriction X be a finite complex is removed. The proof given in [17] also holds for the infinite case.

LEMMA 4.2. *Let h be an s-theory, X be a CW complex, and X^n be its n-skeleton. Let F^r =* Kernel $[i^* : h^*(X) \to h^*(X^{r-1})]$. *Then if $a \in F^r$ and $b \in F^s$,* *we have* $ab \in F^{r+s}$. *Thus for example if* X *is a connected complex of* dim $\leq n$ *and* $a_1, a_2, \cdots, a_{n+1} \in \tilde{h}^*(X)$, *then* $a_1 a_2 \cdots a_{n+1} = 0$. *As another example, if* $x \in \tilde{h}^*(RP^n)$, $\tilde{h}^*(CP^n)$, *or* $\tilde{h}^*(HP^n)$, *then* $x^{n+1} = 0$.

In the real case, Theorem 4.1 has a special implication; namely that the orientability of line bundles implies that the cohomology theory h is strictly commutative.

THEOREM 4.3. *If* h *is an s-theory, any two of the following are equivalent.*
(1) $\mathbf{1} + \mathbf{1} = 0$.
(2) *If* X *is a CW complex and* $a \in h^*(X)$, *then* $a + a = 0$.
(3) *The cohomology theory* h *is strictly commutative, i.e. if* X *is a CW complex and* $a, b \in h^*(X)$, *then* $ab = ba$.
(4) *The Hopf bundle* ξ_1 *over* S^1 *is h-orientable.*

Proof. It is obvious that (1) $\Leftrightarrow$ (2). Suppose (2) is true and $a \in h^k(X)$, $b \in h^l(X)$ with k and l odd. Then $ab = (-1)^{kl}ba = -ba$. According to (2), $ba = -ba$ and thus (2) $\Rightarrow$ (3).

Suppose (3) is true and show (1). Consider $\mathbf{1}^1 \in \tilde{h}_1(S^1)$ and $\mathbf{1}^1 \wedge \mathbf{1}^1 = -\mathbf{1}^1 \wedge \mathbf{1}^1 \in h^2(S^2)$. Now $\mathbf{1}^2 = \mathbf{1}^1 \cdot \mathbf{1}^1$ is a free basis for $\tilde{h}^*(S^2)$, and since $(\mathbf{1} + \mathbf{1})\mathbf{1}^2 = 0$, it follows that $\mathbf{1} + \mathbf{1} = 0$. Thus (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).

Now suppose (4) is true, i.e., $\rho \in \tilde{h}^1(RP^2)$ is an orientation for ξ_1. According to Theorem 4.1, $\tilde{h}^*(RP^2)$ has a free basis ρ, ρ^2. However $\rho^2 = -\rho^2$ and thus $(\mathbf{1} \not+ \mathbf{1})\rho^2 = 0$ and thus $\mathbf{1} + \mathbf{1} = 0$.

Finally suppose (1), (2), (3) and show (4). Since $\mathbf{1}^1 = -\mathbf{1}^1$, the trivial line bundle over a point has a unique orientation $\mathbf{1}^1 \in \tilde{h}^1(S^1)$. S^1 is the union of two intervals, $S^1 = D_1 \cup D_2$ and the trivial bundles $\xi_1 \mid D_1$ and $\xi_1 \mid D_2$ have unique orientations

$$a_1 \in \tilde{h}^1(T(\xi_1 \mid D_1)) \quad \text{and} \quad a_2 \in \tilde{h}^1(T(\xi_1 \mid D_2)).$$

Since a_1 and a_2 restrict to the same class in $\tilde{h}^1(T(\xi_1 \mid S^0))$, they pull back to an orientation class $\rho \in \tilde{h}^1(T(\xi_1))$. This proves theorem 4.3.

LEMMA 4.4. *Suppose* h *is an s-theory,* $n > 0$, *and* $\rho_n \in \tilde{h}^1(RP^n)$, $\tilde{h}^2(CP^n)$, *or* $\tilde{h}^4(HP^n)$ *is an h-orientation class for* ξ_{n-1}. *If* ξ_n *is h-orientable, then* $\exists$ *an h-orientation class* $\rho_{n+1} \in \tilde{h}^1(RP^{n+1})$, $\tilde{h}^2(CP^{n+1})$, *or* $\tilde{h}^4(HP^{n+1})$ *such that* $i^*(\rho_{n+1}) = \rho_n$.

Proof. The proof is given for the real case. Let $x \in \tilde{h}^1(RP^{n+1})$ be an orientation for ξ_n. By Th. 4.1, $\tilde{h}^*(RP^{n+1})$ has a free basis $x, x^2, \cdots, x^{n+1}$ and $\tilde{h}^*(RP^n)$ has a free basis $i^*(x), i^*(x^2), \cdots, i^*(x^n)$. Thus

$$i^* : \tilde{h}^1(RP^{n+1}) \to \tilde{h}^1(RP^n)$$

is onto and thus $\exists\, \rho_{n+1}$ with $i^*(\rho_{n+1}) = \rho_n$. By Theorem 4.1 again, ρ_{n+1} is an orientation for ξ_n iff ρ_n is an orientation for ξ_{n-1}. This proves 4.4.

THEOREM 4.5. *If h is an r-theory, any two of the following are equivalent.*

(1) $\exists$ *a class* $\rho \in \tilde{h}^1(RP^\infty), \tilde{h}^2(CP^\infty)$, *or* $\tilde{h}^4(HP^\infty) \ni$ *for each* $n > 0$, $\tilde{h}^*(WP^n)$ *is a free h^*-module with free basis* $\rho_n, \rho_n^2, \cdots, \rho_n^n$. Here $\rho_n = i_n^*(\rho)$ *where* $i_n : WP^n \to WP^\infty$ *is the inclusion.*

(2) *For each* $n > 0$, ξ_n *is h-orientable.*

(3) ξ *is h-orientable.*

Proof. The proof is given for the real case. It follows from Theorem 4.1 that (1) $\Rightarrow$ (2). Suppose (2) is true. Let $\rho_1 = \pm 1^1$ be an orientation for ξ_0 and use Lemma 4.4 to construct a maximal tower $\rho_1, \rho_2, \rho_3, \cdots$. This determines a cohomology class ρ satisfying (1). Thus (1) $\Leftrightarrow$ (2). (It will be shown later that the sequence $\rho_1, \rho_2, \cdots$ determines ρ uniquely.)

It is immediate that (3) $\Rightarrow$ (2). Show (1) $\Rightarrow$ (3). Let $\rho \in \tilde{h}^1(RP^\infty)$ satisfy (1), and let $j : RP^\infty \to T(\xi)$ be the inclusion. There exists a homeomorphism $p : T(\xi) \to RP^\infty$ which is isotopic to the identity Id: $T(\xi) \to T(\xi)$, and with $pj = \text{Id}: RP^\infty \to RP^\infty$. It is clear that $\rho^*(\rho)$ is an h-orientation for ξ. This completes 4.5.

DEFINITION. Let W = real, complex, or quaterionic. Then an r-cohomology theory is said to be W-orientable if for each integer $n > 0$, the Hopf bundle ξ_n over WP^n is orientable. A W-orientation for h is a class $\rho \in \tilde{h}^1(RP^\infty)$, $\tilde{h}^2(CP^\infty)$, or $\tilde{h}^4(HP^\infty)$ such that $p^*(\rho)$ is an h-orientation for the Hopf bundle ξ over RP^∞, CP^∞, or HP^∞. Thus ρ is a W-orientation for h iff ρ satisfies (1) of Theorem 4.5

THEOREM 4.6. *Let h be an r-theory and* $\rho \in \tilde{h}^1(RP^\infty)$, $\tilde{h}^2(CP^\infty)$, *or* $\tilde{h}^4(HP^\infty)$ *be a W-orientation for h. Let k be a positive integer and*

$$X = WP^\infty \times WP^\infty \times \cdots \times WP^\infty$$

be the k-fold product. Let $\pi_i : X \to WP^\infty$ *be the i-projection and* $u_i = \pi_i^*(\rho)$. *Then we have the following.*

(1) $h^*(X)$ *has no phantom classes.*

(2) *any* $y \in h^t(X)$ *can be written uniquely as an infinite power series in the* $u_i : y = a_0 + a_1 u_1 + \cdots a_n u_n + a_{n+1} u_1 u_2 + \cdots$. *Here* $a_i \in h^s$ *where* $s = s(i)$ *is such that the total degree of the term involving a_i is t. Furthermore each such power series determines a unique y.* Thus $h^{**}(X)$ *is a power series module with variables* $u_1, u_2, \cdots, u_k$ *and scalar "ring"* h^{**}.

