ON THE ASYMPTOTIC BEHAVIOR OF THE SPECTRAL FUNCTION OF ELLIPTIC PSEUDO-DIFFERENTIAL OPERATORS

BY

BUI AN TON[1]

Let A be an elliptic pseudo-differential operator of order $\alpha > 0$ on a bounded open set Ω of R^n with symbol $\tilde{A}(x, \xi)$. Let $\tilde{A}_j(x^j, \xi)$ be the principal part of the symbol of A in a local coordinates system and suppose that $\tilde{A}_j(x^j, \xi)$ admits a Wiener-Hopf type of factorization:

$$\tilde{A}_j(x^j, \xi) = \tilde{A}_j^+(x^j, \xi)\tilde{A}_j^-(x^j, \xi)$$

for $x_n^j = 0$ where $\tilde{A}_j^+(x^j, \xi)$ is homogeneous of order k in ξ, (k is a non-negative integer independent of x^j), analytic in Im $\xi_n > 0$; $\tilde{A}_j^-(x^j, \xi)$ is homogeneous of order $\alpha - k$ in ξ, analytic in Im $\xi_n \leq 0$.

Let B_r; $r = 1, \cdots, k$ (if $k > 0$) be a system of pseudo-differential operators of orders α_r, $0 \leq \alpha_r < \alpha$ and $\tilde{B}_{rj}(x^j, \xi)$ be the symbol of the principal part of B_r in a local coordinates system.

Suppose

(i) $\tilde{A}_j^+(x^j, \xi) + t$; $\tilde{B}_{rj}(x^j, \xi)$ satisfy a Shapiro-Lopatinskii type of condition for each j and for all $t \geq t_0 > 0$,

(ii) A_2 as an operator on $L^2(\Omega)$ defined by

$$D(A_2) = \{u : u \text{ in } H_+^\alpha(\Omega);\ B_r u = 0 \text{ on } \partial\Omega;\ r = 1, \cdots, k\}$$

with

$$A_2 u = Au \quad \text{if} \quad u \in D(A_2)$$

is self-adjoint.

(iii) $\alpha > n$

Then it can be shown that

$$(1) \qquad t^{-n/\alpha} e(x, y, t) = t^{-n/\alpha} \textstyle\sum_{\lambda_j \leq t} \varphi_j(x)\overline{\varphi_j(y)} \to 0$$

as $t \to +\infty$; $x \neq y$

$$e(x, x, t) \sim (2\pi)^{-n} t^{n/\alpha} \alpha(n\pi)^{-1} \sin(n\pi/\alpha) \int_{R^n} (\tilde{A}(x, \xi) + 1)^{-1} d\xi$$

as $t \to +\infty$, x in Ω

If $k = 0$, then

$$(2) \qquad N(t) = \sum_{\lambda_j \leq t} 1 \sim (2\pi)^{-n} t^{n/\alpha} \alpha(n\pi)^{-1} \sin(n\pi/\alpha) \int_\Omega \int_{\tilde{A}(x,\xi)<1} d\xi\, dx$$

as $t \to +\infty$. λ_j, φ_j are the eigenvalues and eigenfunctions of A_2.

Received May 22, 1968.

[1] Research sponsored by the Air Force Office of Scientific Research, Office of Aerospace Research, United States Air Force.

The above results are well known in the case of elliptic differential operators; cf. Carleman [5], Garding [8], Browder [4], Agmon [1], [2], the writer [10]. For a more complete bibliography, we refer to [6].

The elliptic pseudo-differential operators considered in this paper are those studied recently by Eskin-Visik [7].

In Section 1, the notations, the definitions (which are essentially those of Eskin-Visik [7]) and the main assumption of the paper are given. In Section 2, the asymptotic behavior of the Green's function associated with $\{A + tI; B_r; r = 1, \cdots, k\}$ is studied. Finally in Section 3, by a standard argument, the asymptotic behavior of the spectral function is obtained and in the special case when $k = 0$, the asymptotic distribution of the eigenvalues is studied.

Section 1

Let s be an arbitrary real number and $H^s(R^n)$ be the Sobolev-Slobodetskii space of generalized functions f such that

$$\| f \|_s^2 = \int_{R^n} (1 + |\xi|^2)^s |\tilde{f}(\xi)|^2 d\xi < \infty$$

where $\tilde{f}$ is the Fourier transform of f.

Let Ω be a bounded open set of R^n with a smooth boundary $\partial\Omega$. $H^s(\Omega)$ denotes the restriction to Ω of functions in $H^s(R^n)$ with the norm

$$\| u \|_s = \inf \| v \|_{H^s(R^n)}; \qquad v = u \quad \text{on} \quad \Omega; \qquad s \geq 0.$$

By $H_+^s(\Omega)$, we denote the space of functions f defined on all of R^n, equal to 0 on $R^n/\text{cl}\,\Omega$ and coinciding in cl Ω with functions in $H^s(\Omega)$.

$H^s(\partial\Omega)$ is defined as the completion of $C^\infty(\partial\Omega)$ with respect to

$$\| f \|_{H^s(\partial\Omega)} = \{\textstyle\sum_j \| \varphi_j f \|_{H^s(R^{n-1})}^2\}^{1/2}$$

where $\| \varphi_j f \|_{H^s(R^{n-1})}$ is taken in local coordinates and the φ_j are those functions of a finite partition of unity whose supports intersect the boundary $\partial\Omega$. One may show that with different φ_j, one gets equivalent norms.

