ON SEMI-PERFECT RINGS

BY

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1. Main results

A ring is called semi-perfect if every finitely generated R-right-module has a projective cover. Equivalent conditions are: $\bar{R} = R/J$, J the Jacobson-radical, is semi-simple artinian and idempotents can be lifted modulo J; or every simple R-right-module is of the form eR/eJ, $e = e^2 \in R$. These rings have been studied recently by numerous people (e.g. Bass [1], Lambek [7], Mares [9], Kasch and Mares [5], Wu and Jans [11]), and most of the classical structure theory for artinian rings can be obtained for them. It is well known that for a semi-perfect ring R, every primitive idempotent e is local (eRe is a local ring, a ring with unique maximal ideal). Apparently it has not been observed that this property characterizes semi-perfect rings (cf. Lambek [8, §3.7, Prop. 3]).

THEOREM 1. *The following are equivalent for any ring R: (1) R is semi-perfect; (2) the unit $1 \in R$ is the sum of orthogonal local idempotents; (3) every primitive idempotent is local and there doesn't exist an infinite set of orthogonal idempotents in R.*

The (up to isomorphism finitely many) local rings eRe determine the structure of a semi-perfect ring R to a large extent. As an illustration we show

THEOREM 2. *A semi-perfect ring R is left-perfect, respectively semi-primary, if and only if all the local rings eRe are left-perfect, respectively semi-primary.*

The theorem of Kaplansky [4] that every projective module over a local ring is free, generalizes to semiperfect rings as follows:

THEOREM 3. *Every projective module over a semi-perfect ring is the direct sum of primitive ideals.*

2. Semi-perfect rings are generalized matrix-rings over local rings

Starting from a semi-perfect ring R and a decomposition $1 = e_1 + \cdots + e_n$ into primitive orthogonal idempotents we construct an additive category (cf. Mitchell [10]) as usual: Let $1, \cdots, n$ be the objects, $e_i R e_k$ the set of maps from i to k, composition of maps by ring-multiplication. Conversely beginning with an additive category with finitely many objects $1, \cdots, n$ whose endomorphism-rings are local, and sets X_{ik} of maps from i to k, we construct a generalized matrix-ring whose elements are matrices $(x_{ik})_{i,k=1}^{n}$, $x_{ik} \in X_{ik}$.

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Since the X_{ii} are local rings, this matrix-ring is semi-perfect, by Theorem 1. Since any two decompositions of the unit 1 of a semi-perfect ring are related by an inner automorphism, we obtain

THEOREM 4. *The above constructions yield a one-to-one correspondence between the isomorphism-types of semi-perfect rings, and of additive categories with finitely many objects whose endomorphism-rings are local.*

In such a category, the multiplication mappings

$$X_{ii} \times X_{ik} \to X_{ik}, \qquad X_{ik} \times X_{kk} \to X_{ik}$$

turn the X_{ik} $(i \neq k)$ into $X_{ii} - X_{kk}$-bimodules, and the $X_{ij} \times X_{jk} \to X_{ik}$ $(i \neq j, j \neq k)$ factor over the tensor-products, producing bimodule-homomorphisms

$$f_{ijk} : X_{ij} \otimes_{X_{jj}} X_{jk} \to X_{ik}$$

satisfying appropriate associativity conditions. It follows that a semi-perfect ring is describable, in an essentially unique way, by a system $(X_{ii}, X_{ik}, f_{ijk})$ of local rings, bimodules over these rings and bimodule-homomorphisms (cf. Chase [2], Harada [3]).

For example, taking $X_{ii} = D_i$ division-rings, X_{ik} arbitrary $D_i - D_k$-bimodules and all $f_{ijk} = 0$, the associativity conditions are certainly satisfied, and we obtain precisely the self-basic semi-perfect rings R with $J^2 = 0$ and $eJe = 0$ for all primitive idempotents e (cf. Zaks [12]).

3. Remark on a paper by K. Koh

The content of this paper is a characterization of those rings for which every simple right-module has a projective cover. For commutative R this is shown to be equivalent to $\bar{R} = R/J$ being semi-simple artinian and idempotents being liftable, in other words with R being semiperfect. For general R a seemingly weaker condition is given: $\bar{R}$ semi-simple artinian, and for every non-zero idempotent ε in $\bar{R}$ there exists a non-zero idempotent e in R with $\bar{e}\varepsilon = \bar{e}$.

