STIELTJES-VOLTERRA INTEGRAL EQUATIONS

BY

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This paper extends the work of D. B. Hinton [1] who contributed to an integral equation theory developed by J. S. Mac Nerney [2] and [3] and H. S. Wall [4]. In each case a unique reversible function $\mathcal{E}$ is established which maps a class of functions $\mathfrak{F}$ onto a class of functions $\mathcal{G}$ in such a manner that each member F of $\mathfrak{F}$ together with its image $\mathcal{E}(F)$ satisfies a certain linear integral equation. I have extended Hinton's theory by changing the underlying space S from a number interval to a non-degenerate set with any linear ordering and by relaxing the axioms used to define the class $\mathfrak{F}$ of functions. The last is best illustrated by thinking of S as a number interval and defining a neighborhood of the diagonal of $S \times S$ to be the union of any finite collection of squares which cover the diagonal. Then each function in the class investigated in this paper will agree with a function in Hinton's class on some neighborhood of the diagonal but may be different outside of the neighborhood.

The theory presented in this paper is in fact a generalization of the study of the existence and the properties of functions U which satisfy equations of the form $U(t) = K + (L) \int_c^t dF[t, I] \cdot U$. A discussion of this may be found at the end of Section 5.

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1. Left and right integrals

Throughout this paper S will denote a non-degenerate set which is linearly ordered by $\leq$ with $<$ having its usual meaning and $(N, +, \cdot, |\cdot|)$ will denote a complete normed ring with unity 1. The letter "I" will denote the identity function whose range of definition will be clear from the context. To a large extent, definitions and theorems which are analogous to those of Hinton [1] are stated with the same letters and wording which Hinton used. It is hoped that this will facilitate the reading of both papers.

The statement that $[a, b]$ is an interval of S means that a and b are in S, $a < b$, and $[a, b]$ is the set to which x belongs only in case $a \leq x \leq b$. Suppose $(M, +, \cdot, |\cdot|)$ is a normed ring and g is a function mapping S into M. The function g is said to be uniformly quasi-continuous on an interval of S only in case it is the uniform limit of step functions on that interval. If $[a, b]$ is an interval of S and g is of bounded variation on $[a, b]$ then either $\int_a^b |dg|$ or $\int_b^a |dg|$ will denote the total variation of g on $[a, b]$, and if c is in S then $\int_c^c |dg|$ is the number 0. Suppose $(K, \oplus, \odot, \|\cdot\|)$ is a normed ring and f is a function mapping S into K. The statement that s is an $(f - g - \varepsilon)$ chain means that

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ε is a positive number, there is a positive integer n such that s is a monotone sequence from the integers in $[0, 2n]$ to S, if p is an integer and $s_{2p} < x \leq s_{2p+1}$ then

$$\int_x^{s_{2p+1}} \| df \| < \varepsilon/n,$$

and if p is an integer and $s_{2p+1} \leq x \leq y < s_{2p+2}$ then

$$| g(x) - g(y) | < \varepsilon.$$

If ε is a positive number and $[a, b]$ is an interval of S such that f is of bounded variation on $[a, b]$ and g is uniformly quasi-continuous on $[a, b]$ then it can be shown that there are $(f - g - \varepsilon)$ chains $\{s\}_0^{2n}$ and $\{r\}_0^{2n}$ such that $s_0 = r_{2n} = a$ and $s_{2n} = r_0 = b$. Since g is uniformly quasi-continuous on $[a, b]$, there is an increasing sequence $\{t\}_0^n$ such that $t_0 = a$, $t_n = b$, and if p is a non-negative integer less than n and $t_p < x < y < t_{p+1}$ then $| g(x) - g(y) | < \varepsilon$. For each integer p in $[0, n]$ define s_{2p} to be t . For each non-negative integer p less than n define s_{2p+1} to be a member y of S such that $s_{2p} < y \leq s_{2p+2}$ and if $s_{2p} < x \leq y$ then $\int_x^y \| df \| < \varepsilon/n$. The sequence r may be defined in an analogous manner. This result will help to establish the existence of certain integrals which are defined next.

Suppose (a, b) is in $S \times S$; $\{s\}_0^n$ is a subdivision of (a, b), that is, a monotone sequence whose final set is a subset of S such that $s_0 = a$ and $s_n = b$; and each of f and g is a function mapping S into N. $(L) \sum_s df \cdot g$ is defined by

$$\sum_1^n [f(s_p) - f(s_{p-1})] \cdot g(s_{p-1}).$$

The integral $(L) \int_a^b df \cdot g$, when it exists, is the member of N which is approximated by sums of the form $(L) \sum_s df \cdot g$ in the manner of successive refinements of subdivisions. Each of $(L) \int_a^b g \cdot df$, $(R) \int_a^b df \cdot g$, and $(R) \int_a^b g \cdot df$ is defined in a similar way with

$$(R) \sum_s df \cdot g = \sum_1^n [f(s_p) - f(s_{p-1})] \cdot g(s_p).$$

THEOREM 1.1. *If each of f and g is a function from S to N, $[a, b]$ is an interval of S such that $| g |$ is bounded by K on $[a, b]$ and f is of bounded variation on $[a, b]$, s is an $(f - g - \varepsilon)$ chain which is a subdivision of either (a, b) or (b, a), and t refines s then each of*

$$| (L) \textstyle\sum_t df \cdot g - (L) \sum_s df \cdot g | \quad \textit{and} \quad | (L) \sum_t g \cdot df - (L) \sum_s g \cdot df |$$

is less than $\{2K + \int_a^b | df |\}\varepsilon$.

Each term of $(L) \sum_s df \cdot g$ can be expressed in the form

$$\sum_{p=u}^v [f(t_p) - f(t_{p-1})] \cdot g(t_{u-1}).$$

By observing that the first term of the sum is the same as the corresponding term of $(L) \sum_t df \cdot g$, the required inequality is readily obtained. Similar remarks hold for $(L) \sum_s g \cdot df$ and $(L) \sum_t g \cdot df$.

COROLLARY 1.1. *If each of f and g is a function from S to N and $[a, b]$ is an interval of S such that g is uniformly quasi-continuous on $[a, b]$ and f is of bounded variation on $[a, b]$ then each of the following integrals exists:*

$$(L)\int_a^b df\cdot g,\quad (L)\int_a^b g\cdot df,\quad (L)\int_b^a df\cdot g,\quad (L)\int_b^a g\cdot df.$$

THEOREM 1.2. *If a and b are in S and each of f and g is a function from S to N then $(L)\int_a^b df\cdot g$ exists only in case $(R)\int_a^b f\cdot dg$ exists, in which case*

$$(L)\int_a^b df\cdot g + (R)\int_a^b f\cdot dg = f(b)\cdot g(b) - f(a)\cdot g(a).$$

The theorem becomes clear upon considering approximating sums for the integrals with respect to a common subdivision.