Proof. Consider only the case $W = R$. Consider first the case $k = 1$. It follows from 4.1 that $h^*(RP^{n+1}) \to h\ (RP^n)$ is onto. Thus by 3.1, $h^*(RP^\infty)$ has no phantom classes. Now consider an expression $a_0 + a_1 u_1 + a_2 u_1^2 + \cdots$. From the fact that $u_1 = \rho$ lives in reduced cohomology, and from 4.2, it follows that this sum is finite when restricted to any RP^n. Thus the expression does determine a class in $h^t(RP^\infty)$ and since there are no phantom classes, it determines a unique class. Conversely, any $y \in h^t(RP^\infty)$ when restricted to RP^n, is a polynomial in ρ, and thus y determines a power series in ρ. This proves 4.6 for $k = 1$.

Now suppose $k > 1$. Let $X_n = RP^n \times RP^n \times \cdots \times RP^n$, the k-fold product. It follows from 2.1 that $h^*(X_n)$ is a polynomial algebra with coefficients in h^* and over variables $u_1, u_2, \cdots, u_k$ with $u_i^j = 0$ for $i = 1, 2, \cdots, k$ and $j > n$. It follows from 3.1 that $h^*(X)$ has no phantom classes. The rest of the proof is as for the case $k = 1$ above. This proves 4.6.

Note that in general, the direct product h^{**} is not a ring. However, if $h^i = 0$ for $i > 0$, then h^{**} and $h^{**}(X)$ are rings and $h^{**}(X)$ is a power series ring.

5. The splitting principle

Let α^n be a real, complex, or quaterionic bundle over X with total space $E(\alpha)$. If $P(\alpha)$ is the space of lines in $E(\alpha)$, then $\pi : P(\alpha) \to X$ is a fibre bundle with fibre WP^{n-1} Let $l(\alpha)$ be the line bundle over $P(\alpha)$ which, when restricted to a fibre, is ξ_{n-1}. Then $P(\alpha) \to X$ pulls back α to $l(\alpha) \oplus$ (an $(n - 1)$-bundle).

THEOREM 5.1. *Let α^n be a W-bundle over X and h be a W-orientable r-theory· Then $\pi^* : h^*(X) \to h^*(P(\alpha))$ is monic. There exists a space Y and map $f : Y \to X$ such that $f^*(\alpha)$ is the sum of line bundles and $f^* : h^*(X) \to h^*(Y)$ is monic. Also the map*

$$Y_n = WP^n \times WP^n \times \cdots \times WP^n \to BW_n$$

which classifies $\xi \times \xi \times \cdots \times \xi$ induces a monic map $h^*(BW_n) \to h^*(Y_n)$. (W is used for R, C, or H, and correspondingly, for O, U, or Sp.)

Proof. Consider the real case. Let $\rho \in \tilde{h}^1(RP^\infty)$ be a real-orientation for h. Let $u : P(\alpha) \to RP^\infty$ classify $l(\alpha)$ and $w_1 = u^*(\rho)$. Then from 2.1 it follows that $1, w_1, w_1^2, \cdots, w_1^{n-1}$ is a free $h^*(X)$-basis for $h^*(P(\alpha))$. Thus π^*: $h^*(X) \to h^*(P(\alpha))$ is monic. The second statement follows from induction. Let $g : V \to BO_n$ be a splitting map for γ_n. This map factors as $V \to Y_n \to BO_n$ and thus proves the last statement in 5.1.

6. The characteristic classes

THEOREM 6.1. *Suppose h is an r-cohomology theory and $\rho \in \tilde{h}^1(RP^\infty)$, $\tilde{h}^2(CP^\infty)$, or $\tilde{h}^4(HP^\infty)$ is a W-orientation for h. Then the following hold.*
I. *There exists a unique transformation assigning to each real, complex, or quaterionic vector bundle α over a CW complex, an element*

$$w(\alpha) = 1 + w_1(\alpha) + w_2(\alpha) + \cdots \in h^*(X),$$

$$c(\alpha) = 1 + c_1(\alpha) + c_2(\alpha) + \cdots \in h^*(X),$$

or

$$q(\alpha) = 1 + q_1(\alpha) + q_2(\alpha) + \cdots \in h^*(X)$$

satisfying the following.

(0) $w_i(\alpha) \in h^i(X)$, $c_i(\alpha) \in h^{2i}(X)$, *or* $q_i(\alpha) \in h^{4i}(X)$, *where* X *is the base of* α. *If* $i > \dim \alpha$, *then* w_i, c_i, *or* $q_i = 0$. *If* $i > \dim X$, *then* $w_i = 0$. *If* $2i > \dim X$, *then* $c_i = 0$. *If* $4i > \dim X$, *then* $q_i = 0$.

(1) *If* β *is a real, complex, or quaterionic bundle over* Y, *and* $f : X \to Y$ *is covered by a bundle map from* α *to* β, *then*

$$f^*(w(\beta)) = w(\alpha), \quad f^*(c(\beta)) = c(\alpha) \quad \text{or} \quad f^*(q(\beta)) = q(\alpha).$$

(2) *If* α *and* β *are real, complex, or quaterionic bundles over* X, *then*

$$w(\alpha \oplus \beta) = w(\alpha)w(\beta), \quad c(\alpha \oplus \beta) = c(\alpha)c(\beta) \quad \text{or} \quad q(\alpha \oplus \beta) = q(\alpha)q(\beta).$$

(3) *If* ξ *is the real, complex, or quaterionic Hopf bundle, then*

$$w(\xi) = \mathbf{1} + \rho \in h^*(RP^\infty), \quad c(\xi) = \mathbf{1} + \rho \in h^*(CP^\infty), \quad q(\xi) = \mathbf{1} + \rho \in h^*(HP^\infty).$$

(4) *If* X *is connected and* α *is a bundle over* X, *then* $w_i(\alpha)$, $c_i(\alpha)$, *or* $q(\alpha)$ *lives in the reduced cohomology for* $i > 0$.

II. *Let* n *and* k *be positive integers. Denote (ambiguously) by* γ_n *the classifying bundle over* BW_n. *Let* Y_k *be the k-fold product*

$$Y_k = BW_n \times BW_n \times \cdots \times BW_n,$$

and $\pi_i : Y_k \to BW_n$ *the projection on the i-factor. Then*

(1) $h^*(Y_k)$ *has no phantom classes,*

(2) $h^{**}(Y_k)$ *is a power series module with variables*

$$\pi_1^*(w_1(\gamma_n)), \cdots, \pi_k^*(w_1(\gamma_n)), \pi_1^*(w_2(\gamma_n)), \cdots, \pi_k^*(w_n(\gamma_n))$$

and coefficient "ring" h^{**} ($\pi_1^*(c_1(\gamma_n)) \cdots$ *or* $\pi_1^*(q_1(\gamma_n)) \cdots$ *resp.*).

III. (1) *For each integer* t *and positive integer* n,

$$0 \leftarrow h^t(BW_{n-1}) \leftarrow h^t(BW_n) \leftarrow \tilde{h}^t(MW_n) \leftarrow 0$$

is split exact.

(2) *If* BW *is the union* $BW_1 \subset BW_2 \subset \cdots = BW$ *and* $y \in h^t(BW)$ *is a class such that for each inclusion* $i_n : BW_n \to BW$, $i_n^*(y) = 0$, *then* $y = 0$.