Let $\tilde{f}(\xi)$ be a smooth decreasing function. The operator $\prod^+$ is defined by

$$\textstyle\prod^+ \tilde{f}(\xi) = \frac{1}{2}f(\xi', \xi_n) + i(2\pi)^{-1} \text{ v.p.} \int \tilde{f}(\xi', \eta_n)(\xi_n - \eta_n)^{-1} d\eta_n$$

where $\xi' = (\xi_1, \cdots, \xi_{n-1})$. For any $\tilde{f}$, the above relation is understood as the result of the closure of the operator $\prod^+$ defined on the set of smooth and decreasing functions.

Set

$$\xi_- = \xi_n - i|\xi'|; \qquad \xi_+ = \xi_n + i|\xi'|$$

DEFINITION 1.1. $\tilde{A}(\xi)$ is in E_α iff

(i) $\tilde{A}(\xi)$ is a homogeneous function of order α in ξ,

(ii) $\tilde{A}(\xi) \neq 0$ for $|\xi| \neq 0$,

(iii) $\tilde{A}\xi)$ has for $|\xi'| \neq 0$, continuous first order derivatives bounded if $|\xi| = 1, |\xi'| \neq 0$.

DEFINITION 1.2. $\tilde{A}_+(\xi)$ is in C_k^+ iff

(i) $\tilde{A}_+(\xi)$ is homogeneous of order k in ξ, is continuous for $|\xi| \neq 0$ and has an analytic continuation with respect to ξ_n in Im $\xi_n > 0$ for each ξ',

(ii) $\tilde{A}_+(\xi) \neq 0$ for $|\xi| \neq 0$ and for any integer $p \geq 0$, there is an expansion

$$\tilde{A}_+(\xi) = \sum_{s=0}^{p} c_s(\xi')\xi_+^{k-s} + R_{k,p+1-k}(\xi', \xi_n)$$

where all the terms are homogeneous of order k in ξ, with analytic continuation in Im $\xi_n > 0$ and

$$|R_{k,p+1-k}(\xi', \xi_n)| \leq C|\xi'|^{p+1}(|\xi'| + |\xi_n|)^{k-p-1}.$$

DEFINITION 1.3. $\tilde{A}(x, \xi)$ is in D_α^0 iff

(i) $\tilde{A}(x, \xi)$ is infinitely differentiable in x and ξ, $|\xi| \neq 0$,

(ii) $\tilde{A}(x, \xi)$ is homogeneous of order α in ξ for x in R^n,

(iii) $$\frac{\partial^k}{(\partial\xi')^k}\tilde{A}(x, 0, -1) = (-1)^k \exp(-i\pi\alpha)\frac{\partial^k}{(\partial\xi')^k}\tilde{A}(x, 0, 1),$$
$$0 \leq |k| < \infty.$$

DEFINITION 1.4. $\tilde{A}(x, \xi)$ is in $\hat{D}_{\alpha,1}^1$ iff the following hold.

(i) $|D_x^p\tilde{A}(x, \xi)| \leq C_p(1 + |\xi|)^\alpha, \quad 0 \leq |p| < \infty.$

(ii) For any x in R^n and for any $s \geq -\alpha$, there is a decomposition

$$(\xi_- - i)^s\tilde{A}(x, \xi) = \tilde{A}_-(x, \xi) + R(x, \xi),$$

$\tilde{A}_-(x, \xi)$; $R(x, \xi)$ are infinitely differentiable with respect to x, $\tilde{A}_-(x, \xi)$ is analytic in Im $\xi_n < 0$ and, for $0 \leq |p| < \infty$,

$$|D_x^p\tilde{A}_-(x, \xi)| \leq C_p(1 + |\xi|)^{\alpha+s}; \qquad |D_x^p D_\xi\tilde{A}_-(x, \xi)| \leq c_p(1 + |\xi|)^{\alpha+s-1}$$

$$|D_x^p R(x, \xi)| \leq C_p(1 + |\xi'|)^{\alpha+s+1}(1 + |\xi|)^{-1};$$

$$|D_x^p D_\xi R(x, \xi)| \leq c_p(1 + |\xi'|)^{s+\alpha}(1 + |\xi|)^{-1}.$$

Let $\{\varphi_j\}$ be a finite partition of unity corresponding to an open covering $\{N_j\}$ of cl Ω. Let $\{\psi_j\}$ be the infinitely differentiable functions with compact supports in $\{N_j\}$ and such that $\varphi_j\psi_j = \varphi_j$.

P^+ denotes the restriction operator of (generalized) functions from R^n to Ω and γ denotes the passage to $\partial\Omega$.