THEOREM 1.3. *If $[a, b]$ is an interval of S, h is a non-decreasing function from $[a, b]$ to the non-negative numbers, and n is a positive integer then, for each x in $[a, b]$,*

$$(L)\int_a^x h^n\cdot dh \le (h^{n+1}(x) - h^{n+1}(a))/(n+1).$$

With respect to a subdivision s of (a, b), $h^{n+1}(x) - h^{n+1}(a)$ can be written as a sum of terms of the form $h^{n+1}(s_p) - h^{n+1}(s_{p-1})$. By factoring $h(s_p) - h(s_{p-1})$ from each term the desired result may be obtained for approximating sums for $(L)\int_a^x h^n\cdot dh$.

THEOREM 1.4. *If $[a, b]$ is an interval of S, each of h and m is a function from $[a, b]$ to the numbers such that h is non-decreasing and m is bounded above, and K is a non-negative number such that*

$$m(x) \le K + (L)\int_a^x m\cdot dh$$

for each x in $[a, b]$ then

$$m(x) \le K\cdot\exp\,(h(x) - h(a))$$

for each x in $[a, b]$.

Without loss of generality it may be assumed that $h(a)$ is 0. By repeated application of the inequality in the hypothesis and the one in the preceding theorem, a series expansion for $K\cdot\exp\,[h]$ is obtained which bounds m.

It is evident that the statements of the preceding two theorems are true if h is non-increasing instead of non-decreasing and the limits of the integrals are from b to x instead of from a to x. These alternatives as well as the theorems themselves will be useful.

2. The class $\mathfrak{F}$ of functions

DEFINITION 2.1. Suppose $\{s\}_0^n$ is an increasing sequence with final set a subset of S and n is a positive integer. The statement that r is a triple refinement of s means that

(i) r is a sequence from the integers in $[0, 3n]$ to S,
(ii) $r_{3p} = s_p$ for each integer p in $[0, n]$,
(iii) if p is an integer in $[0, n]$ and there is no member of S between s_p and s_{p+1} then $r_{3p+1} = s_p$ and $r_{3p+2} = s_{p+1}$, and
(iv) if p is a non-negative integer less than n and there is a member of S between s_p and s_{p+1} then $s_p < r_{3p+1} \leq r_{3p+2} < s_{p+1}$

If r is a triple refinement of s then $A(r)$ will denote the set to which t belongs only in case there is an integer p such that either $t = s_p$ or $r_{3p+1} \leq t \leq r_{3p+2}$.

DEFINITION 2.2. The statement that s is an $[a, x, b]$ F-chain means that F maps $S \times S$ into N, $[a, b]$ is an interval of S such that $a \leq x \leq b$, and s is the minimal increasing subdivision t of (a, b) with x in its final set such that if c is a positive number and r is a triple refinement of t then there is a y less than x—if x is not the first member of S—and a z greater than x—if x is not the last member of S—such that if w is in $A(r)$ and either $y \leq u \leq v < x$ or $x < u \leq v \leq z$ then $\int_u^v |dF[w, I]| < c$.

DEFINITION 2.3. The statement that r is an s-complete $[a, x, b]$ F-chain means that F maps $S \times S$ into N, $[a, b]$ is an interval of S such that $a \leq x \leq b$, r is an increasing subdivision of $[a, b]$, s is a subsequence of r, there is an $[a, x, b]$ F-chain which is a subsequence of r, if y is in the final set of r and $a < y \leq x$ then there is an $[a, y, y]$ F-chain which is a subsequence of r, and if y is in the final set of r such that $x \leq y < b$ then there is a $[y, y, b]$ F-chain which is a subsequence of r.

DEFINITION 2.4. The statement that g is a super function for F on $[a, b]$ means that F maps $S \times S$ into N, $[a, b]$ is an interval of S, g is a non-decreasing function from $[a, b]$ to the numbers, and there is an increasing subdivision $\{s\}_0^n$ of (a, b) such that if p is a non-negative integer less than n and each of (x, u) and (x, v) is in $[s_p, s_{p+1}] \times [s_p, s_{p+1}]$ then

$$|F(x, u) - F(x, v)| \leq |g(u) - g(v)|.$$

THEOREM 2.1. *Suppose F is a function from $S \times S$ to N such that, for each interval $[a, b]$ of S and each x in $[a, b]$, there is an $[a, x, b]$ F-chain. If there is a super function g for F on $[a, b]$, x is in $[a, b]$, and s is a non-decreasing finite sequence with final set a subset of $[a, b]$ then there is an s-complete $[a, x, b]$ F-chain.*

Proof. Let F, $[a, b]$, x, g, and s be as in the hypothesis of the theorem. Suppose $x < b$. Let t be an increasing subdivision of (x, b) such that if p is an integer and each of (y, u) and (y, v) is in

$$[t_{p-1}, t_p] \times [t_{p-1}, t_p]$$

then

$$|F(y, u) - F(y, v)| \leq |g(u) - g(v)|.$$

Define the sequence R as follows: R_0 is the increasing sequence whose final set is the subset of $[x, b]$ to which y belongs only in case y is in the final set of either s or t or the $[x, x, b]$ F-chain. If p is a non-negative integer then R_{p+1} is the increasing sequence whose final set is the set to which y belongs only in case y is in the final set of R_p or there is a z in the final set of R_p such that y is in the final set of the $[z, z, b]$ F-chain. By observing that if $t_{q-1} < z < t_q$ and r is the $[z, z, b]$ F-chain then $t_q \leq r_1$, an inductive argument may be used to demonstrate that if y is in the final set of R_{p+1} and y is in $[t_0, t_{p+1}]$ then y is in the final set of R_p. Therefore, if m is an integer not less than n then R_m is R_n. If $a < x$ then a subdivision analogous to R_n may be constructed for (x, a). Combining these results one obtains an s-complete $[a, x, b]$ F-chain.

In the definition which follows, $F[I, x+]$ is the function f from S to N such that $f(t)$ is the limit of $F(t, h)$ as h approaches x from the right. The analogous definition of $F[I, x-]$ is evident.