(3) *Let* Y_k *be the k-fold product* $Y_k = BW \times BW \times \cdots \times BW$. *Then* $h^*(Y_k)$ *has no phantom classes.*

(4) *Let* $w_i(\gamma) \in \tilde{h}^i(BO)$ *be the unique class which, when restricted to* BO_n *gives* $w_i(\gamma_n)$. (*Analogously for* $c_i(\gamma)$ *and* $q_i(\gamma)$). *Then* $h^{**}(BW)$ *is a power series module with variables* $w_1(\gamma)$, $w_2(\gamma)$, $\cdots$, *and coefficient "ring"* h^{**}. *Also* $h^{**}(Y_k)$ *is a power series module with variables*

$$\pi_1^*(w_1(\gamma)), \pi_2^*(w_1(\gamma)) \cdots \pi_k^*({}_1w(\gamma)), \pi_1^*(w_2(\gamma)) \cdots.$$

(*If* $W = C$ *or* H, *use* c_i *or* q_i *resp.*)

IV. *There is a unique function* V *which assigns to each real, complex, or quaterionic bundle* α^n *over a connected base, an h-orientation class* $V(\alpha) \in \tilde{h}^{wn}(T(\alpha))$, ($w = 1, 2$, or 4), *and satisfies the following.*

(a) $V(\xi) = p^*(\rho)$ *where* $p : T(\xi) \to WP^\infty$ *is the homeomorphism which is homotopic to the identity* Id : $T(\xi) \to T(\xi)$.

(b) *If α^n is a bundle over X then $V(\alpha)$ maps to $w_n(\alpha)$ under*

$$\tilde{h}^n(T(\alpha)) \approx h^n(E(\alpha), S(\alpha)) \to h^n(E(\alpha)) \approx h^n(X)$$

or $V(\alpha)$ maps to $c_n(\alpha)$ under $\tilde{h}^{2n}(T(\alpha)) \to h^{2n}(X)$ or $V(\alpha)$ maps to $q_n(\alpha)$ under $\tilde{h}^{4n}(T(\alpha)) \to h^{4n}(X)$.

(c) *If α^n and β^m are bundles over X and Y resp.,*

$$V(\alpha \times \beta) = V(\alpha) \wedge V(\beta) \in \tilde{h}^{n+m}(T(\alpha \times \beta) = T(\alpha) \wedge T(\beta)).$$

(d) *If α^n is a bundle over X and θ is the trivial line bundle over X,*

$$V(\theta \oplus \alpha) = V(\theta) \wedge V(\alpha) = \pm \mathbf{1}^w \wedge V(\alpha) \in \tilde{h}^{w(n+1)}(T(\theta \oplus \alpha) = S^w \wedge T(\alpha)),$$

where $w = 1, 2,$ or 4.

(e) *If $f : X \to Y$ is covered by a bundle map from α to β then $T(f)^*(V(\beta)) = V(\alpha)$.*

V. *Let $\bar{\rho} \in \tilde{h}^1(RP^\infty), \tilde{h}^2(CP^\infty)$, or $\tilde{h}^4(HP^\infty)$ be defined by $\bar{\rho} = \rho + a_2\rho^2 + a_3\rho^3 + \cdots$ where $a_i \in h^{n(i)}$ is such that the dimension of the term involving a_i is 1, 2, or 4. Then $\bar{\rho}$ is an orientation for h. Furthermore, if $\bar{\rho}$ is an orientation for h, $\pm\bar{\rho}$ may be written in this form.*

Let $\bar{\rho} = \rho + a_2\rho^2 + \cdots$ and $\tilde{w}_i$, $\tilde{c}_i$, or $\tilde{q}_i$ be the corresponding characteristic classes. Then for any bundle α over X, $\tilde{w}_i(\alpha) = w_i(\alpha) + y$ where y has the form $y = b_1 y_1 + b_2 y_2 + \cdots$. Here $b_j \in h^{m(j)}(X)$, the dimension of $b_j y_j$ is i, $2i$, or $4i$, and $m(j) > i$, $2i$, or $4i$ resp. (Note: If X is connected, $y_j \in \tilde{h}^{m(j)}(X)$.) Furthermore, if $\bar{\rho} = -\rho$, then $\tilde{w}_i(\alpha) = (-1)^i w_i(\alpha)$ for all α.

If $\bar{\rho} = \rho + a_2\rho^2 + \cdots$, then $\tilde{w}_i(\alpha) = w_i(\alpha)$ when restricted to X^i, $\tilde{c}_i(\alpha) = c_i(\alpha)$ when restricted to X^{2i}, or $\tilde{q}_i(\alpha) = q_i(\alpha)$ when restricted to X^{4i}.

VI. *Suppose $\bar{h}$ is an r-theory, $\bar{\rho}$ is a W-orientation for h, and $T : \bar{h} \to h$ is a natural transformation of r-theories with $T(\bar{\rho}) = \rho$. If α is any real, complex, or quaterionic bundle, then $T(\bar{w}_i(\alpha)) = w_i(\alpha)$, $T(\bar{c}_i(\alpha)) = c_i(\alpha)$, or $T(\bar{q}_i(\alpha)) = q_i(\alpha)$.*

Proof of I. The proof of all parts will be given only for the real case. Define $w(\xi) = \mathbf{1} + \rho$. Since ξ over RP^∞ is the classifying bundle for real line bundles, this defines the classes for line bundles. Let α^n be a real vector bundle over X and $\pi : P(\alpha) \to X$ be the bundle with fibre RP^{n-1}. Here $\pi^*(\alpha) = l(\alpha) \oplus$ [an $(n-1)$-bundle] where $l(\alpha)$ is the line bundle which gives ξ_{n-1} when restricted to a fibre RP^{n-1}. Let $w_1 = w_1(l(\alpha)) \in h^1(P(\alpha))$. By the Thom-Dold isomorphism, $h^*(P(\alpha))$ is a free $h^*(X)$-module with basis $\mathbf{1}, w_1, w_1^2, \cdots w_1^{n-1}$. The classes $w_i(\alpha)$ are defined to be the unique classes which satisfy

$$w_1^n = w_1^{n-1}\pi^*(w_1(\alpha)) - w_1^{n-2}\pi^*(w_2(\alpha)) + \cdots (-1)^{n-1}\pi^*(w_n(\alpha)).$$

The proof of I follows in the standard manner (see, for example, [17, p. 47], [29], or [27]).

Proof of II. The proof is given for $k = 1$. Suppose

$$f : Y_n = RP^\infty \times \cdots \times RP^\infty \to BO_n$$

classifies $\xi \times \xi \times \cdots \times \xi$. According to Theorem 4.6, $h^*(Y_n)$ has no phantom classes. By the splitting principle, $f^* : h^*(BO_n) \to h^*(Y_n)$ is monic, and thus $h^*(BO_n)$ has no phantom classes.

Let $u_i = \pi_i^*(\rho)$ where $\pi_i : Y_n \to RP^\infty$ is the i-projection. By 4.6, any element in $h^t(Y_n)$ is a unique power series in the u_i. Now

$$w(\xi \times \xi \times \cdots \times \xi) = (1 + u_1)(1 + u_2) \cdots (1 + u_n),$$

i.e. w_i is the i-symmetric function in the u_j. Any map $Y_n \to Y_n$ which simply interchanges factors, is covered by a bundle map, and thus if $y \in h^t(BO_n)$, $f^*(y) \in h^t(Y_n)$ is a symmetric power series in the u_i, i.e. is a power series in the w_i. Since f^* is monic, this proves II for $k = 1$. The proof for $k > 1$ follows from the Thom-Dold isomorphism.

Proof of III, *part* (1). The total space $E(\gamma_n)$ has the homotopy type of BO_n, and the sphere bundle $S(\gamma_n)$ has the homotopy type of BO_{n-1}. Thus

$$h^t(BO_n, BO_{n-1}) \approx \tilde{h}^t(T(\gamma_n) = MO_n).$$

The map $BO_{n-1} \to BO_n$ pulls γ_n back to $\gamma_{n-1} \oplus \theta$ and thus pulls back SW classes. It follows from II (2) that $h^t(BO_n) \to h^t(BO_{n-1})$ is onto. The splitting map $h^t(BO_{n-1}) \to h^t(BO_n)$ sends a power series in $w_i(\gamma_{n-1})$ to the corresponding series in $w_i(\gamma_n)$. The image $\tilde{h}^*(MO_n) \to h^*(BO_n)$ is the ideal generated by $w_n(\gamma_n)$. Note that

$$h^*(BO_n) \approx h^*(BO_{n-1}) \oplus \tilde{h}^*(MO_n)$$

is an h^*-module isomorphism but not, of course, a ring isomorphism.

The proof of (2) is immediate from (1) above and Theorem 3.1 part (1).

Proof of (3) *for* $k = 1$. If $y \in h^t(BO)$ is zero when restricted to $(BO)^s$, then it is zero when restricted to $(BO_n)^s$. Then by II (1), y is zero when restricted to (BO_n). By III (2), y is zero. The proof for $k > 1$ is similar.