Let $\tilde{A}(\xi)$ be in E_α, $(\alpha > 0)$, and u be an element of $H^s(R_+^n)$ with $u(x) = 0$ for $x_n < 0$. We define

$$Au = F^{-1}(\tilde{A}(\xi)\tilde{u}(\xi))$$

where the inverse Fourier transform is understood in the sense of the theory of distributions. Let $\tilde{A}(x, \xi)$ be in E_α for x in cl Ω and $\tilde{A}(x, \xi)$ be infinitely differentiable with respect to x and ξ. We extend $\tilde{A}(x, \xi)$ with respect to x

to R^n with preservation of homogeneity with respect to ξ. We expand $\tilde{A}(x, \xi)$ in the Fourier series

$$\tilde{A}(x, \xi) = \sum_{k=-\infty}^{\infty} \psi_0(x) \exp(-i\pi kx/p)\tilde{L}_k(\xi), \qquad k = (k_1, \cdots, k_n)$$

and

$$\tilde{L}_k(\xi) = (2p)^{-n} \int_{-p}^{p} \exp(-i\pi kx/p)\tilde{A}(x, \xi)\, dx$$

$\psi_0(x) \in C_c^\infty(R^n); \psi_0(x) = 1$ for $|x| \leq p - \varepsilon, \psi_0(x) = 0$ for $|x| \geq p$.

We have $|\tilde{L}_k(\xi)| \leq C|\xi|^\alpha(1 + |k|)^{-M}$ for large positive M.

For u in $H_+^\alpha(\Omega)$, we define

$$P^+Au = P^+(\sum_{k=-\infty}^{\infty} \psi_0(x) \exp(ikx\pi/p)L_k u).$$

DEFINITION 1.5. (1) Let

$$P^+A = \sum_j P^+\varphi_j A\psi_j + \sum_j P^+\varphi_j A(1 - \psi_j)$$

be an elliptic pseudo-differential operator of order α on Ω with the following properties:

(a) If $\varphi_j A_j \psi_j$ is the principal part of $\varphi_j A \psi_j$ in a local coordinates system, then $\tilde{A}_j(x^j, \xi) \in E_\alpha$ and for $x_n^j = 0$ admits the factorization

$$\tilde{A}_j(x^j, \xi) = \tilde{A}_j^+(x^j, \xi)\tilde{A}_j^-(x^j, \xi)$$

where $\tilde{A}_j^+ \in C_k^+$; k is a non-negative integer independent of x^j and A_j^- is homogeneous of order $\alpha - k$ in ξ with an analytic continuation with respect to ξ_n in $\operatorname{Im} \xi_n \leq 0$.

(b) $\tilde{A}_j(x^j, \xi) \in D_\alpha^0 \cap \hat{D}_{\alpha,1}^1$ for $x \in N_j \cap \partial\Omega \neq \emptyset$.

(2) If $k > 0$, let

$$P^+B_r = \sum_j P^+\varphi_j B_r\psi_j + \sum_j P^+\varphi_j B_r(1 - \psi_j); \qquad r = 1, \cdots, k$$

be a system of pseudo-differential operators of orders α_r with $0 \leq \alpha_r < \alpha$ having the following properties:

If $\varphi_j B_{rj} \psi_j$ is the principal part of B_r in a local coordinate system, then $\tilde{B}_{rj}(x^j, \xi) \in D_{\alpha_r}^0 \cap \hat{D}_{\alpha_r,1}^1$ for $x \in N_j \cap \partial\Omega \neq \emptyset$.

The elliptic problem $\{P^+A; \gamma P^+B_r; r = 1, \cdots, k\}$ is said to be uniformly regular on Ω if

$$\operatorname{Det}\, (b_{rs}(x^j, \xi')) \neq 0$$

for all $x^j \in N_j \cap \partial\Omega \neq \emptyset$ where b_{rs} are determined by

$$\textstyle\prod^+ \tilde{B}_{rs}(x^j, \xi)\xi_n^{s-1}(\tilde{A}_j^+(x^j, \xi))^{-1} = R_{rs}(x^j, \xi) + ib_{rs}(x^j, \xi')\xi_+^{-1},$$

ord $(b_{rs}) = \alpha_r + k - s; r, s = 1, \cdots, k$

The main assumption of the paper is the following condition.

ASSUMPTION (I). *Let $\{P^+A; \gamma P^+B_r; r = 1, \cdots, k\}$ be a uniformly regular elliptic problem on Ω in the sense of Definition* 1.5. *We assume*

(i) $\tilde{A}_j(x^j, \xi) + t \neq 0$ *for all* $t \geq t_0 > 0$ *and all* j;

(ii) *if* $k > 0$, Det $(b_{rs}(x^j, \xi', t)) \neq 0$ *for all* x^j *and all* $t \geq t_0 > 0$ *where* $b_{rs}(x^j, \xi', t)$ *are given by*

$$\prod^{+} \tilde{B}_{rs}(x^j, \xi)\xi_n^{s-1}(A_j^{+}(x^j, \xi, t))^{-1} = R_{rs}(x^j, \xi, t) + ib_{rs}(x^j, \xi', t)(\xi_{+}^{t})^{-1}$$

with

$$\tilde{A}_j(x^j, \xi) + t = \tilde{A}_j^{+}(x^j, \xi, t)\tilde{A}_j^{-}(x^j, \xi, t) \quad and \quad \xi_{+}^{t} = \xi_n - i(|\xi'| + t^{1/\alpha}).$$

DEFINITION 1.6. Let A_2 be the operator on $L^2(\Omega)$ defined as follows:

$$D(A_2) = \{u : u \text{ in } H_{+}^{\alpha}(\Omega) \text{ and } \gamma P^{+}B_r u = 0 \text{ if } k > 0; \quad r = 1, \cdots, k\},$$

$$A_2 u = P^{+}Au \quad \text{if } u \text{ is in } D(A_2)$$

Section 2

First, we have the following theorem.