Definition 2.5. $\mathfrak{F}$ denotes the set to which the function F from $S \times S$ to N belongs only in case

(i) if x is in S then $F(x, x) = 1$,

(ii) if x is in S and $[a, b]$ is an interval of S then each of $F[I, x]$ and $F[I, x+]$ —if x is not the last member of S—and $F[I, x-]$—if x is not the first member of S—is uniformly quasi-continuous on $[a, b]$,

(iii) if $[a, b]$ is an interval of S then there is a number K such that if x is in $[a, b]$ then $\int_a^b |dF[x, I]| < K$,

(iv) if $[a, b]$ is an interval of S then there is a super function for F on $[a, b]$, and

(v) if $[a, b]$ is an interval of S and x is in $[a, b]$ then there is an $[a, x, b]$ F-chain.

Theorem 2.2. *If F is in $\mathfrak{F}$, $[a, b]$ is an interval of S, Q is a function from $[a, b]$ to N which is uniformly quasi-continuous, x is a or x is b, X is L or X is R, and the function P from $[a, b]$ to N is defined by $P(t) = (X) \int_x^t dF[t, I] \cdot Q$ then P is uniformly quasi-continuous.*

Proof. The conclusion is true if Q is a step function. Since Q is the uniform limit of a sequence of step functions and $\int_a^b |dF[t, I]|$ is uniformly bounded for t in $[a, b]$, P is the uniform limit of a sequence of uniformly quasi-continuous functions. Consequently, P is uniformly quasi-continuous.

3. The mapping $\mathfrak{K}$

Theorem 3.1. *If F is in $\mathfrak{F}$ then there is only one function M from $S \times S$ to N such that*

(i) *M is bounded on each square of $S \times S$ and*

(ii) *$M(t, x) = 1 + (L) \int_x^t dF[t, I] \cdot M[I, x]$ for each (t, x) in $S \times S$.*

Moreover, if x is in S then $M[I, x]$ is uniformly quasi-continuous on each interval of S.

Proof. Let $[a, b]$ be an interval of S. Define the sequence G each value of which is a function from $[a, b] \times [a, b]$ to N as follows: $G_0 = 1$ and if p is a non-negative integer then

$$G_{p+1}(t, x) = (L) \int_x^t dF\,[t, I]\cdot G_p[I, x].$$

Let g be a super function for F on $[a, b]$ such that $g(a) \geq 1$, and let $\{s\}_0^n$ be a subdivision of (a, b) such that if p is a positive integer not greater than n and (x, u) and (x, v) are in

$$[s_{p-1}, s_p] \times [s_{p-1}, s_p]$$

then

$$|F(x, u) - F(x, v)| \leq |g(u) - g(v)|.$$

Let K be a number such that $1 + \int_a^b |dF[x, I]| \leq K$ for each x in $[a, b]$. Suppose p is a positive integer not greater than n. Define the function k from $[s_{p-1}, b]$ to the numbers as follows: (i) if y is in $[s_{p-1}, s_p]$ then $k(y) = g(y)$; (ii) if q is an integer such that $p \leq q < n$ and $s_q < y \leq s_{q+1}$ then

$$k(y) = k(s_q) + K\cdot\exp\,(k(s_q) + g(s_{q+1}) - g(s_q)).$$

By observing that if y is a number not less than one and m is a positive integer then $y^{m-1}/(m-1)! \leq \exp\,(y)^m/m!$ and by employing an induction argument involving both q and m one may ascertain that if m is a positive integer,

$$x \leq t,\ s_{p-1} \leq x \leq s_p,\ \text{and}\ s_{p-1} \leq s_q \leq t \leq s_{q+1}$$

then

$$|G_m(t, x)| \leq (k^m(t) - k^m(x))/m\,!.$$

Since a similar result holds for $t \leq x$, the sequence $\sum G$ converges to P uniformly on $[a, b] \times [a, b]$. For each non-negative integer p and each x in $[a, b]$, G_p is bounded and $G_p[I, x]$ is uniformly quasi-continuous on $[a, b]$. Therefore, P is bounded and $P[I, x]$ is uniformly quasi-continuous on $[a, b]$ for each x in $[a, b]$.

Suppose Q is a bounded function from $[a, b] \times [a, b]$ to N such that, for each (t, x) in $[a, b] \times [a, b]$,

$$Q(t, x) = 1 + (L) \int_x^t dF\,[t, I]\cdot Q[I, x].$$

Suppose $s_{p-1} \leq x < s_p$ and the function m from $[a, b]$ to the non-negative numbers is defined by $m(t) = |P(t, x) - Q(t, x)|$. By using an inductive argument one can show that if q is an integer and $x \leq t \leq s_q$ then

$$m(t) \leq (L) \int_x^t dg\cdot m.$$

Consequently, $m = 0$ on $[x, b]$. By employing similar methods it is evident that if $a \leq t \leq x$ then $m(t) = 0$. Hence $m = 0$ on $[a, b]$ and $P = Q$. If M denotes the union of all such functions P then M is the desired function.

Definition 3.1. $\mathfrak{K}$ will denote the function to which the ordered pair

(F, M) belongs only in case F is in $\mathfrak{F}$ and M is the function from $S \times S$ to N which is bounded on each square of $S \times S$ and such that, for each (t, x) in $S \times S$,

$$M(t, x) = 1 + (L) \int_x^t dF\,[t, I] \cdot M[I, x].$$

THEOREM 3.2. *If (F, M) is in $\mathfrak{K}$ and $[a, b]$ is an interval of S then there is a super function for M on $[a, b]$.*

Proof. Let g and s be as in the proof of the preceding theorem and suppose that $s_{p-1} \leq u \leq v \leq s_p$. If $v \leq t \leq s_p$ then

$$| M(t, u) - M(t, v) | \leq \left| (L) \int_u^v dF\,[t, I] \cdot M[I, u] \right| + \left| (L) \int_v^t dF\,[t, I] \cdot \{M[I, u] - M[I, v]\} \right|.$$

Let K denote an upper bound for $| M |$ on $[a, b] \times [a, b]$. If $v = t = s_p$ then $| M(t,u) - M(t, v | \leq K \cdot (g(v) - g(u))$. Suppose $v < s_p$ and m denotes the function from $[v, s_p]$ to the non-negative numbers defined by $m(t) = | M(t, u) - M(t, v) |$. For each t in $[v, s_p]$,

$$m(t) \leq K \cdot (g(v) - g(u)) + (L) \int_v^t dg \cdot m,$$

hence $m(t) \leq K \cdot (g(v) - g(u)) \exp\,(g(t) - g(v)) \leq K \cdot \exp\,(g(b) - g(a)) \cdot (g(v) - g(u))$.
A similar result holds when $s_{p-1} \leq t \leq u$.

