A MAPPING PROPERTY OF REGRESSIVE ISOLS

BY
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1. Introduction

Let ε^*, ε, Λ, Λ_R and Λ^* denote the collections of all integers, non-negative integers, isols, regressive isols and isolic integers respectively. Let $f(x_1, \cdots, x_n)$ be a recursive function, and let f_Λ denote the canonical extension of f to a mapping from Λ^n into Λ^* [11], [12]. A. Nerode proved in [12] that f_Λ maps Λ^n into Λ if and only if f is almost recursive combinatorial. In [3], J. Barback proved that if $f(x)$ is a recursive function of one variable, f_Λ maps Λ_R into Λ_R if and only if f is eventually increasing. However the author showed in [10] that the class of recursive functions of two variables mapping $\Lambda_R \times \Lambda_R$ into Λ_R is rather limited—trivial cases aside, this class consists of functions eventually of the form

$$\min\ (f(x), g(y)) + \sum_{i=0}^{x} \sum_{j=0}^{y} d(i, j)$$

where min (x, y) is the minimum function, $f(x)$ and $g(y)$ are eventually increasing recursive, and $d(i, j) = 0$ for all but finitely many pairs (i, j). The restrictive nature of this last result is not surprising in view of the fact that Λ_R is not closed under addition or multiplication. J. Barback suggested that it is more natural to ask which recursive functions $f(x_1, \cdots, x_n)$ of more than one variable have the property

$$(*) \qquad A_1 + A_2 + \cdots + A_n \in \Lambda_R \Rightarrow f_\Lambda(A_1, A_2, \cdots, A_n) \in \Lambda_R .$$

The main theorem of this paper characterizes the class of recursive functions with the property $(*)$ as follows: Let $\leq$ be the partial ordering of ε^n obtained by setting

$$(x_1, \cdots, x_n) \leq (y_1, \cdots, y_n) \quad \text{iff} \quad x_i \leq y_i, \qquad i = 1, \cdots, n.$$

A function $f(x_1, \cdots, x_n)$ is called *increasing* if

$$(x_1, \cdots, x_n) \leq (y_1, \cdots, y_n) \Rightarrow f(x_1, \cdots, x_n) \leq f(y_1, \cdots, y_n)$$

and *eventually increasing* if there exists a number $k \in \varepsilon$ such that

$$f(x_1 + k, \cdots, x_n + k)$$

is increasing. A function g of fewer than n variables is called a *proper specification* of $f(x_1, \cdots, x_n)$ if g can be obtained from f by substitution of constants for some of the variables $x_1, \cdots, x_n$ of f. f is called *almost increasing* if f and every proper specification of f is eventually increasing.

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THEOREM. *Let $f(x_1, \cdots, x_n)$ be recursive. f has the property $(*)$ if and only if f is almost increasing.*

We note that for $n = 1$ the preceding theorem reduces to the theorem in [3] for functions of one variable. For the sake of simplicity, we shall give the proof only for $n = 2$; the proof for $n > 2$ requires no essential modification of the techniques to be presented here.

The mapping theorem for Λ in [12] can be used to prove the truth in Λ of certain $\forall\exists$ sentences with recursive combinatorial Skolem functions. In the final section of this paper we shall apply our main theorem in a similar manner to determine the truth or falsehood in Λ_R of various sentences with recursive Skolem functions.

2. Preliminaries

We assume that the reader is familiar with the basic concepts and results of [1], [3], [6], [7], [10] and [14]. The following definitions and theorems concerning functions of two variables are the natural analogues of the concepts and results in Section 2 of [14] and Section 4 of [1]. The theorems can be proved using the methods of those papers, and will be stated without proof.