THEOREM 2.1. *Let* $\{P^{+}A; \gamma P^{+}B_r; r = 1, \cdots, k\}$ *be a uniformly regular problem on* Ω *in the sense of Definition* 1.5.

Suppose that

(i) *Assumption* (I) *is satisfied,*

(ii) $\alpha > n$, *is the order of* A.

Then for $t \geq t_0 > 0$, $(A_2 + tI)^{-1}$ *exists and is of Hilbert-Schmidt type*

$$(A_2 + tI)^{-1}f(x) = \int_{\Omega} G(x, y, t)f(y)\, dy,$$

f *in* $L^2(\Omega)$ *and* $G(x, y, t) \in L^2(\Omega) \times L^2(\Omega)$

Proof. In [12], the writer has proved that under the hypotheses of the theorem, $(A_2 + tI)^{-1}$ exists and is a bounded linear mapping from $L^2(\Omega)$ into $H_{+}^{\alpha}(\Omega)$. The following estimate was established:

$$\| u \|_{\alpha} + t \| u \|_0 \leq C \| (A_2 + tI)u \|_0 \quad \text{for all } u \text{ in } D(A_2)$$

Since $\alpha > n$ and Ω is a bounded open set of R^n with a smooth boundary, the injection mapping of $H_{+}^{\alpha}(\Omega)$ into $L^2(\Omega)$ is compact. Hence by a standard argument, it follows that $(A_2 + tI)^{-1}$ is of Hilbert-Schmidt type and

$$(A_2 + tI)^{-1}f(x) = \int_{\Omega} G(x, y, t)f(y)\, dy,$$

f in $L^2(\Omega)$, $G(x, y, t)$ in $L^2(\Omega) \times L^2(\Omega)$ Q.E.D.

In the remainder of this section we shall study the asymptotic behavior of $G(x, y, t)$ as $t \to +\infty$.

LEMMA 2.1. *Let* $\tilde{A}(\xi)$ *be in* E_{α}, $\alpha > 0$ *and such that* $\tilde{A}(\xi) + t \neq 0$ *for* $t \geq t_0 > 0$. *Suppose that* $\alpha > n$. *Then*

$$E(x, y, t) = (2\pi)^{-n} \int_{R^n} \exp(-i < x - y, \xi >)(\tilde{A}(\xi) + t)^{-1}\, d\xi$$

is infinitely differentiable for $x \neq y$. *Moreover*

$$|E(x, y, t)| \leq Mt^{-1+n/\alpha}(1 + t^{N/\alpha}|x - y|^N)^{-1},$$

$$|D_x^\beta E(x, y, t)| \leq Mt^{-\varepsilon/\alpha}|x - y|^{-n-\varepsilon-|\beta|+\alpha}(1 + t^{N/\alpha}|x - y|^N)^{-1}$$

for $-n + \alpha \leq |\beta|$; $0 < \varepsilon < 1$ *and* N *is any positive number.* $E(x, y, t)$ *is a fundamental solution of* $P^+(A + tI)$; i.e., $P^+(A + tI)E = \delta_y$, y *in* Ω.

Proof. Cf. Garding [8]

LEMMA 2.2. *Let* P^+A *be an elliptic pseudo-differential operator of order* α *on* Ω *with symbol* $\tilde{A}(x, \xi)$ *infinitely differentiable in* x *and* ξ. *Let* P^+A_z *be the operator* P^+A *with symbol evaluated at* z. *Let* $E_z(x, z, t)$ *be the fundamental solution of* $P^+(A_z + tI)$. *Set*

(i) $w(x, z, t) = P^+(A - A_z)E_z(x, z, t)$

(ii) $Tv(x, z, t) = \int_\Omega w(x, y, t)v(y, z, t)\, dy.$

Then the integral equation $v + Tv + w = 0$ *may be solved by the Neumann series for large* t. *Moreover*

$$v(x, z, t) = 0(1)t^{-\varepsilon/\alpha}|x - z|^{-n+1-\varepsilon}(1 + t^{N/\alpha}|x - z|^N)^{-1}$$

where $0 < \varepsilon < 1$ *and* N *is a large positive number.*

Proof. The proof is easy and follows from the previous lemma and the definition of P^+A.

THEOREM 2.2. *Suppose the hypotheses of Lemmas* 2.1, 2.2 *are satisfied. Then*

$$E(x, z, t) = E_z(x, z, t) + \int_\Omega E_y(x, y, t)v(y, z, t)\, dy$$

where v *is the solution of the integral equation of Lemma* 2.2 *and* z *in* Ω; *is a fundamental solution of* $P^+(A + tI)$

Proof. We have to verify that $P^+(A + tI)E(x, z, t) = \delta_z$; z in Ω.

$E(\cdot, z, t)$ is in $L^2(R^n)$, so $(A + tI)E(x, z, t)$ is well defined as an element of $H^{-\alpha}(R^n)$.