Define the function h from $[a, b]$ to the numbers by

$$h(t) = K \cdot \exp\,(g(b) - g(a)) \cdot g(t).$$

Suppose q is a positive integer not greater than n, (t, u) and (t, v) are in $[s_{q-1}, s_q] \times [s_{q-1}, s_q]$, and $u < v$. If either $t \leq u$ or $v \leq t$ then

$$| M(t, u) - M(t, v) | \leq h(v) - h(u).$$

If $u < t < v$ then

$$\begin{aligned} | M(t, u) - M(t, v) | &\leq | M(t, u) - M(t, t) | + | M(t, t) - M(t, v) | \\ &\leq h(t) - h(u) + h(v) - h(t) \\ &= h(v) - h(u). \end{aligned}$$

Therefore h is a super function for M on $[a, b]$.

THEOREM 3.3. *If (F, M) is in $\mathfrak{K}$ and $[a, b]$ is an interval of S then there is a number R such that if x is in $[a, b]$ then $\int_a^b | dM[x, I] | \leq R$.*

Proof. Let K' *be an upper bound for* $|M|$ *on* $[a, b] \times [a, b]$ and let K be a number such that, for each x in $[a, b]$,

$$K \geq 1 + K' + \int_a^b |dF[x, I]|.$$

Let g denote a super function for F and M on $[a, b]$. Let $\{s\}_0^n$ be an increasing subdivision of (a, b) such that if p is a positive integer not greater than n and (t, u) and (t, v) are in $[s_{p-1}, s_p] \times [s_{p-1}, s_p]$ then

$$|F(t, u) - F(t, v)| \leq |g(u) - g(v)|,$$

$$|M(t, u) - M(t, v)| \leq |g(u) - g(v)|.$$

Define the non-decreasing number sequence $\{L\}_0^n$ as follows: $L_0 = 0$, $L_1 = g(s_1) - g(s_0)$, and if q is a positive integer less than n then

$$L_{q+1} = (g(s_{q+1}) - g(s_q) + K^2 + KL_q) \cdot \exp\ (g(s_{q+1}) - g(s_q)).$$

If $s_0 \leq t \leq s_1$ and r is a subdivision of $[a, t]$ then

$$\textstyle\sum_r |dM[t, I]| \leq L_1.$$

Suppose q is a positive integer less than n such that if p is a positive integer not greater than q, $s_{p-1} \leq t \leq s_p$, and r is a subdivision of $[a, t]$ then

$$\textstyle\sum_r |dM[t, I]| \leq L_p.$$

Suppose $s_q < t \leq s_{q+1}$ and r is a subdivision of (a, t) for which $\{s\}_0^q$ is a subsequence. Define the function m from $[a, t]$ to the non-negative numbers as follows:

(i) if $a \leq x \leq r_1$ then $m(x) = 0$;
(ii) if $r_1 < x \leq t$ and k is the largest integer p such that $r_p \leq x$ then

$$m(x) = \textstyle\sum_1^k |M(x, r_p) - M(x, r_{p-1})| + |M(x, x) - M(x, r_k)|.$$

Note that $m \leq L_q$ on $[a, s_q]$. Let e be an integer such that $r_e = s_q$. Suppose x is in $[s_q, t]$. If f is the function from $[a, x]$ to the non-negative numbers defined by $f(y) = \int_a^y |dF[x, I]|$ then

$$\begin{aligned} m(x) &\leq g(s_{q+1}) - g(s_q) + \sum_1^e |M(x, r_p) - M(x, r_{p-1})| \\ &< g(s_{q-1}) - g(s_q) + \sum_1^e \left| (L) \int_{r_{p-1}}^{r_p} dF[x, I] \cdot M[I, r_{p-1}] \right| \\ &\quad + \sum_{p=1}^e \left| (L) \int_{r_p}^{r_e} dF[x, I] \cdot (M[I, r_p] - M[I, r_{p-1}]) \right| \\ &\quad + \sum_{p=1}^e \left| (L) \int_{s_q}^{x} dF[x, I] \cdot (M[I, r_p] - M[I, r_{p-1}]) \right| \end{aligned}$$

$$\leq g(s_{q+1}) - g(s_q) + K^2$$

$$+ \sum_{p=1}^{e} \sum_{i=p+1}^{e} (L) \int_{r_{i-1}}^{r_i} df \cdot | M[I, r_p] - M[I, r_{p-1}] |$$

$$+ (L) \int_{s_q}^{x} dg \sum_{1}^{e} | M[I, r_p] - M[I, r_{p-1}] |$$

$$\leq g(s_{q+1}) - g(s_q) + K^2$$

$$+ \sum_{i=2}^{e} (L) \int_{r_{i-1}}^{r_i} df \sum_{p=1}^{i-1} | M[I, r_p] - M[I, r_{p-1}] |$$

$$+ (L) \int_{s_q}^{x} m\, dg$$

$$\leq g(s_{q+1}) - g(s_q) + K^2 + (L) \int_a^{s_q} df \cdot m + (L) \int_{s_q}^{x} dg \cdot m$$

$$\leq (g(s_{q+1}) - g(s_q) + K^2 + KL_q) + (L) \int_{s_q}^{x} m \cdot dg.$$

Therefore, for each x in $[s_q, t]$,

$$m(x) \leq (g(s_{q+1}) - g(s_q) + K^2 + KL_q) \exp (g(x) - g(s_q))$$
$$\leq L_{q+1} \leq L_n .$$

In a similar manner it can be shown that if t is in $[a, b]$ then

$$\int_x^b | dM[t, I] | \leq L_n .$$

Therefore, $2L_n$ is a number such that, for each t in $[a, b]$,

$$\int_a^b | dM[t, I] | \leq 2L_n .$$

THEOREM 3.4. *If (F, M) is in $\mathcal{K}$, $[c, b]$ is an interval of S, and a is in $[c, b]$ then there is a $[c, a, b]$ M-chain.*

Proof. Suppose $[a, b]$ is an interval of S. Let K' denote an upper bound for $| F | + | M |$ on $[a, b] \times [a, b]$ and let K be a number such that

$$K \geq 1 + K' + \int_a^b \left| dF[x, I] \right| + \int_a^b \left| dM[x, I] \right|$$

for each x in $[a, b]$. Let g be a super function for F and M on $[a, b]$, and let s' be a subdivision of (a, b) such that if (x, u) and (x, v) are in $[s'_{p-1}, s'_p] \times [s'_{p-1}, s'_p]$ then

$$| F(x, u) - F(x, v) | \leq | g(u) - g(v) |,$$
$$| M(x, u) - M(x, v) | \leq | g(u) - g(v) |.$$

Let $\{s\}_0^n$ be an s'-complete $[a, a, b]$ F-chain. If r is a non-decreasing finite sequence with final set a subset of $[a, b]$ then m_r will denote the function from $[a, b]$ to the non-negative numbers defined by $m_r(x) = \sum_r | dM[x, I] |$.