By a number-theoretic function of n variables we shall mean a function mapping ε^n into ε^*. Every number-theoretic function f can be written as the difference of two combinatorial functions f^+ and f^-, called the positive and negative parts of f. A number-theoretic function f is called recursive if the functions f^+ and f^- are recursive. For a recursive number-theoretic function $f(x_1, \cdots, x_n)$, we can employ the usual canonical extension procedure to define f_Λ, i.e., for any n-tuple of isols $(x_1, \cdots, x_n)$,

$$f_\Lambda(x_1, \cdots, x_n) = f_\Lambda^+(x_1, \cdots, x_n) - f_\Lambda^-(x_1, \cdots, x_n).$$

Let $f(x, y)$ be recursive and number-theoretic. For $T, U \in \Lambda_R$ we define

$$\textstyle\sum^*_{(T,U)} f(x, y) = \sum_{(T,U)} f^+(x, y) - \sum_{(T,U)} f^-(x, y).$$

By the *partial sum function* of $f(x, y)$ we mean the function

$$S_f(x, y) = \textstyle\sum_{i<x} \sum_{j<y} f(i, j)$$

$(S_f(x, y) = 0$ if $x = 0$ or $y = 0)$.

PROPOSITION 1. *Let $f(x, y)$ and $g(x, y)$ be recursive number-theoretic functions and $T, U \in \Lambda_R$. Then*

$$\textstyle\sum^*_{(T,U)} f(x, y) \pm \sum^*_{(T,U)} g(x, y) = \sum^*_{(T,U)} (f(x, y) \pm g(x, y)).$$

THEOREM 1. *Let $f(x, y)$ be recursive and number-theoretic. Then for all T, $U \in \Lambda_R$, $\sum^*_{(T,U)} f(x, y) = (S_f)_\Lambda(T, U)$.*

For any recursive function $f(x, y)$, we define

$$\begin{aligned}
\hat{f}(x, y) &= 0, && \text{if } x = 0 \text{ or } y = 0 \\
&= f(x-1, y-1), && \text{otherwise.} \\
\Delta_x f(x, y) &= f(x+1, y) - f(x, y), \\
\Delta_y f(x, y) &= f(x, y+1) - f(x, y), \\
Df(x, y) &= \Delta_x \Delta_y \hat{f}(x, y), \\
Df^+(x, y) &= Df(x, y), && Df(x, y) \geq 0 \\
&= 0, && \text{otherwise} \\
Df^-(x, y) &= -Df(x, y), && Df(x, y) \leq 0 \\
&= 0, && \text{otherwise.}
\end{aligned}$$

THEOREM 2. *Let $f(x, y)$ be recursive. Then for T, $U \in \Lambda_R$,*

$$f_\Lambda(T, U) = \sum_{(T+1,U+1)} Df^+ - \sum_{(T+1,U+1)} Df^- = \sum^*_{(T+1,U+1)} Df$$

In particular, for n, $k \in \varepsilon$,

$$f(n, k) = \sum_{i=0}^n \sum_{j=0}^k Df^+(i, j) - \sum_{i=0}^n \sum_{j=0}^k Df^-(i, j).$$

Throughout the remainder of this paper we shall use the notation and terminology introduced in Section 2 of [14]. We shall also use the notations $j(\alpha, \beta)$ for

$$\{j(x, y) \mid x \in \alpha \text{ and } y \in \beta\}$$

and $\alpha \mid \beta$ for "α is separable from β by disjoint r.e. sets".

3. Proof of the main theorem for $n = 2$

LEMMA 1. *Let $f(x, y)$ be an increasing recursive function. Let m, p, k, l be numbers such that $0 \leq m \leq p$ and $0 \leq k \leq l$. Define*

$$W = \{(i, j) \mid (i, j) \leq (p, l) \quad \text{and} \quad (i, j) \nleq (m, k)\}.$$

Then $\sum_{(i,j) \in W} Df(i, j) \geq 0$.

Proof.

$$\begin{aligned}
\sum_{(i,j) \in W} Df(i, j) &= \sum_{i=0}^p \sum_{j=0}^l Df(i, j) - \sum_{i=0}^m \sum_{j=0}^k Df(i, j) \\
&= f(p, l) - f(m, k) \geq 0.
\end{aligned}$$

LEMMA 2. *Let A, $B \in \Lambda_R - \varepsilon$. Let $f(x, y)$ be increasing and recursive. If $A + B \in \Lambda_R$, then $f_\Lambda(A, B) \in \Lambda$.*

