THE Z_2 COHOMOLOGY OF A CANDIDATE FOR $B_{\mathrm{Im}(J)}$

BY

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1. Introduction

The investigation of Hurewicz fiber spaces with fiber of the homotopy type of a sphere leads to the study of "realizations of the image of the stable Whitehead J homomorphism". In [3] we explained that such spaces have applications in at least two directions. First, they can be used to help study the cohomology of BSF, the classifying space for stable oriented spherical fibrations. Second, they can be used to study certain characteristic classes for such fibrations.

By a 2-primary realization of Im (J) we mean a homotopy commutative diagram

$$\begin{array}{ccc} BSO & \xrightarrow{\quad f \quad} & BSF \\ & {\scriptstyle g}\searrow \quad \nearrow{\scriptstyle h} & \\ & BW & \end{array}$$

satisfying the following two conditions.

(1) f is the classifying map of the 2-primary component of the standard inclusion $SO \to SF$. (Recall that the standard inclusion induces the J-homomorphism $\pi_k(SO) \to \pi_k(SF) = \pi_k^S$ for $k \geqq 1$.)

(2) BW is a space whose homotopy groups for $n \geqq 4$ are given in the table below:

n mod 8	0	1	2	3	4	5, 6, 7
$\pi_n(BW)$	$Z_{\lambda(n)}$	Z_2	$Z_2 + Z_2$	Z_2	Z_8	0
Generators	ρ_n	ρ_{n-1}	$\rho_{n-}\eta\eta$, μ_n	$\mu_{n-1}\eta$	ξ_n	

For various purposes we allow $\pi_n(BW)$ to take on different values as explained in [3] if $1 \leqq n \leqq 3$. Here, $\lambda(n)$ is the Milnor-Kervaire number, i.e. $\lambda(n) = 2^m$ where m is the largest integer k such that 2^k divides $2n$. In [3] we explained in detail why these conditions are the appropriate ones to impose upon a "2-primary realization of Im (J)".

It is not known whether a 2-primary realization of Im (J) exists. However, M. E. Mahowald conjectures that such a diagram exists and J. D. Stasheff conjectures that BSF splits into the product $BW \times BY$ where BW is a suit-

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able space in a realization of Im (J) and BY is another space. The purpose of this paper is to present likely candidates for BW and compute their cohomologies. Here are our main results:

1.1 THEOREM. *There is an H-space SX with classifying space BSX and a map* $g : BSO \to BSX$ *satisfying condition* (2). *The map*

$$g_* : \pi_k(BSO) \to \pi_k(BSX)$$

is epic if $k = 0, 1,$ *or* 4 (mod 8) *and monic otherwise. Also* $\pi_1(BSX)$ $\pi_2(BSX) = Z_2$.

1.2 THEOREM. *As* Z_2*-algebras, we have*

$$H^*(SX; Z_2) = H^*(SO; Z_2) \otimes H^*(BSO; Z_2);$$
$$H^*(BSX; Z_2) = H^*(BSO; Z_2) \otimes H^*(BBSO; Z_2).$$

1.3 PROPOSITION. $H^*(BBSO; Z_2) = E[e_3, e_4, e_5, \cdots]$, *the* Z_2 *exterior algebra on one generator of each degree* $\geqq 3$.

We prove 1.1 in Section 3 and we prove 1.2 and 1.3 in Section 4. Section 2 presents preliminary calculations and machinery.

Our preoccupation with the prime 2 stems from the fact that Stasheff has handled the problem for all odd primes [7]. We wish to thank Stasheff for many valuable discussions and J. F. Adams for his suggestion that our candidates for BX are the appropriate spaces to consider. This paper is a revised version of the author's thesis at Northwestern University under M. E. Mahowald.

1.4 LIST OF CONVENTIONS. In this paper, the word "space" will always mean a topological space with basepoint of the same based homotopy type of a connected CW complex.

We denote by Z the integers, by Q the rational numbers, by Z_2 the field $Z/2Z$, and by Q_2 the set of rationals which can be written without a power of 2 in their denominators.

If S is a commutative ring with unit and Y is a set of indeterminates over S, we denote by $SP[Y]$ and $SE[Y]$ the polynomial and exterior algebras over S with the elements of Y as generators. If $S = Z_2$, we suppress the S and write $P[Y]$ and $E[Y]$.

If X is a space and S is a commutative ring with unit, we write $H_*(X; S)$ for the homology of X with coefficients in S. If $S = Z_2$, we suppress the S and write $H_*(X)$. A similar convention holds for cohomology.