We may write

$$P^+(A + tI)E(x, z, t) = P^+(A_z + tI)E_z(x, z, t) + P^+(A - A_z)E_z(y, z, t) + P^+(A + tI)\int_\Omega E_y(x, y, t)v(y, z, t)\, dy$$

Let $\varphi \in C_c^\infty(\Omega)$, then we have

$$\left(\left(P^+(A + tI)\left(\int_\Omega E_y(\cdot, y, t)v(y, z, t)\, dy\right), \varphi\right)\right) = \left(\left((A + tI)\left(\int_\Omega E_y(\cdot, y, t)v(y, z, t)\, dy\right), \varphi\right)\right).$$

(1) We show that

$$(*)\quad \left(\left((A)\left(\int_\Omega E_y(\cdot, y, t)v(y, z, t)\,dy\right), \varphi\right)\right) = \sum_{s=-\infty}^{\infty}\left(\left(\psi(\cdot)e^{i\pi s\cdot/p}L_s\left(\int_\Omega E_y(\cdot, y, t)v(y, z, t)\,dy\right), \varphi\right)\right)$$

We have

$$\left|\left(\left(\sum_{s=k+1}^{\infty}\psi e^{i\pi s\cdot/p}L_s\left(\int_\Omega E_y(\cdot, y, t)v(y, z, t)\,dy\right), \varphi\right)\right)\right| \leq M\,\|\varphi\|_{H^\alpha(R^n)}\cdot\sum_{s=k+1}^{\infty}\left\|L_s\left(\int_\Omega E_y(\cdot, y, t)v(y, z, t)\,dy\right)\right\|_{-\alpha} \leq Mt^{-e/\alpha}\sum_{s=k+1}^{\infty}1/(1+s)^m$$

for some large positive m. Similarly for:

$$\left(\left(\sum_{s=-\infty}^{-k-1}\psi e^{i\pi s\cdot/p}L_s\left(\int_\Omega E_y(\cdot, y, t)v(y, z, t)\,dy\right), \varphi\right)\right)$$

It follows that $(*)$ holds.

(2) Next, we show

$$\left(\left(L_s\left(\int_\Omega E_y(\cdot, y, t)v(y, z, t)\,dy\right), \varphi\right)\right) = \int_\Omega v(y, z, t)((L_s E_y(\cdot, y, t), \varphi))\,dy$$

Taking Fourier transform, we obtain

$$\left(\left(L_s\left(\int_\Omega E_y(\cdot, y, t)v(y, z, t)\,dy\right), \varphi\right)\right) = \left(\left(\tilde{L}_s(\xi)F\left(\int_\Omega E_y(\cdot, y, t)v(y, z, t)\,dy\right), \tilde{\varphi}\right)\right) = \left(\left(F\left(\int_\Omega E_y(\cdot, y, t)v(y, z, t)\,dy\right), F(L_s\varphi)\right)\right)$$

since $\tilde{L}_s(\xi)\tilde{\varphi}(\xi)$ is in $\mathcal{S}$; $\tilde{L}_s(\xi)$ being infinitely differentiable, $|D^\beta\tilde{L}_s(\xi)| \leq C(1+|\xi|^\alpha)$, α is a positive integer.
It is also equal to

$$\left(\left(\int_\Omega E_y(\cdot, y, t)v(y, z, t)\,dy, L_s\varphi\right)\right) = \int_{R^n}\int_\Omega E_y(x, y, t)v(y, z, t)L_s\varphi(x)\,dy\,dx$$

since $\int_\Omega E_y(\cdot, y, t)v(y, z, t)\,dy$ is in $L^1(R^n) \cap L^2(R^n)$ and $L_2\varphi$ is in $\mathcal{S}$.

By the Fubini theorem, the right hand side integral may be written as

$$\int_\Omega v(y, z, t) \int_{R^n} E_y(x, y, t)L_s\varphi(x)\,dx\,dy.$$

We have

$$\begin{aligned}\int_{R^n} E_y(x, y, t)L_s\varphi(x)\,dx &= (FE_y(\cdot, y, t), F(L_s\varphi)) \\ &= ((\tilde{L}_s(\xi)FE_y(\cdot, y, t), \tilde{\varphi})) \\ &= ((F(L_sE_y(\cdot, y, t)), \tilde{\varphi})) \\ &= ((L_sE_y(\cdot, y, t), \varphi)).\end{aligned}$$

Hence

$$\left(\left(L_s\left(\int_\Omega E_y(\cdot, y, t)v(y, z, t)\,dy\right), \varphi\right)\right) = \int_\Omega v(y, z, t)((L_sE_y(\cdot, y, t), \varphi))\,dy$$

(3) Combining (1) and (2), we get

$$\left(\left((A + tI)\left(\int_\Omega E_y(\cdot, y, t)v(y, z, t)\,dy\right), \varphi\right)\right) = \int_\Omega v(y, z, t)(((A + tI)E_y(\cdot, y, t), \varphi))\,dy$$

The right hand side may be written as

$$\int_\Omega v(y, z, t)(((A_y + tI)E_y(\cdot, y, t), \varphi))\,dy + \int_\Omega v(y, z, t)(((A - A_y)E_y(\cdot, y, t), \varphi))\,dy.$$