Proof. By Theorem 2, we need only prove that

$$\sum_{(A+1,B+1)} Df^+ \geq \sum_{(A+1,B+1)} Df^- \tag{2.1}$$

Let α and β be sets belonging to $A + 1$ and $B + 1$ respectively, and let a_n and b_n be regressive functions ranging over α and β respectively. Since $A + B \in \Lambda_R$, $(A + 1)\cdot(B + 1) \in \Lambda_R$. Let u_n be any regressive function ranging over $j(\alpha, \beta)$. Define

$$\delta^+ = \bigcup_{(i,j)\in\varepsilon^2} j_3(a_i, b_j, \nu(Df^+(i, j)))$$

$$\delta^- = \bigcup_{(i,j)\in\varepsilon^2} j_3(a_i, b_j, \nu(Df^-(i, j)))$$

Clearly (2.1) will follow if we show the existence of a one-to-one partial recursive function $p(x)$ such that

(2.2) $\delta^- \subset \text{domain } (p), \quad p(\delta^-) \subset \delta^+ \quad \text{and} \quad p(\delta^-) \mid \delta^+ - p(\delta^-).$

We shall prove the existence of a one-to-one correspondence $z \leftrightarrow g(z)$ which associates with each member z of δ^- a member $g(z)$ of δ^+ in such a manner that

(2.3) given z we can effectively find $g(z)$ and vice-versa, and

(2.4) $g(\delta^-) \mid \delta^+ - g(\delta^-)$.

This will complete the proof by Proposition 1 of [6].

The description of the correspondence $z \leftrightarrow g(z)$ requires the following series of definitions.

Let q_1, q_2 and r be regressing functions of a_n, b_n and u_n respectively. Let ζ be any finite subset of $j(\alpha, \beta)$. Define

$$\zeta_1 = \{k(x) \mid x \in \zeta\}, \qquad \zeta_2 = \{l(x) \mid x \in \zeta\}.$$

Let $\zeta_1 = \{a_{i(1)}, a_{i(2)}, \cdots, a_{i(s)}\}$ and $\zeta_2 = \{b_{j(1)}, \cdots, b_{j(t)}\}$ where $i(1) < i(2) < \cdots < i(s)$ and $j(1) < j(2) < \cdots < j(t)$.

Define

$$P(\zeta) = \{j(a_i, b_j) \mid i < i(s) \quad \text{and} \quad j < j(t)\},$$

$$R(\zeta) = \{r^k(j(a_i, b_j)) \mid a_i \in \zeta_1 \quad \text{and} \quad b_j \in \zeta_2 \quad \text{and} \quad k \in \varepsilon\},$$

$$B(\zeta) = P(R(\zeta)), \qquad B_x(\zeta) = \bigcup_{k=1}^{\infty} B^k(\zeta).$$

We note that $\zeta \subset B_x(\zeta)$ and that $B^i(\zeta) \subset B^{i+1}(\zeta)$ for $i \geq 1$. It is clear that given ζ we can effectively find $B^i(\zeta)$ for any $i \geq 1$. Thus $B_x(\zeta)$ is an r.e. subset of the isolated product set $j(\alpha, \beta)$. It follows immediately that given a finite set ζ we can find $B_x(\zeta)$ by generating the sequence of sets $B(\zeta)$, $B^2(\zeta)$, $\cdots$ until a repetition appears. Furthermore, the definition of P yields the existence of a pair $(a_k, b_l) \in B_x(\zeta)$ such that

$$B_x(\zeta) = \{j(a_i, b_j) \mid (i, j) \leq (k, l)\}. \tag{2.5}$$

We now define sequences of sets $\{S_n\}$ and $\{W_n\}$ as follows:

$$S_0 = B_x(\{j(a_0, b_0)\}),$$

$$S_{n+1} = B_x(S_n \cup \{j(a_k, b_m)\})$$

where $j(a_k, b_m) = u_t$ and $t = (\mu y)[u_y \notin S_n]$,

$$W_0 = S_0, \qquad W_{n+1} = S_{n+1} - S_n,$$
$$\tilde{W}_n = \{(i, j) \mid j(a_i, b_j) \in W_n\}.$$