Finally, part (4) follows from III (2) and the computation of $h^i(BO_n)$ in II (2).

Proof of IV. Define $V(\gamma_n) \in \tilde{h}^n(MO_n)$ to be the unique class which maps to $w_n(\gamma_n)$ under $\tilde{h}^n(MO_n) \to h^n(BO_n)$. Since the BO_n are the classifying spaces, this defines $V(\alpha)$ for any bundle α. Consider the following commutative diagram.

$$\begin{array}{llll}
w_n(\gamma_n) \otimes w_m(\gamma_m) \to & w_n(\gamma_n) \times w_m(\gamma_m) = & w_{n+m}(\gamma_n \times \gamma_m) \leftarrow & w_{n+m}(\gamma_{n+m}) \\
h^n(BO_n) \otimes h^m(BO_m) & \to h^{n+m}(BO_n \times BO_m) & \longleftarrow & h^{n+m}(BO_{n+m}) \\
\uparrow \qquad\qquad \uparrow & \quad \uparrow g^* & & \uparrow \\
h^n(MO_n) \otimes h^m(MO_m) & \to \tilde{h}^{n+m}((MO_n \wedge MO_m) & & \\
 & \quad = M(O_n \times O_m)) & \xleftarrow{f^*} & \tilde{h}^{n+m}(MO_{n+m}) \\
V(\gamma_n) \otimes V(\gamma_m) \to & V(\gamma_n) \wedge V(\gamma_m) & V(\gamma_n \times \gamma_m) \leftarrow & V(\gamma_{n+m})
\end{array}$$

It follows from II above and the Thom-Dold isomorphism that g^* is monic. This implies that $V(\gamma_n) \wedge V(\gamma_m) = V(\gamma_n \times \gamma_m)$.

It is immediate that $V(\gamma_1)$ is an orientation class for γ_1, namely $V(\gamma_1) = p^*(\rho)$. Assume inductively that $V(\gamma_n)$ and $V(\gamma_m)$ are orientations. Since

$$1^n \wedge 1^m = 1^{n+m} \in \tilde{h}^{n+m}(S^n \wedge S^m = S^{n+m}),$$

it follows that $f^*(V(\gamma_{n+m})) = V(\gamma_n) \wedge V(\gamma_m)$ is an orientation for $\gamma_n \times \gamma_m$, from which it follows that $V(\gamma_{n+m})$ is an orientation for γ_{n+m}. This completes the proof of IV.

Proof of V. Any $\tilde{\rho} \in \tilde{h}_1(RP^\infty)$ may be written uniquely as $\tilde{\rho} = a_1\rho + a_2\rho^2 + \cdots$. According to 4.2, $\rho^t = 0$ when restricted to the 1-skeleton for all $t > 1$. If $i : S^1 = RP^1 \to RP^\infty$ is the inclusion, $i^*(\tilde{\rho}) = a_i\, i^*(\rho) = \pm a_1 \mathbf{1}^1$. Thus $\tilde{\rho}$ is an orientation for ξ iff $a_1 = \pm\mathbf{1} \in h^0$.

Now suppose $\tilde{\rho} = \rho + a_2\rho^2 + \cdots$. To prove that $\tilde{w}_i$ has the specified form, it is only necessary to prove it for the classifying bundles, and by the splitting principle it is only necessary to prove it for $\xi \times \xi \times \cdots \times \xi$ over $Y_n = RP^\infty \times \cdots \times RP^\infty$. Let $\tilde{u}_i = \pi_i^*(\tilde{\rho})$ and $u_i = \pi_i^*(\rho)$ so that

$$w(\xi \times \cdots \times \xi) = (1 + u_1)(1 + u_2) \cdots (1 + u_n)$$

and

$$\tilde{w}(\xi \times \cdots \times \xi) = (1 + \tilde{u}_1)(1 + \tilde{u}_2) \cdots (1 + \tilde{u}_n).$$

Since $\tilde{u}_i = u_i + \pi_i^*(a_2\rho^2 + a_3\rho^3 + \cdots)$, the result is immediate. If $\tilde{\rho} = -\rho$, then $\tilde{u}_i = -u_i$ and thus $\tilde{w}_i = (-1)^i w_i$.

Suppose h is a connected theory, i.e. $h^n = 0$ for $n > 0$. This implies $h^n(X^i) = 0$ for $n > i$ and this implies that in the equation $\tilde{w}_i = w_i + y$, $y = 0$ when restricted to X^i. Thus for the connected theory $h(-) = h(-; \overline{MO})$, we have $\tilde{w}_i = w_i$ when restricted to X^i. The general case follows from the universality of $h(-; \overline{MO})$, (the real analogue of Theorem 8.1).

Proof of VI. From the hypothesis it follows that $T(\tilde{w}(\alpha)) = w(\alpha)$ for all line bundles α. Therefore this equality holds if α is the sum of line bundles, and the general result follows from the splitting principle.

7. Extending the Chern classes to K theory

Since K theory is usually restricted to finite complexes, a brief review is in order. All bundles in this section are complex bundles. Consider the Ω-spectrum E defined by $E_{2i} = BU \times Z$ and $E_{2i+1} = U$ [7]. For any topological group G there is a homotopy equivalence $G \to \Omega BG$ and this is used to define the map $E_{2i-1} \to \Omega E_{2i}$. The homotopy equivalence $E_{2i} \to \Omega E_{2i+1}$ is given by the Bott periodicity theorem. This is a ring spectrum and the sphere spectrum maps into it by maps $S^i \to E_i$ defined by the generators of $\pi_{2i}(BU \times Z) \approx Z$ and $\pi_{2i+1}(U) \approx Z$. The r-cohomology theory $h(-; E)$ is denoted by $K^n(X, A) = h^n(X, A; E)$ and $K(X, A) = K^0(X, A)$. Thus $K(X) = [X, BU \times Z]$, based homotopy classes of maps. The base point of $BU \times Z$ is in $BU \times O$. Since BU is 1-connected, the set of based homotopy classes of maps of X into BU is isomorphic to the set of free homotopy classes of maps of X into BU. So $K(X) = \{X, BU \times Z\}$ and $\tilde{K}(X) \approx \{X, BU \times Z\}_0$ where, in the second instance, the maps are required to send the component X_0 of X containing the base point, into $BU \times O$. Thus if X is connected, $\tilde{K}(X) \approx \{X, BU\}$, free homotopy classes of maps.

If X is any complex, let $U(X)$ be the set of isomorphism classes of complex bundles over X. A bundle may have different dimensions over different components of X. $U(X)$ is an abelian semigroup under $\oplus$ and a commutative semiring over $\otimes$. The unit element is the trivial line bundle and is denoted by $\mathbf{1}$. A trivial bundle of dimension k will be denoted by $\mathbf{k}$. There is a natural transformation from U to K. Due to the semiring isomorphism $U(\bigcup X_a) \approx \prod[U(X_a)]$ and the ring isomorphism $K(\bigcup X_a) \approx \prod[K(X_a)]$, it suffices to consider connected X. If X is connected, a bundle α over X has a given dimension n, and the image of α under $U(X) \to K(X)$ is given by the classifying map

$$f_1 : X \to BU_n \times n \subset BU \times Z.$$

Let $f_2 : X \to BU_m \times m$ represent β^m. Then $(f_1) + (f_2) \in K(X)$ is represented by the

$$f_3 : X \to BU_{n+m} \times (n + m) \subset BU \times Z$$

which classifies $\alpha \oplus \beta$. The product $(f_1)(f_2) \in K(X)$ is represented by

$$f_4 : X \to BU_{nm} \times (nm) \subset BU \times Z$$

which classifies $\alpha \otimes \beta$. Thus $U(X) \to K(X)$ is a semiring homomorphism.