We observe first that this condition implies the liftability of idempotents, hence that R is semi-perfect. For $\bar{e}\varepsilon = \bar{e}$ yields $\bar{e} \in \bar{R}\varepsilon$, and if ε is primitive then $\overline{Re} = \bar{R}\varepsilon$ and there is an inner automorphism of $\bar{R}$ mapping e into ε: $\bar{x}\bar{e}\bar{x}^{-1} = \varepsilon$. Then x is invertible in R and xex^{-1} is a lift of ε. The standard procedure of lifting sets of orthogonal idempotents allows then to lift finite orthogonal sets of primitive idempotents, and since each idempotent in $\bar{R}$ is the sum of such a set, all idempotents can be lifted.

This result—*all simple R-right-modules have projective cover if and only if R is semi-perfect*—is very shortly proved as follows. If X is simple, we have a projective extension $0 \to I \to R \to X \to 0$ with a maximal right-ideal I, hence the projective cover is $0 \to I \cap eR \to eR \to X \to 0$ with an idempotent e of R. Since $I \cap eR$ is small in eR hence in R, it is contained in the radical; consequently $I \cap eR = eJ$ and $X \cong eR/eJ$, and R is semi-perfect.

4. Proof of Theorem 1

The non-trivial implication is that from (2) to (1). In $1 = e_1 + \cdots + e_n$ let e_i, e_j be isomorphic idempotents, non-isomorphic to e_k. Then no map $e_i R \to e_k R \to e_j R$ will be an isomorphism and therefore $e_i Re_k Re_j \subset e_i Je_j$ since $e_i Re_j$ is semilinearly isomorphic to $e_i Re_i$ which has the unique maximal submodule $e_i Je_i$. Let e denote the sum of all the idempotents in $1 = e_1 + \cdots + e_n$ that are isomorphic to e_i, and $f = 1 - e$; then we obtain $eRfRe \subset eJe$. This implies that $I = eRf + eJe$ is a right-ideal; and if M were any maximal right-ideal not containing I, we would get
$R = I + M,\ 1 = exf + eje + m,\ e = eje + me \in J + M = M,\ I \subset eR \subset M$; consequently I is contained in every maximal right-ideal and $I \subset J$. Then

$$eRf + eJe = I \subset eJ = eJf + eJe$$

hence $eRf = eJf$ and $e_i Re_k = e_i Je_k$.

Now we consider any $e_i x \in e_i R, \notin e_i J$. Then

$$e_i xf \in e_i Rf = e_i Jf$$

and therefore there exists $e_i xe_j \notin e_i Je_j$. Then $\overline{e_i xe_j}$ will be "invertible" in $e_i Re_j/e_i Je_j$ (which is semi-isomorphic to the division-ring $e_i Re_i/e_i Je_i$): We get

$$\overline{e_i xe_j\, y} = \bar{e_i} \quad \text{and} \quad \overline{e_i xR} = \overline{e_i R},$$

and $e_i R/e_i J$ is simple. It follows immediately that every simple R-right-module is isomorphic to some $e_i R/e_i J$, which means that R is semi-perfect.

5. Proof of Theorem 2

Since eJe is the radical of eRe, one direction is obvious. Suppose now that all $e_i Re_i$ are left-perfect hence all $e_i Je_i$ left-T-nilpotent where

$$1 = e_1 + \cdots + e_n$$

is a decomposition into primitive orthogonal idempotents, and assume J not left-T-nilpotent. Then there exists a sequence $x^{(m)} \in J$ with $x^{(1)} \cdots x^{(m)} \neq 0$ for all m. Set

$$x^{(m)} = \sum_{i_m, k_m = 1}^{n} x_{i_m k_m}^{(m)}, \quad x_{i_m k_m}^{(m)} \in e_{i_m} Je_{k_m};$$

then $\sum x_{i_1 k_1}^{(1)} \cdots x_{i_m k_m}^{(m)} \neq 0$ for all m.

$$A_m = \{ (k_1, \cdots, k_m) \mid \text{there exists } x_{i_1 k_1}^{(1)} \cdots x_{i_m k_m}^{(m)} \neq 0 \}$$

is finite and non-empty; hence by König's Graph Theorem there exists a sequence k_m such that $x_{i_1 k_1}^{(1)} \cdots x_{i_m k_m}^{(m)} \neq 0$ for all m; observe this forces $i_{s+1} = k_s$ hence $x_{k_1 k_2}^{(2)} \cdots x_{k_{m-1} k_m}^{(m)} \neq 0$ for all m. One index k will occur infinitely often in the sequence k_m, and multiplying appropriate factors together we get terms $a^{(j)} \in e_k Je_k$ with $a^{(1)} \cdots a^{(r)} \neq 0$ for all r. This contradicts the left-T-nilpotence of $e_k Je_k$.—The statement for semi-primary rings follows similarly.