Clearly given any member of W_n we can use the given regressing functions to list the sets $S_0, \cdots, S_n$ and thus obtain $W_0, W_1, \cdots, W_n, \tilde{W}_0, \cdots, \tilde{W}_n$. We note that by (2.5) and the definition of W_{n+1}, for each $n \in \varepsilon$ there exist numbers m, p, k, l such that $(m, k) \leq (p, l)$ and

$$\tilde{W}_{n+1} = \{(i, j) \mid (i, j) \leq (p, l) \quad \text{and} \quad (i, j) \nleq (m, k)\}.$$

Hence by Lemma 2, for $n > 0$,

$$\textstyle\sum_{(i,j)\in\tilde{W}_n} Df^+(i, j) - \sum_{(i,j)\in\tilde{W}_n} Df^-(i, j) \geq 0. \tag{2.6}$$

For $n = 0$, the left-hand expression in (2.6) is merely a member of the range of the recursive $f(x, y)$. Hence (2.6) holds for all $n \in \varepsilon$.

We now define the correspondence $z \leftrightarrow g(z)$. Let $z \in \delta^-$. Let n be the unique number such that $j(k_1(z), k_2(z)) \in W_n$. Define

$$\gamma_1^n = \textstyle\bigcup_{(i,j)\in\tilde{W}_n} j_3(a_i, b_j, \nu(Df^+(i, j)))$$
$$\gamma_2^n = \textstyle\bigcup_{(i,j)\in\tilde{W}_n} j_3(a_i, b_j, \nu(Df^-(i, j)))$$

Let $c_1 < c_2 < \cdots < c_s$ and $d_1 < d_2 < \cdots < d_t$ be finite sequences which array in increasing order all elements of the sets γ_1^n and γ_2^n respectively. By (2.6), $s \geq t$. Pair c_i with d_i, $i = 1, \cdots, t$. Since $z \in \gamma_2^n$, $z = d_k$ for some k, $1 \leq k \leq t$. Thus z is paired with c_k. Set $g(z) = c_k$. (Note that $g(z) \in \delta^+$, as required.)

That g has properties (2.3) and (2.4) is a consequence of the following observations. Let $y \in \delta^+ \cup \delta^-$ be given. Let $j(k_1(y), k_2(y)) \in W_n$. Then we can find all members of W_n, $\tilde{W}_n$, γ_1^n and γ_2^n, and list all pairs $(z, g(z))$, $z \in \gamma_2^n$. Examination of this list suffices to determine whether or not $y \in g(\delta^-)$. Furthermore if $y \in \delta^-$ $(y \in g(\delta^-))$, examination of the list suffices to determine the value of $g(y)$ $(g^{-1}(y))$.

LEMMA 3. *Let $A, B \in \Lambda_R - \varepsilon$. If $A \cdot B \in \Lambda_R$ and $h(x, y)$ is a recursive function such that $h(x, y) \geq 0$ for $x, y \in \varepsilon$, then $\sum_{A,B} h(x, y) \in \Lambda_R$.*

Proof. Let $\alpha \in A$ and $\beta \in B$. Let a_n and b_n be regressive functions with ranges α and β respectively. Let $p(x)$ be a regressing function for a_n and $q(x)$ a regressing function for b_n. Let $p^*(x)$ and $q^*(x)$ be partial recursive functions such that $p^*(a_n) = n$ and $q^*(b_n) = n$ for $n \in \varepsilon$. Let u_n be a regressive function ranging over $j(\alpha, \beta)$. Define $\tilde{a}_n = k(u_n)$ and $\tilde{b}_n = l(u_n)$. Then $j(\alpha, \beta) = \{j(\tilde{a}_n, \tilde{b}_n) \mid n \in \varepsilon\}$.