Define $K'(X)$ to be the set of ordered pairs (α_1, α_2) of bundles over X with the equivalence relation $(\alpha_1, \alpha_2) \sim (\beta_1, \beta_2)$ iff $\exists$ bundles $\bar{\alpha}$ and $\bar{\beta}$ such that

$$(\alpha_1 \oplus \bar{\alpha}, \alpha_2 \oplus \bar{\alpha}) \approx (\beta_1 \oplus \bar{\beta}, \beta_2 \oplus \bar{\beta}).$$

If X is a pointed space, $\tilde{K}'(X) \subset K'(X)$ consists of pairs (α_1, α_2) such that α_1 and α_2 have the same dimension over the component of X containing the base point. Addition is coordinatewise Whitney sum and the additive inverse is given by $-(\alpha_1, \alpha_2) = (\alpha_2, \alpha_1)$. $K'(X)$ is a commutative ring with unit element, where multiplication is given by

$$(\alpha_1, \alpha_2)(\beta_1, \beta_2) = (\alpha_1 \otimes \beta_1 \oplus \alpha_2 \otimes \beta_2, \alpha_2 \otimes \beta_1 \oplus \alpha_1 \otimes \beta_2).$$

There is a semiring homomorphism $U(X) \to K'(X)$ given by

$$\alpha \to (\alpha, \mathbf{0}) \sim (\alpha \oplus \mathbf{k}, \mathbf{k}).$$

Two bundles $\alpha, \beta \in U(X)$ map to the same element in $K'(X)$ iff $\exists$ a bundle γ over X with $\alpha \oplus \gamma \approx \beta \oplus \gamma$. In particular, α and β must have the same dimension over any component of X. If each component of X is finite dimensional, then $\exists$ a bundle γ_1 so that $\gamma_2 = \gamma \oplus \gamma_1$ is trivial bundle of some dimension over each component of X. Thus, for such X, α and β map to the same element in $K'(X)$ iff $\exists$ a bundle γ_2 which is a trivial bundle of some dimension over each component of X and $\alpha \oplus \gamma_2 \approx \beta \oplus \gamma_2$. Note that $\mathbf{0} \in U(X)$ is the only element mapping to $\mathbf{0} \in K'(X)$.

A ring homomorphism $K'(X) \to K(X)$ is defined by

$$(\alpha_1, \alpha_2) \to (\alpha_1) - (\alpha_2) \in K(X).$$

Here (α_i) is the image of α_i under the map $U(X) \to K(X)$. If X is finite-dimensional on each of its components, an inverse ring homomorphism $K(X) \to K'(X)$ will be defined. Due to the ring isomorphism $K'(\bigcup X_a) \approx \prod [K'(X_a)]$, it suffices to consider X connected and of finite dimension. If $f: X \to BU \times Z$ represents $(f) \in K(X)$ then f factors as

$$f_1 : X \to BU_n \times k \subset BU \times Z$$

and thus determines an element $(f^*(\gamma_n) \oplus \mathbf{k}, \mathbf{n})$. Thus for such X, $K'(X) \approx K(X)$.

A "third possible K-theory" is LK. Define $LK(X) =$ Inverse limit $K(X^n)$ where X^n is the n-skeleton. This definition depends only on the homotopy type of X and is, in particular, independent of the CW structure of X. There is a projection homomorphism $K(X) \to LK(X)$ and it is obvious that this map is onto and that its kernel is the set of phantom elements plus $\mathbf{0}$. Thus $LK(X)$ is the quotient of $K(X)$ and the ideal of all phantom classes plus $\mathbf{0}$. Since $K(X^n) \approx K'(X^n)$, an equivalent definition of LK is $LK(X) =$ Inverse limit $K'(X^n)$. The projection $K'(X) \to LK(X)$ is merely the composition $K'(X) \to K(X) \to LK(X)$.

An augmentation is a (semi) ring homomorphism from $U(X)$, $K'(X)$, $K(X)$, or $LK(X)$ to $\{X, Z\}$. Due to the ring isomorphism $\{\bigcup X_a, Z\} \approx \prod\{X_a, Z\}$, it suffices to consider connected X. Suppose X is connected and identify $\{X, Z\}$ with Z. Then Aug : $U(X) \to Z$ is given by Aug $(\alpha) = \dim \alpha$. Aug : $K'(X) \to Z$ is given by

$$\text{Aug } (\alpha_1, \alpha_2) = \text{Aug } (\alpha_1) - \text{Aug } (\alpha_2).$$

If $f : X \to BU \times k \subset BU \times Z$ represents $(f) \in K(X)$, then Aug $(f) = k$. If $a \in LK(X)$ and $Y \subset X$ is any connected finite subcomplex and $i : Y \to X$ is the inclusion, then Aug $(a) =$ Aug $(i^*(a))$ does not depend upon the choice of Y and thus defines Aug : $LK(X) \to Z$.

Summary. There are contravariant functors U, K', K, and LK defined on the category of CW complexes and homotopy classes of maps. The functor U maps to the category of commutative semirings with unit and semiring homomorphisms. The functors K', K, and LK map to the category of commutative rings with unit and ring homomorphisms. If X is the union of disjoint open sets, $X = \bigcup X_a$, then $F(X) \approx \prod F(X_a)$ for each of $F = U, K', K$, and LK. There are five natural transformations which are related by the following commutative diagram:

$$\begin{array}{ccccc} & & K'(X) & & \\ & \nearrow & \Big| & \searrow & \\ U(X) & & \Big| & & LK(X) \\ & \searrow & \downarrow & \nearrow & \\ & & K(X) & & \end{array}$$

Each of these functors admits a natural transformation

$$\text{Aug} : U, K', K, LK \to \{\quad, Z\}$$

and the above diagram commutes with these augmentations. The map $K(X) \to LK(X)$ is onto for all X and its kernel is the ideal of all phantom elements in $K(X)$ plus **0**. If each component of X is finite-dimensional, $K'(X) \approx K(X) \approx LK(X)$ Finally, if X is any complex and α_1 and α_2 are bundles over X, then $\alpha_1 - \alpha_2 \in K(X)$ will denote the image of (α_1, α_2) under $K'(X) \to K(X)$.

Now suppose h is an r-theory and $\rho \in \tilde{h}^2(CP^\infty)$ is a complex orientation for h. We wish to extend the domain of the Chern classes from $U(X)$ to $K(X)$, i.e. for any $(f) \in K(X)$, we wish to define $c(f) \in \prod_{i \geq 0} h^{2i}(X) \subset h^{**}(X)$. Let $f_1 : X \to BU$ be the composition

$$X \xrightarrow{f} BU \times Z \to BU$$

and $c_i(f) = f_1^*(c_i(\gamma)) \in h^{2i}(X)$. It is obvious that

$$c((f_1) + (f_2)) = c(f_1)c(f_2)$$

when restricted to any finite skeleton and it will be seen that they are actually equal. Let $A : BU \times BU \to BU$ be the map which represents addition, that is

$$A \mid BU_n \times BU_n : BU_n \times BU_n \to BU_{2n} \subset BU$$

classifies the bundle $\gamma_n \oplus \gamma_n$. Let

$$d_i = \pi_1^*(c_i(\gamma)) \quad \text{and} \quad e_i = \pi_2^*(c_i(\gamma)) \in \tilde{h}^{2i}(BU \times BU).$$

Then $A^*(c_i(\gamma)) = d_0 e_i + d_1 e_{i-1} + \cdots + d_i e_0$ (because $\tilde{h}^{2i}(BU \times BU)$ has no phantom classes). The map which determines $(f_1) + (f_2)$ is the composition

$$f_3 : X \xrightarrow{f_1 \times f_2} BU \times BU \xrightarrow{A} BU.$$

Then

$$\begin{aligned} f_3^*(c_i(\gamma)) &= (f_1 \times f_2)^*(d_0 e_i + \cdots + d_i e_0) \\ &= f_1^*(c_0(\gamma))f_2^*(c_i(\gamma)) + \cdots + f_1^*(c_i(\gamma))f_2^*(c_0(\gamma)). \end{aligned}$$

This proves the product formula.

THEOREM 7.1. *Suppose h is an r-cohomology theory and $\rho \in \tilde{h}^2(CP^\infty)$ is a complex orientation for h.*

I. *There exists a unique transformation assigning to each CW complex X and $x \in K(X)$, an element*

$$c(x) = \mathbf{1} + c_1(x) + c_2(x) + \cdots \in h^{**}(X)$$

satisfying the following.

(0) $c_i(x) \in h^{2i}(X)$. *If X is finite-dimensional,* $c_i(x) = 0$ *when* $2i > \dim X$.

(1) *If* $g : X \to Y$, $y \in K(Y)$, *and* $g^*(y) = x$ *where* $g^* : K(Y) \to K(X)$, *then* $g^*[c_i(y)] = c_i(x)$ *for all* $i \geq 0$ *where* $g^* : h^{2i}(Y) \to h^{2i}(X)$.