We denote by KO and KU the real and complex K-theories. We denote by RP and CP the real and complex infinite-dimensional projective spaces. The KO and KU theoretic operations ψ^3 of Adams will be denoted by ψ_*.

If X is a space and n is a positive integer, we denote by $X[n]$ the space such that there is a map $f : X[n] \to X$ satisfying the following conditions:

(1) $f_* : \pi_k(X[n]) \to \pi_k(X)$ is an isomorphism if $k \geqq n$;
(2) $X[n]$ is $(n - 1)$-connected.

The space $X[n]$ is unique up to homotopy type.

If X is a contractible space, we frequently write $X = *$.

Two maps $f_1, f_2 : X \to Y$ of spaces are said to be *weakly homotopic* or *weakly equal* if and only if whenever $g : K \to X$ is a map where K is a finite CW complex it is true that $f_1 \circ g$ and $f_2 \circ g$ are homotopic.

2. Half exact functors

In this section we shall study some of the properties of certain half exact functors. Our study will lead to the definition of a space BP and a map closely related to the map $\psi : BO \to BO$. This will lead to the definition of BSJ in Section 3.

2.1 THEOREM (Edgar H. Brown[2]). (1) *Let K be a contravariant functor from the category of spaces with basepoint which have the based homotopy type of a finite CW complex to the category of sets with basepoint. Then* (a) $\leftrightarrow$ (b) *where*

(a) $\tilde{K}$ *is a half exact functor;*

(b) $\tilde{K}$ *is naturally isomorphic to* $[\cdot\, ; B]$ *for some space B.*

(2) *Let* $\tilde{K}_1$ *and* $\tilde{K}_2$ *be half exact functors and let* $S : \tilde{K}_1 \to \tilde{K}_2$ *be a natural transformation. Then there is a unique (up to weak basepoint homotopy) map* $T : B_1 \to B_2$ *such that*

$$S = T_* : [\cdot\ ; B_1] \to [\cdot\ ; B_2].$$

2.2 DEFINITION. Define a half exact functor $\tilde{K}P$ by $\tilde{K}P = \tilde{K}O \otimes Q_2$ Let BP denote the classifying space for $\tilde{K}P$ whose existence is asserted by 2.1.

2.3 BOTT PERIODICITY THEOREM FOR BP. $\Omega^8 BP = BP$, *where we denote by* ΩX *the basepoint component of the space more often called* ΩX.

This follows immediately from 2.1 and the Bott periodicity theorem for BO.

The natural transformation $\psi_* : \tilde{K}O \to \tilde{K}O$ is induced by a map $\psi : BO \to BO$ because BO is the classifying space for the half exact functor $\tilde{K}O$. Define $\psi_* : \tilde{K}P \to \tilde{K}P$ by

$$\psi_* = \psi_* \otimes Q_2 : \tilde{K}O \otimes Q_2 \to \tilde{K}O \otimes Q_2 .$$

Let $\psi : BP \to BP$ denote a map which induces the natural transformation ψ_*. Let $T_* : \tilde{K}O \to \tilde{K}P$ be the natural transformation $x \to x \otimes 1$ and let $T : BO \to BP$ be a map which induces T_*. T is obviously a 2-primary homotopy equivalence, and, by 2.1, there is a weak homotopy commutative diagram

2.4

$$\begin{array}{ccc} BO & \xrightarrow{\psi} & BO \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle T} \\ BP & \xrightarrow{\psi} & BP \end{array}$$

2.5 LEMMA. $\Omega^8 \psi = 81\, \psi : BO \to BO$ *(weak equality).*

Proof. By [1, Corollary 5.3, page 618], there is a commutative diagram

$$\begin{array}{ccc} \widetilde{KO}(X) & \xrightarrow{S^8} & \widetilde{KO}(S^8X) \\ \downarrow{\scriptstyle 81(\psi_*)} & & \downarrow{\scriptstyle \psi_*} \\ \widetilde{KF}(X) & \xrightarrow{S^8} & \widetilde{KO}(S^8X). \end{array}$$

This diagram together with 2.1 and Bott periodicity give the result, Q.E.D.

2.6 PROPOSITION. *We can divide by* 81 *in* $\widetilde{KP}$. *Let* $\varphi_* = \psi_*/81$. *Then* $\Omega^8\varphi = \psi : BP \to BP$ *(weak equality).*

Proof. By 2.5 and definition of φ respectively, we know that $\Omega^8\psi = 81\,\psi$ weakly and $81\,\varphi = \psi$ weakly. Then $81\,\Omega^8\varphi = \Omega^8\,81\,\varphi = \Omega^8\psi = 81\,\psi$ weakly. Then $81\,((\Omega^8\varphi)_*) = (81\,\Omega^8\varphi)_* = (81\,\psi)_* = 81\,(\psi_*)$. Then $(\Omega^8\varphi)_* = \psi_*$. Then $\Omega^8\varphi = \psi$ weakly, Q.E.D.