Define

$$\gamma = \textstyle\bigcup_{n=0}^{\infty} j_3(\tilde{a}_n, \tilde{b}_n, \nu[h(p^*(\tilde{a}_n), q^*(\tilde{b}_n))])$$

Then $\gamma \in \sum_{A,B} h(x, y)$. We array the elements of γ in the following fashion:

$$\begin{array}{ll} j_3(\tilde{a}_0, \tilde{b}_0\, 0), & \cdots, j_3(\tilde{a}_0, \tilde{b}_0, h(p^*(\tilde{a}_0), q^*(\tilde{b}_0)) - 1) \\ \vdots & \\ j_3(\tilde{a}_n, \tilde{b}_n, 0) & \cdots, j_3(\tilde{a}_n, \tilde{b}_n, h(p^*(\tilde{a}_n), q^*(\tilde{b}_n)) - 1) \\ j_3(\tilde{a}_{n+1}, \tilde{b}_{n+1}, 0), & \cdots, j_3(\tilde{a}_{n+1}, \tilde{b}_{n+1}, h(p^*(\tilde{a}_{n+1}), q^*(\tilde{b}_{n+1})) - 1) \\ \vdots & \end{array}$$

It can easily be shown that we can "regress" through this array by proceeding from right to left in each row, and from the $(n + 1)^{\text{st}}$ row to the n^{th}.

THEOREM 3. *Let $A, B \in \Lambda_R - \varepsilon$, and let $f(x, y)$ be an increasing recursive function. Then*

$$A + B \in \Lambda_R \Rightarrow f_\Lambda(A, B) \in \Lambda_R .$$

Proof. Observe that the function

$$h(x, y) = f(x, y) + \textstyle\sum_{i=0}^{x} \sum_{j=0}^{y} Df^{-}(i, j)$$

is a recursive function with the property that $Dh(x, y) \geq 0$ for all $(x, y) \in \varepsilon^2$. Extending the defining equation of h to Λ_R, we obtain

$$h_\Lambda(A, B) = f_\Lambda(A, B) + \textstyle\sum_{(A+1,B+1)} Df^{-}(x, y).$$

By Lemma 2, $f_\Lambda(A, B) \in \Lambda$, and by Lemma 3 $\sum_{(A+1,B+1)} Df^{-}(x, y) \in \Lambda_R$ and $h_\Lambda(A, B) \in \Lambda_R$.

Hence $f_\Lambda(A, B) \leq h_\Lambda(A, B)$, and $f_\Lambda(A, B) \in \Lambda_R$.

COROLLARY. *Let $f(x, y)$ be recursive and almost increasing. Then*

$$(*) \qquad A + B \in \Lambda_R \Rightarrow f_\Lambda(A, B) \in \Lambda_R .$$

Proof. Let A and B be regressive isols such that $A + B \in \Lambda_R$. Let k be a number such that $f(x + k, y + k)$ is increasing. We distinguish two cases.

Case 1. Either A or B is finite. Suppose A is finite. Define $g(y) = f(A, y)$ for $y \in \varepsilon$. Then $g_\Lambda(B) = f_\Lambda(A, B)$. Since g is a proper specification of f, g is eventually increasing. Hence by Theorem 4 of [3], $f_\Lambda(A, B) \in \Lambda_R$. The proof is completely similar if B is finite.

Case 2. A and B are both infinite isols. Let $g(x, y) = f(x + k, y + k)$. Then $f_\Lambda(A, B) = g_\Lambda(A - k, B - k)$. Since $g(x, y)$ is recursive and increasing, Theorem 3 yields $A + B \in \Lambda_R$.

Notation. We shall write $(x, y) < (z, w)$ if $(x, y) \leq (z, w)$ and $(x, y) \neq (z, w)$.

LEMMA 4. *Let $f(x, y)$ be a recursive function which is not eventually increasing. Then there exist infinite regressive isols A and B such that $A + B \in \Lambda_R$ and $f_\Lambda(A, B) \in \Lambda^* - \Lambda$.*

Proof. Since $f(x, y)$ is not eventually increasing, we can effectively generate a strictly increasing sequence

$$\{ (n_i, m_i, s_i, t_i) \mid i = 0, 1, \cdots \}$$

of four-tuples such that for $i \in \varepsilon$,

$$(2.7) \quad \begin{array}{c} (n_i, m_i) < (s_i, t_i) < (n_{i+1}, m_{i+1}), \qquad f(n_i, m_i) > f(s_i, t_i), \\ \lim_{i\to\infty} n_i = \lim_{i\to\infty} m_i = \infty . \end{array}$$