Hence it is equal to

$$\int_\Omega \varphi(y)v(y, z, t)\,dy + \int_\Omega v(y, z, t) \int_\Omega P^+(A - A_y)E_y(x, y, t)\varphi(x)\,dx\,dy.$$

Taking into account the definition of v, we obtain

$$P^+(A + tI)E(x, y, t) = \delta_y, \qquad y \text{ in } \Omega, \quad \text{Q.E.D.}$$

The main result of this section is the following theorem:

THEOREM 2.3. *Let $\{P^+A; \gamma P^+B_r; r = 1, \cdots, k\}$ be a uniformly regular elliptic problem on Ω in the sense of Definition* 1.5 *and satisfying Assumption* (I). *Let $G(x, z, t)$ be the Green's function associated with the boundary problem*

$$\{P^+(A + tI); \gamma P^+B_r; r = 1, \cdots, k\}.$$

Then $G(x, z, t) = E(x, z, t) - u(x, z, t)$ *where* $E(x, z, t)$ *is the fundamental solution of Theorem* 2.2 *and* $u(x, z, t)$ *is the unique solution of the boundary problem*

$$P^+(A + tI)u(x, z, t) = 0 \quad \text{on } \Omega,$$
$$P^+B_r\, u(x, z, t) = P^+B_r\, E(x, z, t) \quad \text{for } r = 1, \cdots, k.$$

$G(x, z, t)$ *is a continuous function of* x *and* $\lim_{t\to+\infty} t^{1-n/\alpha}u(x, z, t) = 0$ *for any* x *in* Ω, z *in* Ω.

Proof. If u is the solution of the boundary-value problem

$$P^+(A + tI)u = 0 \text{ on } \Omega; \quad \gamma P^+B_r\, u = \gamma P^+B_r\, E \text{ on } \partial\Omega; \quad r = 1, \cdots, k,$$

then it is clear that $G(x, z, t) = E(x, z, t) - u(x, z, t)$ is the Green's function associated with

$$\{P^+(A + tI); \quad P^+B_r; \quad r = 1, \cdots, k\}$$

In [12], generalizing a result of Agranovich-Visik [3], we have shown that the above boundary-value problem has a unique solution u and the following estimate holds:

$$\sum_{s=0}^{\alpha} t^{1-s/\alpha} \| u \|_s \le M \sum_{r=1}^{k} \{\| \gamma P^+B_r\, E(\cdot, z, t) \|'_{\alpha-\alpha_r-1/2} + t^{1-(\alpha_r+1/2)/\alpha} \cdot \| \gamma P^+B_r\, E(\cdot, z, t) \|'$$

where M is independent of z, t.

Since $\alpha > n$, using the Sobolev imbedding theorem, we get

$$t^{1-n/\alpha} \,| u(x, z, t) | \le M \sum_{r=1}^{k} \{\| \gamma P^+B_r\, E(\cdot, z, t) \|'_{\alpha-\alpha_r-1/2} + t^{1-(\alpha_r+1/2)/\alpha} \| \gamma P^+B_r\, E(\cdot, z, t) \|'.$$

We study the expressions inside of the bracket. We have

$$B_r\, E(x, z, t) = B_r\, E_z(x, z, t) + B_r\left(\int_\Omega E_y(x, y, t)v(y, z, t)\, dy\right).$$

Using the expansion of B_r, we consider

$$B_{rs}\, E_z(x, z, t) \quad \text{and} \quad B_{rs}\left(\int_\Omega E_y(x, y, t)v(y, z, t)\, dy\right)$$

where the symbol $\tilde{B}_{rs}(\xi)$ of B_{rs} is a homogeneous function of order α_r in ξ with

$$| \tilde{B}_{rs}(\xi) | \le C\xi^{\alpha_r}(1 + | s |)^{-M}$$

(1) By an easy computation, we get

$$| B_{rs}\, E_z(x, z, t) | \le Ct^{-2+(1+\alpha_r-\varepsilon)/\alpha}(1 + | s |)^{-M} | x - z |^{-n-\alpha-\varepsilon+1}/(1 + t^{N/\alpha} | x - z |^N)$$

where $0 < \varepsilon < 1$, $N \ge 0$.

Let $d(z) = \text{dist}\,(z, \partial\Omega)$; for $t \ge d(z)^{-\varepsilon(n+\alpha+\varepsilon-1)/\alpha}$, we have

$$| \gamma P^+B_{rs}\, E_z(x, z, t) | \le Ct^{-2+(1+\alpha_r)/\alpha}(1 + | s |)^{-M}(1 + t^{N/\alpha} | x - z |^N)^{-1}$$

where C is independent of x, z, t, s. So

$$|\gamma P^{+} B_r E_z(x, z, t)| \leq C t^{-2+(1+\alpha_r)/\alpha} (1 + t^{N/\alpha} |x - z|^N)^{-1}.$$

(2) Next, we show that

$$B_{rs}\left(\int_\Omega E_y(x, y, t) v(y, z, t)\, dy\right) = \int_\Omega B_{rs} E_y(x, y, t) v(y, z, t)\, dy.$$