We remark that $\psi : BO \to BO$ does not deloop 8 times. For suppose $\Omega^8\varphi = \psi : BO \to BO$. Then $81(\varphi_*) = (\psi_*)$. But ψ_* is not 81 times an operation by Adam's calculation of ψ on spheres [1, Corollary 5.2, page 617].

We are told that the following result is due to Atiyah and is well known. Since we are unable to find a proof in the literature, however, we present one.

2.7 PROPOSITION. *The map* $\psi^* : H^{2k}(BU;\,Q) \to H^{2k}(BU;\,Q)$ *is multiplication by* 3^k.

Proof. Let $\gamma : BU(n) \to CP \times \cdots \times CP$ be the canonical map. A routine calculation shows that, if $u(n)$ denotes the universal n-plane bundle, and n denotes the trivial n-plane bundle over $BU(n)$, then

$$\gamma^*\,\mathrm{ch}\,(u(n) - n) = (e^{\alpha_1} - 1) + \cdots + (e^{\alpha_n} - 1);$$

and

$$\gamma^*\psi^*\,\mathrm{ch}\,(u(n) - n) = (e^{3\alpha_1} - 1) + \cdots + (e^{3\alpha_n} - 1)$$

where α_i is $\pi_i^*\,\alpha$, $\pi_i : CP \times \cdots \times CP \to CP$ being the ith projection and $\alpha \in H^2(CP;\,Q)$ being the canonical (up to sign) generator.

In degree $2k$, $\gamma^*\,\mathrm{ch}\,(u(n) - n)$ is therefore

$$(\alpha_1^k + \cdots + \alpha_n^k)/k\,!$$

and $\gamma^*\,\mathrm{ch}\,(u(n) - n)$ is

$$3^k(\alpha_1^k + \cdots + \alpha_n^k)/k\,!.$$

If $\beta_k \in H^{2k}(BU(n);\,Q)$ is the unique preimage under γ^* of the symmetric polynomial $(\alpha_1^k + \cdots + \alpha_n^k)/k\,!$, then $\gamma_*\psi_*\beta_k = 3^k\gamma^*\beta_k$. Thus $\psi^*\beta_k = 3^k\beta_k$ since γ^* is monic and $H^*(CP \times \cdots \times CP;\,Q)$ is torsion free. By [9, Problem 6, page 82], we know that

$$H^*(BU(n);\,Q) = QP[\beta_1, \beta_2, \beta_3, \cdots].$$

Then ψ^* is multiplication by 3^k in degree 2^k, Q.E.D.

This proposition implies that the map

$$(\psi - 1)^* : H^*(BU; Z) \to H^*(BU; Z)$$

multiplies each primitive element by an even integer. Problem 4, page 82 of [9] and the fact that the first Chern class which the above map does not annihilate mod 2 is sent to a mod 2 primitive imply that the map

$$(\psi - 1)^* : H^*(BU) \to H^*(BU)$$

is trivial.

2.8 Corollary. *The maps*

$$(\psi - 1)^* : H^*(BO) \to H^*(BO)$$

and

$$(\psi - 1)^* : H^*(BP) \to H^*(BP)$$

are trivial.

Proof. This is because $\psi - 1$ commutes with complexification and because $BO \to BU$ sends the n^{th} Chern class to the square of the n^{th} Stiefel-Whitney class, Q.E.D.

2.9 Proposition. *The map* $\psi_* : \pi_{4n}(BP) \to \pi_{4n}(BP)$ *is multiplication by* 3^{2n}.

Proof. By [1, Corollary 5.2, page 617],

$$\psi_* : \pi_{4n}(BO) \to \pi_{4n}(BO)$$

is multiplication by 3^{2n}. Since $\psi_* : \pi_{4n}(BP) \to \pi_{4n}(BP)$ is a map $Q_2 \to Q_2$ and, such, is completely determined by its values on Z, the result follows from diagram 2.4, Q.E.D.