(2) *If* $x_1, x_2 \in K(X)$, *then*

$$c(x_1 + x_2) = c(x_1)c(x_2) \quad \text{and} \quad c(x_1 - x_2) = c(x_1)/c(x_2).$$

(3) *If* ξ *is the Hopf bundle over* CP^∞ *and* $k \geq 0$, *then*

$$c(\xi - \mathbf{k}) = \mathbf{1} + \rho \in h^*(CP^\infty).$$

(4) *If X is connected,* $c_i(x) \in \tilde{h}^{2i}(X)$ *for all* $i > 0$. *Also if*

$$x_1 \oplus x_2 \in \tilde{K}(X) \oplus K(\text{pt}) = K(X),$$

then $c(x_2) = \mathbf{1}$ *and* $c(x_1 \oplus x_2) = c(x_2)$.

II. *Part* I *above holds with K replaced by K'.*

Proof. The existence for I follows from the definition and discussion preceding the theorem. If $y \in K'(X)$ is mapped by $K'(X) \to K(X)$ to $x \in K(X)$, then define $c_i(y) = c_i(x)$. Thus the existence for II is immediate. The uniqueness for II follows from the uniqueness in Theorem 6.1 part I. Thus uniqueness for I holds for finite-dimensional X. Let $\bar{c}$ denote a set of Chern classes satisfying I. Let Id : $BU \to BU$ be the identity, (Id) $\in \tilde{K}(BU)$. To prove uniqueness, it is only necessary to show $\bar{c}_i(\text{Id}) = c_i(\gamma) \in \tilde{h}^{2i}(BU)$. This is true on any finite skeleton, and since $\tilde{h}^{2i}(BU)$ has no phantom classes, it is true.

8. The universality of $h(-; MU)$

THEOREM 8.1. *Let* $h(-) = h(-; E)$ *be an r-cohomology theory and* $\rho \in \tilde{h}^2(CP^\infty)$ *be a positive complex orientation for h (i.e.* ρ *restricted to* S^2 *gives* $+\mathbf{1}^2$ *and not* $-\mathbf{1}^2$). *Let* ρ_{MU} *be the canonical complex orientation for* $h(-; MU)$. *Then there exists a unique natural transformation of s-theories*

$$T : h(-; MU) \to h(-)$$

satisfying $T'(\rho_{MU}) = \rho$. *Also there is a natural transformation*

$$T : h(-; \overline{MU}) \to h(-)$$

of r-theories. If $\bar{c}_i \in h^{2i}(-; \overline{MU})$ *represent the Chern classes determined by the canonical orientation and* $c_i \in h^{2i}(-)$ *represent the Chern classes determined by* ρ, *then for each* $x \in K(X)$, $T(\bar{c}_i(x)) = c_i(x)$.

Proof. Define a spectrum map $T' : MU \to E$ by letting $T'_i : MU_i \to E_{2i}$ represent the class $V(\gamma_i) \in \tilde{h}^{2i}(MU_i)$. This is a spectrum map because $S^2 \wedge MU_i \to MU_{i+1}$ pulls back $V(\gamma_{i+1})$ to $\mathbf{1}^2 \wedge V(\gamma_i)$. It is a map of ring spectra because $MU_s \wedge MU_t \to MU_{s+t}$ pulls back $V(\gamma_{s+t})$ to $V(\gamma_s) \wedge V(\gamma_t)$. The map preserves the unit element, i.e.

$$\begin{array}{ccc} T'_i : MU_i & \xrightarrow{\hspace{2em}} & E_{2i} \\ {}_{f_1}\nwarrow & & \nearrow_{f_2} \\ & S^{2i} & \end{array}$$

is homotopy commutative because $f_1^*(V(\gamma_i)) = \mathbf{1}^{2i}$ which is represented by $f_2 : S^{2i} \to E_{2i}$. The canonical complex orientation $\rho_{MU} \in \tilde{h}^2(CP^\infty; MU)$ is given by the inclusion $i : CP^\infty \to MU_1$ and $T(\rho_{MU}) = i^*(V(\gamma_1)) = \rho$.

To show uniqueness, let $I_n \in \tilde{h}^{2n}(MU_n ; MU)$ be the class represented by $\mathrm{Id} : MU_n \to MU_n$. Then the requirement that $T'(\rho_{MU}) = \rho$ determines $T'(I_1)$. Since T' is multiplicative, $T'(I_1 \wedge I_1 \wedge \cdots \wedge I_1) \in \tilde{h}^{2n}(MU_1 \wedge \cdots \wedge MU_1)$ is also uniquely determined. Let $f : MU_1 \wedge \cdots \wedge MU_1 \to MU_n$ be the ring map of MU and note that $f^*(I_n) = I_1 \wedge \cdots \wedge I_1$. From the diagram

$$\begin{array}{ccc} \tilde{h}^{2n}(MU_n ; MU) & \xrightarrow{T'} & \tilde{h}^{2n}(MU_n) \\ f^* \Big\downarrow & & f^* \Big\downarrow \text{ monic} \\ \tilde{h}^{2n}(MU_1 \wedge \cdots \wedge MU_1 ; MU) & \xrightarrow{T'} & \tilde{h}^{2n}(MU_1 \wedge \cdots \wedge MU_1) \end{array}$$

it is seen that $T'(I_n)$ is determined and this implies uniqueness of T'.

Finally, the map of spectra $T' : MU \to E$ induces a map of the associated Ω-spectra $T : \overline{MU} \to \bar{E}$, but since $\bar{E} = E$, this completes Theorem 8.1. T' is, of course, merely the composition $MU \to \overline{MU} \to E$.

Note that if $E = \overline{MSO}$ and ρ is the canonical orientation, then $T : \overline{MU} \to \overline{MSO}$ is merely the canonical transformation induced by $MU \subset MSO$. As another example, take $E = K(Z)$. Then $T : \overline{MU} \to K(Z)$ is the canonical transformation induced by $MU \subset K(Z)$ and sends the $\bar{c}_i \in h^{2i}(X, \overline{MU})$ to the ordinary Chern classes $c_i \in H^{2i}(X; Z)$.

9. K-Theory and cobordism

The purpose of this section is to present the theorem of Conner-Floyd, that for connected X, $\tilde{K}(X)$ is an additive direct summand of $\tilde{h}^{2i}(X; \overline{MU})$ for $i = 1, 0, -1, \cdots$.

LEMMA 9.1. *Let $g : \tilde{K}(-) \to \tilde{K}(-)$ be an additive natural transformation such that $g(\xi - 1) = 0$ where $(\xi - 1) \in \tilde{K}(CP^\infty)$. Then if X is connected, $g[\tilde{K}(X)] = 0$.*

Proof. The proof is based upon the fact that K^* is a complex orientable r-cohomology theory. According to the splitting principle, if

$$f : Y_n = CP^\infty \times \cdots \times CP^\infty \to BU_n$$

classifies $\xi \times \xi \times \cdots \times \xi$, then $f^* : K(BU_n) \to K(Y_n)$ is monic. Since
$*f(\gamma_n - \mathbf{n}) = (\xi \times \cdots \times \xi - \mathbf{n}) = (\xi - \mathbf{1}) \times (\xi - \mathbf{1}) \times \cdots \times (\xi - \mathbf{1})$,
it follows that $g(\gamma_n - \mathbf{n}) = 0$. If X is any connected finite-dimensional complex, any $x \in \tilde{K}(X)$ may be written as $x = (\alpha^n - \mathbf{n})$ for some n, and therefore $g(x) = 0$. Now consider

$$\mathrm{Id} : BU \to BU \times O \subset BU \times Z,$$

i.e. $(\mathrm{Id}) \in \tilde{K}(BU)$. The class $g(\mathrm{Id}) \in \tilde{K}(BU)$ is zero when restricted to any finite skeleton. According to 6.1 part II, $\tilde{K}(BU)$ has no phantom classes, and therefore $g(\mathrm{Id}) = 0$ and the lemma follows.