Let ε be a positive number. Suppose q is a non-negative integer less than $n - 1$ such that if t is a triple refinement of $\{s\}_0^{q+1}$ then there is a y in S such that $a < y \leq s_1$ and such that if r is a non-decreasing finite sequence with final set a subset of $(a, y]$ then $m_r < \varepsilon$ on $A(t)$. Evidently 0 is such an integer if $n \neq 1$. Suppose t' is a triple refinement of $\{s\}_0^{q+2}$. Let t be a triple refinement of $\{s\}_0^{q+2}$ such that $A(t')$ is a subset of $A(t)$, $g(t_{3q+4}) - g(t_{3q+3}+) < \varepsilon$, and

$$\sum_{p=0}^{q} \left\{ \int_{t_{3p}+}^{t_{3p+1}} | dF[x, I] | + \int_{t_{3p+2}}^{t_{3p+3}-} | dF[x, I] | \right\} < \varepsilon$$

for each x in $[t_{3q+4}, t_{3q+5}]$. Note that in the preceding $g(t_{3q+3}+)$ and the integrals of the form $\int_{u+}^{v}$ and $\int_{u}^{v-}$ denote the usual limits. Let y be a member of S such that

(i) $a < y \leq s_1$,

(ii) $\int_{a+}^{y} | dM[s_{q+2}, I] | < \varepsilon$,

(iii) if r is a non-decreasing finite sequence with final set a subset of $(a, y]$ then $m_r < \varepsilon/(q + 1)$ on $A(\{t\}_0^{3q+3})$,

(iv) $g(y) - g(a+) < \varepsilon$,

(v) $\sum_1^{q+1} \int_{a+}^{y} | dM[t_{3p} -, I] | < \varepsilon$,

(vi) $\int_{a+}^{y} | dF[x, I] | < \varepsilon$ for each x in $A(t)$.

Suppose $\{r\}_0^e$ is a non-decreasing finite sequence with final set a subset of $(a, y]$ and x is in $[t_{3q+4}, t_{3q+5}]$. In order to abbreviate what follows, $H(i)$ will denote

$$dF[x, I] \cdot (M[I, r_i] - M[I, r_{i-1}]).$$

Observe that

$$\begin{aligned} m_r(x) \leq \sum_{i=1}^{e} \Bigg\{ & \left| (L) \int_{r_{i-1}}^{r_i} dF[x, I] \cdot M[I, r_{i-1}] \right| \\ & + \left| (L) \int_{r_i}^{t_3} H(i) \right| + \left| (L) \int_{t_{3q+3}}^{t_{3q+4}} H(i) \right| + \left| (L) \int_{t_{3q+4}}^{x} H(i) \right| \Bigg\} \\ & + \sum_{p=1}^{q} \sum_{i=1}^{e} \Bigg\{ \left| (L) \int_{t_{3p}}^{t_{3p+1}} H(i) \right| \\ & + \left| (L) \int_{t_{3p+1}}^{t_{3p+2}} H(i) \right| + \left| \int_{t_{3p+2}}^{t_{3p+3}} H(i) \right| \Bigg\}. \end{aligned}$$

Since x is in $A(t)$,

$$\sum_{i=1}^{e} \int_{r_{i-1}}^{r_i} | dF[x, I] | < \varepsilon$$

and hence

$$\sum_{i=1}^{e}\left|(L)\int_{r_{i-1}}^{r_i} dF[x,I]\cdot M[I,r_{i-1}]\right| < \varepsilon K.$$

Since $g(y) - g(a+) < \varepsilon$,

$$\textstyle\sum_{i=1}^{e} |M[I,r_i] - M[I,r_{i-1}]| < \varepsilon$$

on $[r_1, t_3]$ and hence

$$\sum_{i=1}^{e}\left|(L)\int_{r_i}^{t_3} H(i)\right| < K\varepsilon.$$

If p is a positive integer not greater than q then

$$\sum_{i=1}^{e}\left|(L)\int_{t_{3p}}^{t_{3p+1}} H(i)\right| \le m_r(t_{3p})\int_{t_{3p}}^{t_{3p+1}}\left|dF[x,I]\right| + K\int_{t_{3p}+}^{t_{3p+1}}\left|dF[x,I]\right|$$
$$< K\left\{\epsilon/(q+1) + \int_{t_{3p}+}^{t_{3p+1}} |dF[x,I]|\right\}$$

and hence

$$\sum_{p=1}^{q}\sum_{i=1}^{e}\left|(L)\int_{t_{3p}}^{t_{3p+1}} H(i)\right| < 2K\varepsilon.$$

If p is a positive integer not greater than q then

$$\sum_{i=1}^{e}\left|(L)\int_{t_{3p+2}}^{t_{3p+3}} H(i)\right| \le K\int_{t_{3p+2}}^{t_{3p+3}-}\left|dF[x,I]\right|$$
$$+\int_{t_{3p+2}}^{t_{3p+3}}\left|dF[x,I]\right|\sum_{i=1}^{e}\left|M(t_{3p+3}-,r_i)\right.$$
$$\left. - M(t_{3p-3}-,r_{i-1})\right|$$
$$\le K\left\{\int_{t_{3p+2}}^{t_{3p+3}-}\left|dF[x,I]\right| + \int_{a+}^{y}\left|dM[t_{3p+3}-,I]\right|\right\}$$

and hence

$$\textstyle\sum_{p=1}^{q}\sum_{i=1}^{e}\left|(L)\int_{t_{3p+2}}^{t_{3p+3}} H(i)\right| < 2K\varepsilon.$$

Since $m_r < \varepsilon$ on $A(\{t\}_0^{3q+3})$,

$$\sum_{p=1}^{q}\sum_{i=1}^{e}\left|(L)\int_{t_{3p+1}}^{t_{3p+2}} H(i)\right| < \varepsilon\sum_{p=1}^{q}\int_{t_{3p+1}}^{t_{3p+2}}\left|dF[x,I]\right| < \varepsilon K.$$