2.10 Proposition. *The map* $(\psi - 1)_* : \pi_n(BP) \to \pi_n(BP)$ *is multiplication by* $3^{n/2} - 1$ *if* $n = 0 \pmod 4$ *and* 0 *in all other dimensions.*

Proof. The result for $n = 0 \pmod 4$ is just 2.9 together with the fact that addition on $\pi_*(BP)$ is the homotopy functor applied to addition in the H-space BP. The result for $n = 1$ or $2 \pmod 8$, but $n \geqq 9$, follows from 2.9 and the commutative diagram

$$\begin{array}{ccccccc} S^{8k+2} & \xrightarrow{\eta} & S^{8k+1} & \xrightarrow{\eta} & S^{8k} & \xrightarrow{\rho_{8k}} & BP \\ & & & & \downarrow{\scriptstyle 3^{4k}-1} & & \downarrow{\scriptstyle \psi - 1} \\ & & & & S^{8k} & \xrightarrow[\rho_{8k}]{} & BP \end{array}$$

since η is of order 2.

It remains to show that $(\psi - 1)_* : \pi_\varepsilon(BP) \to \pi_\varepsilon(BP)$ if 0 for $\varepsilon = 1$ and 2. We consider $\varepsilon = 1$. Suppose $(\psi - 1)_* : \pi_1(BP) \to \pi_1(BP)$ is the nontrivial map $Z_2 \to Z_2$ so that $(\psi - 1)_* : H^1(BP) \to H^1(BP)$ is nontrivial, contrary to 2.8. Then

$$(\psi - 1)_* : \pi_1(BP) \to \pi_1(BP)$$

is trivial.

A slightly more complicated but similar argument disposes of the case $\varepsilon = 2$, Q.E.D.

3. Candidates for $B_{\mathrm{Im}(J)}$ and their homotopy groups

In this section we define our candidate SX for a 2-primary realization of Im (J) and prove Theorem 1.1. To do this, we construct BSX explicitly. The homotopy groups of SX are almost immediate from 2.10. A little work is required to decide whether $\pi_{8k+1}(SX)$ is $Z_2 + Z_2$ or Z_4.

3.1 LEMMA. *Let $\varphi : BP[9] \to BP[9]$ be a lifting of ψ. Then there is a commutative diagram*

$$\begin{array}{ccccc} & & BP[10] & & \\ & \overset{\varphi - 1}{\nearrow} & \big\downarrow \gamma & & \\ BP[9] & \xrightarrow[\varphi - 1]{} & BP[9] & \xrightarrow[\kappa]{} & K(Z_2, 9) \end{array}$$

where we denote the lifting of $\varphi - 1$ also by $\varphi - 1$.

The proof is routine.

Henceforth, $\chi : BP[m] \to BP[n]$ will denote an appropriate k-fold looping of an appropriate lifting of the map $(\varphi - 1) : BP \to BP$.

3.2 DEFINITION. We define the spaces BX, BSX, $B\hat{X}$, $BS\hat{X}$, $B\bar{X}$ and $BS\bar{X}$ by insisting that the following sequences be fibrations, where the projections are χ.

$$B\hat{X} \to B(BP[4]) \to BBP$$

$$BS\hat{X} \to B(BP[4]) \to B(BP[2])$$

$$BX \to B(BP[2]) \to BBP$$

$$BSX \to B(BP[2]) \to B(BP[2])$$

$$B\bar{X} \to BBP \to BBP$$

$$BS\bar{X} \to BBP \to B(BP[2]).$$

We define W for $W = X, \hat{X}, \bar{X}, SX, S\hat{X}$, and $S\bar{X}$ by $W = \Omega BW$.

Before proving 1.1, we notice that, if $\nu(n)$ denotes the largest integer μ such that 2^μ divides n, then $\nu(3^{2n} - 1) = 3 + \nu(n)$. This is an easy consequence of [1, Lemma 8.1, page 630].

3.3 *Proof of* 1.1. We appeal to the homotopy sequence of the fibrations

$$SX \to BSP \xrightarrow{\chi} BSP$$

to see that $\pi_n(SX)$ is given by the table of Section 1 except for $n = 1 \pmod 8$ in which case we know that $\pi_n(SX) = Z_2 + Z_2$ or Z_4. To see that $\pi_{4k-1}(SX) = Z_{\lambda(n)}$, apply 2.10. From the exact homotopy sequence of the fibration

$$SP \to SX \to BSP$$

we see that the generators of the table in Section 1 generate $\pi_n(SX)$.