Consider the r-cohomology theory K^* and identify K^{2i} with K. Let $\rho = (\xi - \mathbf{1}) \in \tilde{K}(CP^\infty)$ be the canonical complex orientation for K-theory. The natural transformation $T : \tilde{h}^*(-; \overline{MU}) \to \tilde{K}^*(-)$ defines an additive and multiplicative transformation $T : \tilde{h}^{\text{even}}(-; \overline{MU}) \to \tilde{K}(-)$. Since T preserves unit elements,

$$T(\mathbf{1}^{2i}) = (\xi_1 - \mathbf{1}) \wedge (\xi_1 - \mathbf{1}) \wedge \cdots \wedge (\xi_1 - \mathbf{1}).$$

THEOREM 9.2. *Let $K(-)$ and $h(-) = h(-; \overline{MU})$ have their canonical complex orientations, X be a connected complex, and i be a positive integer.*

(1) *The following is commutative.*

$$\begin{array}{ccc} \tilde{K}(X \wedge S^{2i}) & \xleftarrow{T} & \tilde{h}^2(X \wedge S^{2i}) \\ \approx\uparrow & & \approx\uparrow \\ \tilde{K}(X) & \xleftarrow{T} & \tilde{h}^{2-2i}(X) \end{array}$$

(2) *The composition*

$$\tilde{K}(X) \xrightarrow{c_1} \tilde{h}^2(X) \xrightarrow{T} \tilde{K}(X)$$

is the identity.

(3) *The composition of the following four maps is the identity (starting with $\tilde{K}(X)$).*

$$\begin{array}{ccc} \tilde{K}(X \wedge S^{2i}) & \xrightarrow{c_1} & \tilde{h}^2(X \wedge S^{2i}) \\ \approx\uparrow & & \approx\downarrow \\ \tilde{K}(X) & \xleftarrow{T} & \tilde{h}^{2-2i}(X) \end{array}$$

Proof. The isomorphisms in (1) are given by multiplication by

$$(\xi_1 - \mathbf{1}) \wedge (\xi_1 - \mathbf{1}) \wedge \cdots \wedge (\xi_1 - \mathbf{1}) \in \tilde{K}(S^{2i})$$

and $\mathbf{1}^{2i} \in \tilde{h}^{2i}(S^{2i})$. The diagram commutes because T is multiplicative and preserves units.

Part (2) will follow from 9.1 if it holds for $X = CP^\infty$ and the class $(\xi - \mathbf{1}) \in \tilde{K}(CP^\infty)$. From Theorem 6.1, I (3), it follows that $c_1(\xi - \mathbf{1}) =$ the canonical orientation in $\tilde{h}^2(CP^\infty)$. From Theorem 8.1, T sends the canonical orientation for h to the orientation for K. Thus $Tc_1(\xi - \mathbf{1}) = (\xi - \mathbf{1})$.

The proof of (3) follows from (1) and the fact that the composition

$$\tilde{K}(X \wedge S^{2i}) \xrightarrow{c_1} \tilde{h}^2(X \wedge S^{2i}) \xrightarrow{T} \tilde{K}(X \wedge S^{2i})$$

is the identity.

10. Chern classes in K-theory

Chern classes are defined $c_i : K(X) \to h^{2i}(X)$ where h is any r-theory with a complex orientation. In this section we restate the results of the previous sections for the special case $h = K$ and for the Chern classes $c_i : K(X) \to K(X)$. Most of the proofs are immediate and thus omitted.

An orientation for α^n over a connected X is a class $V \in \tilde{K}(T(\alpha))$ which, when restricted to a fibre S^{2n} yields

$$\pm(\xi_1 - \mathbf{1}) \wedge (\xi_1 - \mathbf{1}) \wedge \cdots \wedge (\xi_1 - \mathbf{1}).$$

A complex orientation for K-theory is a class $\rho \in \tilde{K}(CP^\infty)$ such that ρ restricts to $\pm(\xi_1 - \mathbf{1}) \in \tilde{K}(S^2)$. The canonical orientation is $\rho_K = (\xi - \mathbf{1}) \in \tilde{K}(CP^\infty)$.

The following is a special case of the Thom-Dold isomorphism. A fibration $\pi : E \to B$ is a map satisfying Theorem 2.1 parts (1) and (2).

THEOREM 10.1. *Suppose $\pi : E \to B$ is a fibration with fibre F. Suppose $A \subset B$, $E' \subset E$, $F' \subset F$, and $\pi^{-1}(A) \subset E$ are subcomplexes. Suppose $K^1(F, F') = 0$ and $\exists$ classes $c_i \in K(E, E')$ for $i = 1, 2, \cdots, k$ such that for each $b \in B$, $j_b^*(c_1)$, $j_b^*(c_2)$, $\cdots$ $j_b^*(c_k)$ is a free abelian basis for the additive group $K(F, F')$. (Here*

$$j_b : (F, F') \to (\pi^{-1}(b), \pi^{-1}(b) \cap E')$$

is a homotopy equivalence). Then

$$f : K(B, A) \oplus K(B, A) \oplus \cdots \oplus K(B, A) \to K(E, E' \cup \pi^{-1}(A))$$

defined by

$$f(x_1, x_2, \cdots x_k) = \pi^*(x_1)c_1 + \cdots \pi^*(x_k)c_k$$

is an additive isomorphism.

The next theorem is the splitting principle for K-theory (Theorem 5.1).

THEOREM 10.2. *Let α^n be a complex bundle over a CW complex X. Then*

$$\pi^* : K^i(X) \to K^i(P(\alpha))$$

is monic for $i = 1, 2$. There exists a space Y and a map $f : Y \to X$ such that $f^*(\alpha)$ is the sum of line bundles and $f^* : K^i(X) \to K^i(Y)$ is monic for $i = 1, 2$.

Also the map

$$Y_n = CP^\infty \times \cdots \times CP^\infty \to BU_n$$

which classifies $\xi \times \xi \times \cdots \times \xi$ *induces a monic homomorphism* $K(BU_n) \to K(Y_n)$.

Stated next is the analogue of 6.1 and 7.1. The statement is somewhat complicated by the fact that the classes c_i do not determine the total Chern class c unless $K(X)$ has no phantom classes.

THEOREM 10.3. *Let* $\rho \in \tilde{K}(CP^\infty)$ *be a complex orientation for K-theory.*
I. *There exists a unique collection* $\{c, c_i, i = 1, 2, \cdots\}$ *assigning to each CW complex X and* $x \in K(X)$ *elements* $c(x) \in K(X)$ *and* $c_i(x) \in K(X)$ *for* $i = 1, 2, \cdots$ *satisfying the following.*

(0) *The class* $c_i(x) = 0$ *when restricted to* X^{2i-1}. *For any finite-dimensional skeleton* X^j, $c(x)$ *and* $\mathbf{1} + c_1(x) + c_2(x) + \cdots$ *are equal when restricted to* X^j. *If* α *is a complex n-bundle over X,* $c_i(\alpha) = 0$ *for* $i > n$ *and* $c(\alpha) = \mathbf{1} + c_1(\alpha) + \cdots + c_n(\alpha)$.

(1) *If* $g : X \to Y$, $y \in K(Y)$ *and* $g^*(y) = x$ *where* $g^* : K(Y) \to K(X)$, *then* $g^*[c_i(y)] = c_i(x)$ *for all* $i \geq 0$ *and* $g^*[c(y)] = c(x)$.

(2) *If* $x_1, x_2 \in K(X)$ *then* $c(x_1 + x_2) = c(x_1)c(x_2)$ *and*

$$c_n(x_1 + x_2) = c_0(x_1)c_n(x_2) + c_1(x_1)c_{n-1}(x_2) + \cdots + c_n(x_1)c_0(x_2)$$

for $n > 0$.

(3) *If* ξ *is the Hopf bundle over* CP^∞ *and* $k \geq 0$, *then*

$$c(\xi - \mathbf{k}) = \mathbf{1} + \rho \in K(CP^\infty).$$

(4) *If X is connected,* $c_i(x) \in \tilde{K}(X)$ *for all* $x \in K(X)$, $i > 0$. *Also if*

$$x_1 \oplus x_2 \in \tilde{K}(X) \oplus K(pt) = K(X)$$

then $c(x_2) = \mathbf{1}$, $c_i(x_2) = 0$ *for* $i > 0$, $c(x_1 \oplus x_2) = c(x_1)$, $c_i(x_1 \oplus x_2) = c_i(x_1)$ *for* $i > 0$.

I′. *Part* I *above holds if the domain K is replaced by K′, i.e.* $c_i : K'(X) \to K(X)$.