Since $g(t_{3q+4}) - g(t_{3q+3}+) < \varepsilon$,

$$\sum_{i=1}^{e}\left|(L)\int_{t_{3q+3}}^{t_{3q+4}} H(i)\right| \le m_r(t_{3q+3})\int_{t_{3q+3}}^{t_{3q+4}} |dF[x,I]| + K\int_{t_{3q+3}+}^{t_{3q+4}} |dF[x,I]|$$
$$< \varepsilon K + K(g(t_{3q+4}) - g(t_{3q+3}+)) < 2K\varepsilon.$$

Finally,

$$\sum_{i=1}^{e} \left| (L) \int_{t_{3q+4}}^{x} H(i) \right| \leq \int_{t_{3q+4}}^{x} dg \cdot m_r .$$

Therefore,

$$m_r(x) \leq 9K\varepsilon + (L) \int_{t_{3q+4}}^{x} dg \cdot m_r$$

for each x in $[t_{3q+4}, t_{3q+5}]$ and hence

$$m_r \leq 9K\varepsilon \exp \left(g(t_{3q+5}) - g(t_{3q+4})\right)$$

on $[t_{3q+4}, t_{3q+5}]$. Therefore,

$$m_r \leq 9K\varepsilon \exp \left(g(b) - g(a)\right)$$

on $A[\{t\}_0^{3q+6})$ and hence on $A(\{t'\}_0^{3q+6})$. Consequently, if ε is a positive number and t' is a triple refinement of s then there is a y in S such that $a < y$ and if $a < u \leq v \leq y$ and x is in $A(t')$ then $\int_u^v |dM[x, I]| < \varepsilon$.

In a similar manner it may be shown that if ε is a positive number, t' is a triple refinement of s, and a is not the first member of S then there is a z in S such that $z < a$ and if $z \leq u \leq v < a$ and x is in $A(t')$ then $\int_u^v |dM[x, I]| < \varepsilon$. Therefore there is an $[a, a, b]$ M-chain. If $c < a$ then the same techniques establish the existence of a $[c, a, a]$ M-chain. Consequently there is a $[c, a, b]$ M-chain.

THEOREM 3.5. *If each of F and $G + 1$ is in $\mathfrak{F}$ and H is the function from $S \times S$ to N defined by*

$$H(t, x) = (L) \int_x^t dF[t, I] \cdot G[I, x]$$

then $H + 1$ is in $\mathfrak{F}$.

Proof. Suppose $[a, b]$ is an interval of S, the funtion Q from $[a, b]$ to N is uniformly quasi-continuous on $[a, b]$, and P is the function from $[a, b] \times [a, b]$ to N defined by

$$P(w, x) = (L) \int_x^w dF[w, I] \cdot Q$$

when $x < w$ and $P(w, x) = 0$ otherwise. If Q is a simple step function whose only values are 0 and 1 then one can readily ascertain that $P[I, a+]$ is uniformly quasi-continuous on $[a, b]$. Since every step function is a linear combination of simple step functions whose only values are 0 and 1, a similar result is obtained if Q is a step function. Since Q is the uniform limit of a sequence of step functions, $P[I, a+]$ is uniformly quasi-continuous on $[a, b]$. Suppose Q is $G[I, a+]$. By employing the fact that if s is a subdivision of (a, b) and w is in $[a, b]$ then there is a triple refinement t of s such that w is in $A(t)$, one can conclude that $H[I, a+] = P[I, a+]$ on $(a, b]$. Consequently, $H[I, a+]$ is uniformly quasi-continuous on $[a, b]$. The other cases $H[I, a-]$,

$H[I, b-]$, and $H[I, b+]$ can be handled in a similar manner. Since G is in $\mathfrak{F}$ one can use techniques which are analogous, but simpler in nature, to those used in proving some of the preceding theorems in order to show that $H + 1$ satisfies the remaining requirements for membership in $\mathfrak{F}$.

THEOREM 3.6. *If (F, M) is in $\mathfrak{H}$ then M is in $\mathfrak{F}$.*

By the preceding theorem one observes that each value of the sequence of partial sums used to define M in the proof of Theorem 3.1 is the restriction to a square of some member of $\mathfrak{F}$. Since the convergence is uniform, M is in $\mathfrak{F}$.

THEOREM 3.7. *If P is in $\mathfrak{F}$; $[a, b]$ is an interval of S; and each of Q, K, D, and E is a function from $[a, b]$ to N such that Q is of bounded variation, K is uniformly quasi-continuous on $[a, b]$, D is defined by*

$$D(t) = (L) \int_t^b dQ \cdot P[I, t],$$

and E is defined by

$$E(t) = (R) \int_a^t P[t, I] \cdot dK$$

then $(L) \int_a^b dQ \cdot E = (R) \int_a^b D \cdot dK$.

The statement of the theorem can be established for the case where K is a simple step function whose only values are 0 and 1 and hence for the case where K is a step function. The proof can then be completed by observing that K is the uniform limit of a sequence of step functions.

THEOREM 3.8. *If (F, M) is in $\mathfrak{H}$ then (M, F) is in $\mathfrak{H}$. Hence $\mathfrak{H}$ is a reversible function from $\mathfrak{F}$ onto $\mathfrak{F}$.*

Proof. For each (t, x) in $S \times S$,

$$1 + (L) \int_x^t dM[t, I] \cdot F[I, x] = 1 + F(t, x) - M(t, x) - (R) \int_x^t M[t, I] \cdot dF[I, x].$$

Define the function H from $S \times S$ to N by

$$H(t, x) = 1 - (R) \int_x^t M[t, I] \cdot dF[I, x].$$

Using Theorem 3.5 observe that H is in $\mathfrak{F}$. With the aid of the preceding theorem one can show that, for each (t, x) in $S \times S$,

$$H(t, x) = 1 + (L) \int_x^t dF[t, I] \cdot H[I, x].$$

Consequently H is M and by the first equation (M, F) is in $\mathfrak{H}$.