Indeed, let $\varphi \in C_c^\infty(R^n)$ and consider

$$\left(\left(B_{rs}\left(\int_\Omega E_y(\cdot, y, t) v(y, z, t)\, dy\right), \varphi\right)\right).$$

Since $\int_\Omega E_y(\cdot, y, t) v(y, z, t)\, dy$ is an element of $L^2(R^n)$, $B_{rs}(\int_\Omega)$ is in $H^{-\alpha}(R^n)$. Using Plancherel theorem, we obtain

$$\begin{aligned}
&\left(\left(\tilde{B}_{rs}(\xi) F\left(\int_\Omega E_y(\cdot, y, t) v(y, z, t)\, dy\right), \tilde{\varphi}\right)\right) \\
&\qquad = \left(\left(B_{rs}\left(\int_\Omega E_y(\cdot, y, t) v(y, z, t)\, dy\right), \varphi\right)\right) \\
&\qquad = \left(\left(F\left(\int_\Omega E_y(\cdot, y, t) v(y, z, t)\, dy\right), F(B_{rs}\varphi)\right)\right) \\
&\qquad = \left(\left(\int_\Omega E_y(\cdot, y, t) v(y, z, t)\, dy, B_{rs}\varphi\right)\right).
\end{aligned}$$

Since $\int_\Omega E_y(\cdot, y, t) v(y, z, t)\, dy$ is in $L^1(R^n) \cap L^2(R^n)$, we get

$$\begin{aligned}
&\left(\left(\int_\Omega E_y(\cdot, y, t) v(y, z, t)\, dy, B_{rs}\varphi\right)\right) \\
&\qquad = \int_{R^n} B_{rs}\varphi\left(\int_\Omega E_y(x, y, t) v(y, z, t)\, dy\right) dx \\
&\qquad = \int_\Omega v(y, z, t) \int_{R^n} B_{rs}\varphi(x) E_y(x, y, t)\, dx\, dy
\end{aligned}$$

by Fubini's theorem. But the last integral may also be written as

$$\int_{R^n} B_{rs}\varphi(x) E_y(x, y, t)\, dx = \int_{R^n} B_{rs} E_y(x, y, t) \varphi(x)\, dx.$$

Applying the Fubini theorem, we obtain

$$\begin{aligned}
&\left(\left(B_{rs}\left(\int_\Omega E_y(\cdot, y, t) v(y, z, t)\, dy\right), \varphi\right)\right) \\
&\qquad = \int_{R^n} \varphi(x) \int_\Omega B_{rs} E_y(x, y, t) v(y, z, t)\, du\, dx \\
&\qquad = \left(\left(\int_\Omega B_{rs} E_y(\cdot, y, t) v(y, z, t)\, dy, \varphi\right)\right)
\end{aligned}$$

for all φ in $C_c^\infty(R^n)$.

So $B_{rs}(\int_\Omega E_y(x, y, t)v(y, z, t)\, dy) = \int_\Omega B_{rs} E_y(x, y, t)v(y, z, t)\, dy$ in the distribution sense. Since the right hand side of the equality is a continuous function of x for $x \neq z$, the equality is true in the classical sense for $x \neq z$.

We get

$$\left| \gamma P^+ B_{rs} \left(\int_\Omega E_y(x, y, t)v(y, z, t)\, dy \right) \right| \leq Ct^{-2+(1+\alpha_r)/\alpha}(1 + |s|)^{-M}/(1 + t^{N/\alpha} |x - z|^N)$$

for $t \geq d(z)^{-\alpha(n+\alpha+\varepsilon-1)/\varepsilon}$. C is a constant independent of x, z, t.

Therefore

$$\left| \gamma P^+ B_r \left(\int_\Omega E_y(x, y, t)v(y, z, t)\, dy \right) \right| \leq Ct^{-2+(1+\alpha_r)/\alpha}(1 + t^{N/\alpha} |x - z|^N)^{-1}$$

(3) From (1) and (2), we have $t^{1-(\alpha_r+1/2)/\alpha} \| \gamma P^+ B_r E(\cdot, z, t) \|_0'$ less than $Ct^{-1+1/2\alpha}$ for $t \geq d(z)^{-\alpha(n+\alpha+\varepsilon-1)/\varepsilon}$.

(4) Consider $\| \gamma P^+ B_r E(\cdot, z, t) \|_{\alpha-\alpha_r-1/2}' \leq C \| \gamma P^+ B_r E(\cdot, z, t) \|_{\alpha-\alpha_r}'$.

Again, we look at $\| \gamma P^+ B_{rs} E(\cdot, z, t) \|_{\alpha-\alpha_r}'$

By a computation as above, we get

$$|D^{\alpha-\alpha_r} \gamma P^+ E(x, z, t)| \leq Ct^{-\varepsilon/2\alpha}(1 + t^{N/\alpha} |x - z|^N)^{-1}$$

for $t \geq d(z)^{-(n-|+\varepsilon)\alpha/2\varepsilon}$; C is again a constant independent of x, z, t.