II. *Let n and k be positive integers,* Y_k *be the k-fold product*

$$Y_k = BU_n \times BU_n \times \cdots \times BU_n,$$

and $\pi_i : Y_k \to BU_n$ *be the projection on the i-factor. Then*

(1) $K(Y_k)$ *has no phantom classes,*

(2) $K(Y_k)$ *is a power series ring with variables*

$$\pi_1^*(c_1(\gamma_n)), \cdots, \pi_k^*(c_1(\gamma_n)), \pi_1^*(c_2(\gamma_n)), \cdots, \pi_k^*(c_n(\gamma_n)).$$

(3) $K^1(Y_k) = 0$

III. (1) *The sequence*

$$0 \leftarrow K(BU_{n-1}) \leftarrow K(BU_n) \leftarrow \tilde{K}(MU_n) \leftarrow 0$$

is split exact. $K^1(MU_n) = 0$.

(2) *If* $y \in K(BU)$ *is a class such that for each inclusion*

$$i_n : BU_n \to BU,\ i_n^*(y) = 0,$$

then $y = 0$.

(3) *Let* Y_k *be the k-fold product* $Y_k = BU \times BU \times \cdots \times BU$. *Then* $K(Y_k)$ *has no phantom classes.* $K^1(Y_k) = 0$.

(4) *Let* $c_i(\gamma) \in K(BU)$ *be the unique class which, when restricted to* BU_n *gives* $c_i(\gamma_n)$. *Then* $K(BU)$ *is a power series ring with variables* $c_1(\gamma), c_2(\gamma), \cdots$ *and coefficient ring* Z. *Also* $K(Y_k)$ *is a power series ring with variables*

$$\pi_1^*(c_1(\gamma)),\ \pi_2^*(c_1(\gamma)),\ \cdots,\ \pi_k^*(c_1(\gamma)),\ \pi_1^*(c_2(\gamma)),\ \cdots$$

IV. *There is a unique function* V *which assigns to each complex bundle* α^n *over a connected base, a K-orientation class* $V(\alpha) \in \tilde{K}(T(\alpha))$ *and satisfies the following.*

(a) $V(\xi) = p^*(\rho)$ *where* $p : T(\xi) \to CP^\infty$ *is the homeomorphism which is homotopic to the identity* $\mathrm{Id} : T(\xi) \to T(\xi)$.

(b) *If* α_n *is a bundle over* X *then* $V(\alpha)$ *maps to* $c_n(\alpha)$ *under*

$$\tilde{K}(T(\alpha)) \approx K(E(\alpha)) \to K(E(\alpha)) \approx K(X).$$

(c) *If* α *and* β *are complex bundles over* X *and* Y *resp.,*

$$V(\alpha \times \beta) = V(\alpha) \wedge V(\beta) \in \tilde{K}(T(\alpha \times \beta) = T(\alpha) \wedge T(\beta)).$$

(d) $V(\mathbf{1} \oplus \alpha) = V(\mathbf{1}) \wedge V(\alpha)$

$$= \pm(\xi_1 - \mathbf{1}) \wedge V(\alpha) \in \tilde{K}(T(\mathbf{1} \oplus \alpha) = S^2 \wedge T(\alpha)).$$

(e) *If* $f : X \to Y$ *is covered by a bundle map from* α *to* β *then* $T(f)^*[V(\beta)] = V(\alpha)$.

V. *Let* $\bar{\rho} \in \tilde{K}(CP^\infty)$ *be defined by* $\bar{\rho} = \rho + a_2\rho^2 + a_3\rho^3 + \cdots$ *where* $a_i \in Z$. *Then* $\bar{\rho}$ *is a complex orientation for K-theory. Furthermore, if* $\bar{\rho}$ *is an orientation for* K, *then* $\pm\bar{\rho}$ *may be written in this form.*

Let $\bar{\rho} = \rho + a_2\rho^2 + a_3\rho^3 + \cdots$ *and* $\bar{c}$, $\bar{c}_i$ *be the corresponding Chern classes. Then* $c(x) = \bar{c}(x)$ *and* $c_i(x) = \bar{c}_i(x)$ *when restricted to* X^{2i}. *Furthermore, if* $\bar{\rho} = -\rho$, *then* $\bar{c}_i(x) = (-1)^i c_i(x)$ *for* $i \geq 0$.

The canonical Chern classes for K-theory will now be computed. If α^n is a complex bundle over X, let $\lambda^i(\alpha)$ denote the i-exterior power of α. Then $\lambda^0(\alpha) = \mathbf{1}, \lambda^1(\alpha) = \alpha, \cdots, \lambda^n(\alpha) =$ a line bundle, $\lambda^i(\alpha) = 0$ for $i > n$. Also

$$\lambda^r(\alpha \oplus \beta) = \oplus_{p+q=r}[\lambda^p(\alpha) \otimes \lambda^q(\beta)].$$

THEOREM 10.4. *Let* $c, c_i : K(X) \to K(X)$ *be the canonical Chern classes, i.e. those determined by* $\rho = (\xi - \mathbf{1}) \in \tilde{K}(CP^\infty)$. *Then if* α^n *is a complex bundle,*

$$c_0(\alpha) = \lambda^0(\alpha) = \mathbf{1}, \qquad c_1(\alpha) = \lambda^1(\alpha) - \mathbf{n} = \alpha - \mathbf{n},$$

$$c_2(\alpha) = \lambda^2(\alpha) - (n-1)\lambda^1(\alpha) + \mathbf{n}(\mathbf{n} - \mathbf{1})/2,$$

$$\cdots, c_n(\alpha) = \lambda^n(\alpha) - \lambda^{n-1}(\alpha) + \cdots + (-1)^{n-1}\lambda^1(\alpha) + (-\mathbf{1})^n,$$

and in general $c_i(\alpha) = \oplus_{0 \le t \le i} (-1)^t C(n - i + t, t)\lambda^{i-t}(\alpha)$. *The total Chern class is* $c(\alpha) = \lambda^n(\alpha)$. *Finally, if* X *is connected and* $x \in \tilde{K}(X)$, *then* $c_1(x) = x$.

Proof. It is only necessary to prove the formula for the classifying bundle γ_n over BU_n and by the splitting principle, it suffices to consider $\alpha = \xi \times \cdots \times \xi$ over $Y_n = CP^\infty \times \cdots \times CP^\infty$. Let $\alpha_i = \pi_i^*(\xi)$ where $\pi_i : Y_n \to CP^\infty$ is projection on i-factor. By hypothesis, $c_1(\xi) = \rho = (\xi - \mathbf{1})$ and thus $c_1(\alpha_i) = \alpha_i - \mathbf{1}$. Thus

$$c_i(\alpha) = \oplus[(\alpha_{n(1)} - \mathbf{1}) \otimes (\alpha_{n(2)} - \mathbf{1}) \otimes \cdots \otimes (\alpha_{n(i)} - \mathbf{1})]$$

where the sum is over all partitions $1 \le n(1) < n(s) < \cdots < n(i) \le n$, i.e., $c_i(\alpha)$ is the i-symmetric function in the $(\alpha_j - \mathbf{1})$. Note that $\lambda^i(\alpha) =$ the i-symmetric function in the α_j. Expanding out the expression above for $c_i(\alpha)$ gives

$$c_i(\alpha) = \oplus_{0 \le t \le i} (-1)^t C(n - i + t, t)\lambda^{i-t}(\alpha).$$

Now $c(\alpha_i) = \mathbf{1} + c_1(\alpha_i) = \alpha_i$ and thus

$$c(\alpha) = c(\alpha_1)c(\alpha_2) \cdots c(\alpha_n) = \alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_n = \lambda^n(\alpha).$$

Let $(\mathrm{Id}) \in \tilde{K}(BU)$ be the class determined by the identity

$$\mathrm{Id} : BU \to BU \times O \subset BU \times Z.$$

To show $c_1(x) = x$ for all $x \in \tilde{K}(X)$, it suffices to show $c_1(\mathrm{Id}) = (\mathrm{Id})$. The class (Id) is $\gamma_n - \mathbf{n}$ when restricted to BU_n. Since $c_1(\gamma_n - \mathbf{n}) = \gamma_n - \mathbf{n}$, the result follows from Theorem 10.3, III (2). This completes 10.4.

The canonical Chern classes above do not agree with the classes γ^i constructed in [27, p. 163]. They do, however, agree for any $x \in \tilde{K}(X)$ where X is a connected finite complex.

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