4. The non-homogeneous case

In this section N' will denote the complete normed ring of 2×2 matrices over N with norm $\| \cdot \|$ defined by

$$\| A \| = \max \textstyle\sum_{j=1}^{2} | A_{ij} |, \quad 1 \leq i \leq 2.$$

With N' replacing N, $\mathfrak{F}'$ and $\mathcal{H}'$ are defined in a manner analogous to that of $\mathfrak{F}$ and $\mathcal{H}$ respectively. The next two observations will be needed and are stated without proof. If F is in $\mathfrak{F}'$ and the function G from $S \times S$ to N is defined by $G(t, x) = F(t, x)_{11}$, $G(t, x) = F(t, x)_{22}$, $G(t, x) = F(t, x)_{12} + 1$, or $G(t, x) = F(t, x)_{21} + 1$ then G is in $\mathfrak{F}$. If $\{F\}_0^5$ is a sequence with final set a subset of $\mathfrak{F}$ then the function G from $S \times S$ to N' defined by

$$G(t, x) = \begin{vmatrix} F_0(t, x) & F_2(t, x) - F_3(t, x) \\ F_4(t, x) - F_5(t, x) & F_1(t, x) \end{vmatrix}$$

is in $\mathfrak{F}'$.

THEOREM 4.1. *If each of (F, M) and (E, J) is in $\mathcal{H}$ then there is only one function P from $S \times S$ to N, bounded on each square of $S \times S$, such that, for each (t, x) in $S \times S$,*

$$P(t, x) = E(t, x) + (L) \int_x^t dF[t, I] \cdot P[I, x].$$

Moreover, P is in $\mathfrak{F}$ and, for each (t, x) in $S \times S$,

(i) $P(t, x) = E(t, x) - (L) \int_x^t dM[t, I] \cdot E[I, x]$

(ii) $M(t, x) = J(t, x) - (L) \int_x^t dP[t, I] \cdot J[I, x]$.

PROOF. Let F' denote the member of $\mathfrak{F}'$ defined by

$$F'(t, x) = \begin{vmatrix} F(t, x) & F(t, x) - E(t, x) \\ 0 & 1 \end{vmatrix}$$

and let M' denote $\mathcal{H}'(F')$. Then there is a P in $\mathfrak{F}$ such that, for each (t, x) in $S \times S$,

$$M'(t, x) = \begin{vmatrix} M(t, x) & P(t, x) - 1 \\ 0 & 1 \end{vmatrix}.$$

If (t, x) is in $S \times S$ then

$$M'(t, x) = 1 + (L) \int_x^t dF'[t, I] \cdot M'[I, x],$$

where 1 is the multiplicative identity of N'; hence

$$P(t, x) - 1 = (L) \int_x^t dF[t, I] \cdot (P[I, x] - 1) + (L) \int_x^t d(F[t, I] - E[t, I])$$

$$= (L) \int_x^t dF[t, I] \cdot P[I, x] - 1 + E(t, x).$$

Uniqueness can be established by using Theorem 1.4. Since (M', F') is in $\mathfrak{K}'$,

$$F'(t, x) = 1 + (L) \int_x^t dM'[t, I] \cdot F'[I, x]$$

and therefore

$$F(t, x) - E(t, x) = (L) \int_x^t dM[t, I] \cdot (F[I, x] - E[I, x]) + (L) \int_x^t dP[t, I]$$

$$= F(t, x) - (L) \int_t^x dM[t, I] \cdot E[I, x] - P(t, x),$$

for each (t, x) in $S \times S$.

Let each of A and B be the member of $\mathfrak{F}'$ defined by

(i) $$A(t, x) = \begin{vmatrix} J(t, x) & 0 \\ M(t, x) - J(t, x) & 1 \end{vmatrix}$$

(ii) $$B(t, x) = \begin{vmatrix} E(t, x) & 0 \\ 1 - P(t, x) & 1 \end{vmatrix}.$$

Using the foregoing results, observe that (A, B) is in $\mathfrak{K}'$; consequently, (B, A) is in $\mathfrak{K}'$. Proceeding as before, one can deduce that, for each (t, x) in $S \times S$,

$$M(t, x) - J(t, x) = -(L) \int_x^t dP[t, I] \cdot J[I, x].$$

THEOREM 4.2. *If each of* (F, M) *and* (E, J) *is in* $\mathfrak{K}$ *then there is only one function* P *from* $S \times S$ *to* N, *bounded on each square of* $S \times S$, *such that, for each* (t, x) *in* $S \times S$,

$$P(t, x) = E(t, x) - (R) \int_x^t P[t, I] \cdot dF[I, x].$$

Moreover, P *is in* $\mathfrak{F}$ *and, for each* (t, x) *in* $S \times S$,

(i) $P(t, x) = E(t, x) + (R) \int_x^t E[t, I] \cdot dM[I, x]$

(ii) $M(t, x) = J(t, x) + (R) \int_x^t J[t, I] \cdot dP[I, X]$.

The result becomes evident upon substituting (J, E) for (F, M) and (M, F) for (E, J) in the preceding theorem.

5. Invariants of $\mathfrak{K}$

If $[a, b]$ is an interval of S and f is a function from $[a, b]$ to N then the statement that f is uniformly continuous means that if ε is a positive number then there is a subdivision s of (a, b) such that if $s_{p-1} \leq u \leq v \leq s_p$ then $|f(u) - f(v)| < \varepsilon$. Observe that if F is in $\mathfrak{F}$ and $F[I, x]$ is continuous on an interval $[a, b]$ of S then $F[I, x]$ is uniformly continuous on $[a, b]$ since $F[I, x]$ is uniformly quasi-continuous on each interval of S. If each of $[a, b]$ and $[c, d]$ is an interval of S and f is a function from $[a, b] \times [c, d]$ to N then the statement that f is uniformly continuous means that if ε is a positive number then there is a subdivision s of (a, b) and a subdivision t of (c, d) such that if (u, v) and (x, y) are in $[s_{p-1}, s_p] \times [t_{q-1}, t_q]$ then $|f(u, v) - f(x, y)| < \varepsilon$.

The next two theorems are stated without proofs. They may be verified by making rather natural arguments involving approximating sums.

THEOREM 5.1. *Suppose F is in $\mathfrak{F}$, each of $[a, b]$ and $[c, d]$ is an interval of S, Q is a uniformly continuous function from $[c, d] \times [c, d]$ to N, and P is a function from $[a, b] \times [c, d]$ to N defined by either*

(i) $P(t, x) = (L) \int_x^d dF[t, I] \cdot Q[I, x]$ *or*

(ii) $P(t, x) = (L) \int_x^c dF[t, I] \cdot Q[I, x]$.

If F is uniformly continuous on $[a, b] \times [c, d]$ then so is P.