Hence $\| \gamma P^+ B_r E(\cdot, z, t) \|_{\alpha-\alpha_r-1/2}' \leq Ct^{-\varepsilon/2\alpha}$ for $t \geq d(z)^{-(n-|+\varepsilon)\alpha/2\varepsilon}$. Therefore

$$\lim_{t\to+\infty} t^{1-n/\alpha} |u(x, z, t)| \to 0.$$

The theorem is proved.

Section 3

In this section, we apply the Hardy-Littlewood Tauberian theorem to get the wanted results.

THEOREM 3.1. *Suppose the hypotheses of Theorem 2.1 are satisfied. Suppose further that A_2 is self-adjoint. Let λ_j, φ_j be the eigenvalues and eigenfunctions of A_2 respectively. Then*

(i) $t^{-n/\alpha} e(x, y, t) = t^{-n/\alpha} \sum_{\lambda_j \leq t} \varphi_j(x) \overline{\varphi_j(y)} \to 0$ *as* $t \to +\infty$ *for* x, y *in* Ω, $x \neq y$

(ii) $e(x, x, t) \sim (2\pi)^{-n} t^{n/\alpha} \alpha (n\pi)^{-1} \sin(n\pi/\alpha) \int_{R^n} (\tilde{A}(x, \xi) + 1)^{-1} d\xi$ *as* $t \to \infty$; x *in* Ω.

(iii) *If* $k = 0$, *then*

$$N(t) = \sum_{\lambda_j \leq t} 1 \sim (2\pi)^{-n} t^{n/\alpha} \alpha (n\pi)^{-1} \sin(n\pi/\alpha) \int_\Omega \int_{\tilde{A}(x,\xi)<1} d\xi dx$$

as $t \to +\infty$.

Proof. First we note that for $\alpha > n$, the Green's function $G(x, y, t)$ for fixed y in Ω may be represented as a uniformly convergent series:

$$G(x, y, t) = \sum_{j=1} \varphi_j(x)\varphi_j(y)(\lambda_j + t)^{-1}.$$

Applying the Hardy-Littlewood Tauberian theorem [9] and taking into account the results of Theorem 2.3, we get the assertions (i), (ii) of the theorem.

If $k = 0$, since no boundary conditions are required, we have

$$G(x, y, t) = E(x, y, t)$$

and

$$| t^{1-n/\alpha}G(x, x, t) | = | t^{1-n/\alpha}E(x, x, t) | = \left| (2\pi)^{-n} \int (\tilde{A}(x, \xi) + 1)^{-1} \, d\xi \right| \leq M$$

for all x in Ω. By the Lebesgue bounded convergence theorem and the Hardy-Littlewood Tauberian theorem, we obtain

$$N(t) \sim (2\pi)^{-n} t^{n/\alpha} \alpha(n\pi)^{-1} \sin (n\pi/\alpha) \int_\Omega \int_{\tilde{A}(x,\xi)<1} d\xi dx$$

as $t \to +\infty$.

Bibliography

1. S. Agmon, *On kernels and eigenfunctions of operators related to elliptic problems*, Comm. Pure Appl. Math. vol. 18 (1965), pp. 559–579.
2. S. Agmon and Y. Kannai, *On the asymptotic behavior of spectral functions and resolvent kernels of elliptic operators*, Israel J. Math., vol. 5 (1967), pp. 1–30.
3. M. S. Agranovich and M. I. Visik, *Elliptic problems with a parameter and parabolic problems of general type*, Uspehi Mat. Nauk, vol. 19 (1964), pp. 53–161 = Russian Math. Surveys, vol. 19, (1964), pp. 53–159.
4. F. E. Browder, *Asymptotic distribution of eigenvalues and eigenfunctions for nonlocal elliptic boundary value problems I*, Amer. J. Math. vol. 87 (1965), pp. 175–195.
5. T. Carleman, *Proprietes asymptotiques des fonctions fondamentales des membranes vibrantes*, Attonde Skand. Matematikerekongressen, Stockholm, 1934, pp. 34–44.
6. C. Clark, *The asymptotic distribution of eigenvalues and eigenfunctions for elliptic boundary value problems*, SIAM rev., vol. 9 (1967), 627–646.
7. G. I. Eskin and M. I. Visik, *Equations in convolutions in a bounded region*, Uspehi Mat. Nauk, vol. 20 (1965), pp. 90–152 = Russian Math. Surveys, vol. 20 (1965), pp. 85–157.
8. L. Garding, *On the asymptotic distribution of eigenvalues and eigenfunctions of elliptic differential operators*, Math. Scand., vol. 1 (1953), pp. 237–255.
9. G. H. Hardy and T. E. Littlewood, *Notes on the theory of series (XI); On tauberian theorem*, Proc. London Math. Soc. (2), vol. 20 (1930), pp. 23–27.
10. B. A. Ton, *Asymptotic distribution of eigenvalues and eigenfunctions for general linear elliptic boundary value problems*, Trans. Amer. Math. Soc., vol. 122 (1966), pp. 516–546.
11. ———, *Nonlinear elliptic convolution equations of Weiner-Hopf type in a bounded region*, Pacific J. Math., vol. 24 (1968), pp. 577–587.
12. ———, *Boundary value problems for elliptic convolution equations of Wiener-Hopf type in a bounded region*, Pacific J. Math., vol. 26 (1968), pp. 395–413.

University of British Columbia
Vancouver, British Columbia