We note that each of the functions n_i, m_i, s_i and t_i is recursive and increasing. We define recursive functions a_i and b_i as follows:

$$\begin{array}{ll} a_0 = n_0, & b_0 = m_0 \\ a_1 = s_0 - n_0, & b_1 = t_0 - m_0 \\ \vdots & \vdots \\ a_{2k} = n_k - s_{k-1}, & b_{2k} = m_k - t_{k-1} \\ a_{2k+1} = s_k - n_k, & b_{2k+1} = t_k - m_k . \end{array}$$

We note that for $k \in \varepsilon$,

$$(2.8) \quad n_k = \sum_{i=0}^{2k} a_i, \quad s_k = \sum_{i=0}^{2k+1} a_i, \quad m_k = \sum_{i=0}^{2k} b_i, \quad t_k = \sum_{i=0}^{2k+1} b_i .$$

For $T \in \Lambda_R$, we define $A_T = \sum_T a_i$ and $B_T = \sum_T b_i$. Since

$$A_T + B_T = \sum_T (a_i + b_i), A_T + B_T \in \Lambda_R \quad \text{for } T \in \Lambda_R .$$

For $i \in \varepsilon$, define

$$\begin{aligned} W_i &= \{ (x, y) \mid (x, y) \leq (s_i, t_i) \text{ and } (x, y) \nleq (n_i, m_i)\}, \\ S_0 &= \{ (x, y) \mid (x, y) \leq (n_0, m_0)\}, \\ S_{i+1} &= \{ (x, y) \mid (x, y) \leq (n_{i+1}, m_{i+1}) \text{ and } (x, y) \nleq (s_i, t_i)\}. \end{aligned}$$

We note that the sequence of sets $S_0, W_0, S_1, W_1, \cdots$ is an infinite sequence of mutually disjoint sets whose union is ε^2, and that for $i \in \varepsilon$,

$$(2.9) \quad \begin{aligned} S_0 \sqcup W_0 \sqcup \cdots \sqcup W_{i-1} \sqcup S_i &= \{ (x, y) \mid (x, y) \leq (n_i, m_i)\} \\ S_0 \sqcup W_0 \sqcup \cdots \sqcup S_i \sqcup W_i &= \{ (x, y) \mid (x, y) \leq (s_i, t_i)\} \end{aligned}$$

We define a recursive function $g(x)$ mapping ε to ε^* by

$$g(2k) = \sum_{(i,j)\in S_k} Df(i, j), \qquad g(2k + 1) = \sum_{(i,j)\in W_k} Df(i, j).$$

By (2.8) and (2.9),

$$(2.10) \quad \begin{aligned} \sum_{i=0}^{2k} g(i) &= f(n_k, m_k) = f(\sum_{i=0}^{2k} a_i, \sum_{i=0}^{2k} b_i) \\ \sum_{i=0}^{2k+1} g(i) &= f(s_k, t_k) = f(\sum_{i=0}^{2k+1} a_i, \sum_{i=0}^{2k+1} b_i) \end{aligned}$$

Thus for $k \in \varepsilon$,

$$(2.11) \quad \sum_{i=0}^{k} g(i) = f(\sum_{i=0}^{k} a_i, \sum_{i=0}^{k} b_i).$$

By (2.10) and (2.7), for $k \in \varepsilon$,

$$g(2k + 1) = f(s_k, t_k) - f(n_k, m_k) < 0.$$

Thus g assumes negative values infinitely often. Extending (2.11) to Λ_R, we obtain for $T \in \Lambda_R$,

$$\textstyle\sum_{T+1}^{*} g(i) = f_\Lambda(\sum_{T+1} a_i, \sum_{T+1} b_i) = f_\Lambda(A_{T+1}, B_{T+1}).$$

It follows from statement (30) in the proof of Theorem 4 of [3] that there exists a regressive isol $T + 1$ such that $\sum_{T+1}^{*} g(i) \in \Lambda^* - \Lambda$. For such an isol $T + 1$,

$$f_\Lambda(A_{T+1}, B_{T+1}) \in \Lambda^* - \Lambda,$$

while $A_{T+1} + B_{T+1} \in \Lambda_R$.