6. Proof of Theorem 3

We sketch the proof which follows closely Kaplansky's argument. By his results it is sufficient to show that every element x of the projective (right-) module P is contained in a direct summand which is a finite direct sum of primitive ideals. A *quasi-basis* of a module X shall be a family of elements b_α such that there exists a family of primitive idempotents e_α with $b_\alpha e_\alpha = b_\alpha$ and that every $x \in X$ has a unique representation $x = \sum b_\alpha x_\alpha$, $x_\alpha \in e_\alpha R$. The projective module P is direct in a free module, $P \oplus Q = F$; let y' denote the projection of $y \in F$ in P. A free module has a quasi-basis, and we choose such a quasi-basis of F that the given $x \in P$ has a minimal number of non-zero components;

$$x = \sum\nolimits_{\alpha \in B} b_\alpha x_\alpha, \quad x_\alpha \neq 0.$$

We obtain $x = x' = \sum_{\alpha \in B} b'_\alpha x_\alpha$; $b'_\alpha = \sum b_\beta c_{\beta\alpha}$, $c_{\beta\alpha} \in e_\beta R e_\alpha$; hence

$$x_\beta = \sum\nolimits_{\alpha \in B} c_{\beta\alpha} x_\alpha \quad \text{for all } \beta \in B.$$

The minimality condition on the quasibasis implies that e_α is not a left-multiple of $e_\alpha - c_{\alpha\alpha}$ nor of $c_{\beta\alpha}$ ($\beta \neq \alpha$); hence $c_{\alpha\alpha}$ in invertible in the local ring $e_\alpha R e_\alpha$, and $c_{\beta\alpha} \in e_\beta J e_\alpha$ if e_β, e_α are isomorphic. If e_β, e_α are non-isomorphic we also have $c_{\beta\alpha} \in e_\beta R e_\alpha = e_\beta J e_\alpha$ (cf. proof of Theorem 1). Consequently the matrix $C = (c_{\beta\alpha})_{\beta,\alpha \in B}$ has an "inverse" D such that CD, DC have e_α's in the main diagonal, zeros elsewhere. This implies that b'_β ($\beta \in B$), b_α ($\alpha \notin B$) is a quasibasis of F, hence

$$P = (\oplus_{\beta \in B} b'_\beta e_\beta R) \oplus (\oplus_{\alpha \notin B} b_\alpha e_\alpha R \cap P) \quad \text{and} \quad x \in \oplus_{\beta \in B} b'_\beta e_\beta R \cong \oplus_{\beta \in B} e_\beta R.$$

REFERENCES

1. H. BASS, *Finitistic dimension and a homological generalization of semi-primary rings*, Trans. Amer. Math. Soc., vol. 95 (1960), pp. 466–488.
2. ST. U. CHASE, *A generalization of the ring of triangular matrices*, Nagoya Math. J., vol. 18 (1961), pp. 13–25.
3. M. HARADA, *Hereditary semi-primary rings and triangular matrix rings*, Nagoya Math. J., vol. 27 (1966), pp. 463–484.
4. I. KAPLANSKY, *Projective modules*, Ann. of Math., vol. 68 (1958), pp. 372–377.
5. F. KASCH UND E. MARES, *Eine Kennzeichnung semi-perfekter Moduln*, Nagoya Math. J., vol. 27 (1966), pp. 525–529.
6. K. KOH, *On a semiprimary ring*, Proc. Amer. Math. Soc., vol. 19 (1968), pp. 205–208.
7. J. LAMBEK, *On the ring of quotients of a noetherian ring*, Canad. Math. Bull., vol. 8 (1965), pp. 279–290.
8. ———, *Lectures on rings and modules*, Waltham, 1966.
9. E. MARES, *Semi-perfect modules*, Math. Zeitschr., vol. 82 (1963), pp. 347–360.
10. B. MITCHELL, *Theory of categories*, Academic Press, New York, 1965.
11. L. E. T. WU AND J. P. JANS, *On quasi-projectives*, Illinois J. Math., vol. 11 (1967), pp. 439–448.
12. A. ZAKS, *Residue rings of semi-primary hereditary rings*, Nagoya Math. J., vol. 30 (1967), pp. 279–283.

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