THEOREM 5.2. *Suppose F is in $\mathfrak{F}$, each of $[a, b]$ and $[c, d]$ is an interval of S, Q is a uniformly continuous function from $[c. d]$ to N, and P is a function from $[a, b]$ to N defined by either*

(i) $P(t) = (L) \int_c^d dF[t, I] \cdot Q$ *or*

(ii) $P(t) = (L) \int_d^c dF[t, I] \cdot Q$

If for each y in $[c, d]$, $F[I, y]$ is continuous on $[a, b]$ then P is uniformly continuous.

THEOREM 5.3. *Suppose F is in $\mathfrak{F}$; $[c, d]$ is an interval of S; $[a, b]$ is a subinterval of $[c, d]$; Q is a uniformly continuous function from $[a, b] \times [c, d]$ to N; g is a non-decreasing function from $[a, b]$ to the numbers such that if each of u, v, and w is in $[a, b]$ then*

$$|F(w, u) - F(w, v)| \leq |g(u) - g(v)|;$$

and P is the function from $[a, b] \times [c, d]$ to N defined by

$$P(t, x) = (L) \int_x^t dF[t, I] \cdot Q[I, x]$$

if (t, x) is in $[a, b] \times [a, b]$,

$$P(t, x) = (L) \int_a^t dF[t, I] \cdot Q[I, x]$$

if $x < a$, and

$$P(t, x) = (L) \int_b^t dF[t, I] \cdot Q[I, x]$$

if $a < x$. If F is uniformly continuous on $[a, b] \times [c, d]$ then so is P.

Two preliminary observations are in order. (i) Suppose $\{s\}_0^{2n}$ is a non-decreasing $(f - h - \varepsilon)$ chain and $\{t\}_0^m$ is a non-decreasing sequence whose final set is a subset of $[s_0, s_{2n}]$ which contains every member of the final set of s which lies between t_0 and t_m. Then there is an $(f - h - \varepsilon)$ chain r which is a subdivision of (t_0, t_m) such that

$$(L) \textstyle\sum_t df \cdot h = (L) \sum_r df \cdot h;$$

hence, if K is an upper bound for h on $[t_0, t_m]$ then

$$\left| (L) \int_{t_0}^{t_m} df \cdot h - (L) \sum_t df \cdot h \right| \leq \left\{ \int_{t_0}^{t_m} | df | + 2K \right\} \varepsilon.$$

(ii) If ε is a positive number then there is a subdivision s' of (a, b) such that if x is in $[c, d]$ then s' is a $(g - Q[I, x] - \varepsilon)$ chain. This is a consequence of the uniform continuity of Q on $[a, b] \times [c, d]$. Therefore, for each (u, x) in $[a, b] \times [c, d]$, s' is an $(F[u, I] - Q[I, x] - \varepsilon)$ chain.

Let $\{s'\}_0^{2n}$ and ε be as in part (ii) of the foregoing. Let s be a subdivision of (c, d) such that s' is a subsequence of s and if (u, x) and (v, y) are in $[s_{p-1}, s_p] \times [s_{q-1}, s_q]$, for some integer pair (p, q), then

$$| Q(u, x) - Q(v, y) | < \varepsilon \quad \text{and} \quad | F(u, x) - F(v, y) | < \varepsilon/2n.$$

With respect to the partitioning of $[a, b] \times [c, d]$ determined by s, the argument may be completed by examining the approximating sums for $| P(u, x) - P(v, y) |$ which are derived from s in a natural way.

The next theorem can be proven in an analogous manner.

THEOREM 5.4. *Suppose F is in $\mathfrak{F}$, $[a, b]$ is an interval of S, Q is a uniformly continuous function from $[a, b]$ to N, and P is a function from $[a, b]$ to N defined by either*

(i) $P(t) = (L) \int_a^t dF[t, I] \cdot Q$ *or*

(ii) $P(t) = (L) \int_b^t dF[t, I] \cdot Q$.

If, for each y in $[a, b]$, $F[I, y]$ is continuous on $[a, b]$ and there is a super function g for F on $[a, b]$ such that if each of u, v, and w is in $[a, b]$ then

$$| F(w, u) - F(w, v) | \leq | g(u) - g(v) |;$$

then P is uniformly continuous.

The next two theorems are obtained by applying Theorems 5.3 and 5.4 to the by now familiar sequence whose sequence of partial sums converges uniformly on each square of $S \times S$ to $\mathfrak{K}(F)$.

THEOREM 5.5. *If (F, M) is in $\mathfrak{K}$ and F is uniformly continuous on each square of $S \times S$ then M is uniformly continuous on each square of $S \times S$.*

THEOREM 5.6. *If (F, M) is in $\mathfrak{K}$ and F is continuous with respect to its first place then M is continuous with respect to its first place.*

A proof of the next theorem may be obtained by using Theorems 5.5 and 5.6 applied to the ordered pair (F', M') in the proof of Theorem 4.1.

THEOREM 5.7. *Suppose each of F and E is in $\mathfrak{F}$ and P is the member of $\mathfrak{F}$ such that, for each (t, x) in $S \times S$,*

$$P(t, x) = E(t, x) + (L) \int_x^t dF[t, I] \cdot P[I, x].$$

(i) *If each of F and E is uniformly continuous on each square of $S \times S$ then P is uniformly continuous on each square of $S \times S$.*

(ii) *If each of F and E is continuous with respect to its first place then P is continuous with respect to its first place.*

It is the purpose of the next two theorems to place the theory presented in this paper in a more familiar setting. In the context of the foregoing development the proofs are easy to come by and are therefore omitted.

THEOREM 5.8. *If F is in $\mathfrak{F}$, c is in S, and K is in N then there is only one function U from S to N, bounded on each interval of S, such that*

$$U(t) = K + (L) \int_c^t dF[t, I] \cdot U$$

for each t in S. Moreover, if $M = \mathcal{K}(F)$ then $U = M[I, c] \cdot K$.

Note that U is uniformly quasi-continuous on each interval of S and that if F is continuous with respect to its first place then U is continuous.

THEOREM 5.9. *If F is in $\mathfrak{F}$, c is in S, and H is a function from S to N which is uniformly quasi-continuous on each interval of S then there is only one function V from S to N, bounded on each interval of S, such that*

$$V(t) = H(t) + (L) \int_c^t dF[t, I] \cdot V$$

for each t in S. Moreover, if $M = \mathcal{K}(F)$ then, for each t in S,

$$V(t) = H(t) - (L) \int_c^t dM[t, I] \cdot H.$$

Observe that V is uniformly quasi-continuous on each interval of S and that if H is continuous and F is continuous with respect to its first place then V is continuous.

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