COROLLARY. *If $f(x, y)$ is a recursive function which is not eventually increasing, there exist recursive functions a_n and b_n and an isol $T \in \Lambda_R$ such that*

$$f_\Lambda(\textstyle\sum_T a_n, \sum_T b_n) \in \Lambda^* - \Lambda.$$

THEOREM 4. *Let $f(x, y)$ be a recursive function. f has the property*

$$(*) \qquad A + B \in \Lambda_R \Rightarrow f_\Lambda(A, B) \in \Lambda_R$$

if and only if f is almost increasing.

Proof. We have already shown that the second condition implies the first. Suppose that f is not almost increasing. If f is not eventually increasing, f does not have the property $(*)$ by Lemma 4. If there is a number k such that the proper specification $f(x, k)$ is not eventually increasing, by Theorem 4 of [3] there exists a regressive isol T such that $f_\Lambda(T, k) \notin \Lambda_R$. Hence f does not have the property $(*)$. The case in which some specification $f(k, y)$ is not eventually increasing is handled similarly.

4. Applications

Myhill and Nerode have shown that if $\Phi(x_1, \cdots, x_n)$ is a quantifier-free Horn formula built up from equations between almost combinatorial recursive functions of $x_1, \cdots, x_n$ and $\Phi(x_1, \cdots, x_n)$ is true for all natural numbers, then $\Phi(x_1, \cdots, x_n)$ is true for all isols $x_1, \cdots, x_n$ [11], [12]. Using this result and the fact that any recursive function can be expressed as the difference of two recursive combinatorial functions, one can easily prove the following proposition.

PROPOSITION 2. *Let $\mathfrak{A}(x_1, \cdots, x_n)$ be a quantifier free Horn formula built up from equations between almost increasing recursive functions of $x_1, \cdots, x_n$. If $\mathfrak{A}(x_1, \cdots, x_n)$ is true for all natural numbers, then $\mathfrak{A}(x_1, \cdots, x_n)$ is true for all isols $x_1, \cdots, x_n$ such that $x_1 + x_2 + \cdots + x_n \in \Lambda_R$.*

COROLLARY. *Let $\mathfrak{A}(x_1, \cdots, x_n, y)$ be a quantifier free Horn formula built up from equations between almost increasing recursive functions of $x_1, \cdots, x_n, y$. If $(\exists y)\mathfrak{A}(x_1, \cdots, x_n, y)$ is true for all natural numbers $x_1, \cdots, x_n$ and has an* *almost increasing recursive Skolem function* $f(x_1, \cdots, x_n)$, *then* $(\exists y)\mathfrak{A}(x_1, \cdots, x_n, y)$ *is true for all isols* $x_1, \cdots, x_n$ *such that* $x_1 + x_2 + \cdots + x_n \in \Lambda_R$.

Proof. Immediate from Proposition 2 and the fact that the class of almost increasing functions is closed under composition. We leave verification of the latter fact to the reader.

Let $pr(n)$ denote the primitive recursive function which enumerates the prime numbers in increasing order. J. C. E. Dekker proved in [8] that $X^{pr(n)} \equiv X \pmod{pr(n)}$ for $X \in \Lambda, n \in \varepsilon$. J. Barback has shown in [2] that the class of all prime regressive isols has cardinality c and properly contains the class $\{pr_\Lambda(T) \mid T \in \Lambda_R\}$. Hence it is natural to ask if the above congruence holds if $pr(n)$ is replaced by an infinite regressive prime isol. The following corollary describes a class of regressive isols for which the congruence is satisfied.

COROLLARY. *Let* A *and* B *be regressive isols such that* $A + B \in \Lambda_R$. *Let* $P = pr_\Lambda(B)$. *Then* $A^P \equiv A \pmod{P}$.

Proof. Consider the formula

$$\mathfrak{A} = (\exists w)[(x + 1)^{pr(n)} = w \cdot pr(n) + (x + 1)].$$

This sentence is satisfied by all x and n in ε and has the recursive increasing Skolem function

$$W(x, n) = \frac{(x + 1)^{pr(n)} - (x + 1)}{pr(n)}.$$

By the previous corollary $\mathfrak{A}$ is satisfied by all $A, B \in \Lambda_R$ such that $A + B \in \Lambda_R$. This completes the proof in case $A \neq 0$; if $A = 0$, the congruence is clearly satisfied.

J. Barback has shown in [2] that there exist $A, B \in \Lambda_R$ such that

$$\min_\Lambda(A, B) \not\leq A + B,$$

but that the restriction $A + B \in \Lambda_R$ is sufficient to guarantee that $\min_\Lambda(A, B) \leq A + B$. The following proposition shows that this restriction is not sufficient to guarantee that $\min_\Lambda(A, B) \leq \max_\Lambda(A, B)$.

PROPOSITION 3. *There exist isols* A, B *such that*

$$A + B \in \Lambda_R \quad \textit{and} \quad \min_\Lambda(A, B) \not\leq \max_\Lambda(A, B).$$

Proof. Let $g(x, y) = \max(x, y) \dot{-} \min(x, y)$. Then the identity

$$(**) \qquad \min_\Lambda(A, B) + g_\Lambda(A, B) = \max_\Lambda(A, B)$$

holds in Λ^* for $A, B \in \Lambda_R$. However $g(x, y)$ is not eventually increasing. Let $A, B \in \Lambda_R$ be such that

$$A + B \in \Lambda_R \quad \text{and} \quad g_\Lambda(A, B) \in \Lambda^* - \Lambda.$$

Since $A + B \in \Lambda_R$, both $\min_\Lambda (A, B)$ and $\max_\Lambda (A, B)$ are regressive. By the identity (**), $\min_\Lambda (A, B) \lneq \max_\Lambda (A, B)$.

By the corollary to Lemma 4, Proposition 3 may be strengthened as follows.

PROPOSITION 4. *There exist recursive functions a_n and b_n and a regressive isol T such that*

$$\min_\Lambda \left(\textstyle\sum_T a_n, \sum_T b_n\right) \lneq \max_\Lambda \left(\textstyle\sum_T a_n, \sum_T b_n\right).$$

BIBLIOGRAPHY

1. J. BARBACK, *Double series of isols*, Canad. J. Math, vol. 19 (1967), pp. 1–15.
2. ———, *Prime regressive isols*, to appear.
3. ———, *Recursive functions and regressive isols*, Math. Scand., vol. 15 (1964), pp. 29–42.
4. J. C. E. DEKKER, *Closure properties of regressive functions*, Proc. London Math. Soc., vol. XV (1965), pp. 226–238.
5. ———, *Congruences in isols with a finite modulus*, Math. Zeitschr., vol. 70 (1958), pp. 113–124.
6. ———, *Infinite series of isols*, Proceedings of Symposia in Pure Mathematics, vol. 5, Amer. Math. Soc., Providence, R.I., 1962, pp. 77–96.
7. ———, *The minimum of two regressive isols*, Math. Zeitschr., vol. 84 (1964), 345–366.
8. J. C. E. DEKKER AND J. MYHILL, *Recursive equivalence types*, Univ. Calif. Publ. Math (N. S.), vol. 3 (1960), pp. 67–214.
9. M. HASSETT, *Some theorems on regressive isols and isolic groups*, Doctoral Thesis, Rutgers, The State University, 1966.
10. J. MYHILL, *Recursive equivalence types and combinatorial functions (Part I)*, Bull. Amer. Math. Soc., vol. 64 (1958), pp. 373–376.
11. ———, *Recursive equivalence types and combinatorial functions (Part II)*, Proc. Internat. Congress Math. (Stanford, 1962), Stanford Univ. Press, Stanford, 1960, pp. 46–55.
12. A. NERODE, *Extensions to isols*, Ann. Math., vol. 73 (1961), pp. 362–403.
13. ———, *Extensions to isolic integers*, Ann. of Math., vol. 75 (1962), pp. 419–448.
14. F. J. SANSONE, *A mapping of regressive isols*, Illinois J. Math, vol. 9 (1965), pp. 726–735.

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