We will be finished if we can prove that $\pi_{8k+1}(SX) \neq Z_4$ for $k \geqq 1$. Suppose on the contrary that some $\pi_{8k+1}(SX) = Z_4$. Take a lifting

$$\chi : BP[8k + 1] \to BP[8k + 1].$$

The fiber of this lifting is $SX[8k]$, so that $\pi_{8k+1}(SX[8k]) = Z_4$ by our assumption. Since

$$\chi_* : \pi_{8k+1}(BP[8k + 1]) \to \pi_{8k+1}(BP[8k + 1])$$

is the trivial map $Z_2 \to Z_2$ by 2.10, we know that

$$\chi^* : H^{8k+1}(BP[8k + 1]) \to H^{8k+1}(BP[8k + 1])$$

is 0. Thus in the stable range we have

$$H^*(SX[8k]) = (A/AS_q^2)b_{8k} \otimes_{Z_2} (A/AS_q^2)b_{8k+1}$$

for example by, [8, Theorem A, page 538], applied to the Serre sequence of our fibration. Now η applied to the fundamental class of

$$\pi_{8k+1}(BP[8k + 1]) = Z_2$$

is nontrivial by what we have already shown. Thus $S_q^2 b_{8k} = 0$. When we build a Postnikov tower for $SX[8]$, it must therefore be true that the first step is to take a map

$$\kappa : K(Z_2, 8k) \to K(Z_4, 8k + 2)$$

such that $\kappa^* b_{8k+2} = S_q^2 b_{8k}$. Since $S_q^1 b_{8k+2} = 0$ in $H^*(K(Z_4, 8k + 2))$ whereas $S_q^1 S_q^2 b_{8k} = S_q^3 b_{8k} \neq 0$ in $H^*(K(Z_2, 8k))$, a map κ does not exist. This implies that our assumption that $\pi_{8k+1}(SX) = Z_4$ is false, Q.E.D.

4. Calculation of the cohomology algebras

4.1 PROPOSITION. *In the Z_2-cohomology spectral sequences $E^*(\rho_1)$ of the fibrations*

$$SP \xrightarrow{\iota_1} W \xrightarrow{\rho_1} BP[n], \qquad n = 1, 2, \text{ or } 4,$$

Obtained from 3.2, *all differentials are* 0 *for* $W = SX$, $S\hat{X}$, *and* $S\bar{X}$. *Thus* $H^*(W) = H^*(SO) \otimes H^*(BO[n])$ *as algebras.*

Proof. Consider the map of fibrations

$$\begin{array}{ccccc} SP & \xrightarrow{\iota_1} & W & \xrightarrow{\rho_1} & BP[n] \\ \downarrow{\scriptstyle 1} & & \downarrow & & \downarrow{\scriptstyle \chi} \\ SP & \xrightarrow[\iota_2]{} & * & \xrightarrow[\rho_2]{} & BSP. \end{array}$$

We know that
$H^*(SP) = P[x_1, x_3, x_5, \cdots]$, $H^*(BSP) = P[w_2, w_3, w_4, \cdots]$, and $\tau_2 x_i = w_{i+1}$ where subscripts of x_i and w_i denote degree and τ_j is the transgression of $E^*(\rho_j)$. Since $\chi^* = 0$ by 2.8, the result follows from functoriality of E^*, Q.E.D.

4.2 PROPOSITION. *In the Z_2-cohomology Serre spectral sequence of all of the fibrations $BP[m] \to BW \to B(BP[n])$, all differentials are* 0. *Therefore*

$$H^*(BW) = H^*(BP[m]) \otimes H^*(B(BO[n])).$$

Proof. For $m = 2, n = 1$, consider the map of fibrations

$$\begin{array}{ccccc} S\bar{X} & \xrightarrow{\kappa_1} & * & \xrightarrow{\pi_1} & BS\bar{X} \\ {\scriptstyle \rho_1}\downarrow & & \downarrow & & \downarrow{\scriptstyle B\rho_1} \\ BP & \xrightarrow{\kappa_2} & * & \xrightarrow{\pi_2} & BBP. \end{array}$$

By [5, page 253], $E_*(\pi_2)$ has fiber $P[a_1, a_2, a_3, \cdots]$ and base

$$P[b_2, b_3, b_5, \cdots, b_{2k+1}, \cdots]$$

with $\sigma_2 a_i = b_{i+1}$ if i is even or $i = 1$. If $m = 2$, $\rho_{1*} : H_*(W) \to H_*(BP)$ is epic by 4.1, and, we conclude that

$$B\rho_{1*} : H_*(BSX) \to H_*(BBP)$$

is epic. Thus the differentials of $E_*(\pi_1)$ are 0, and, by duality, so are those of $E^*(\pi_1)$.

For $m = 2$ and $n = 1$, 2 or 4 we deduce the result from functoriality of E^* and the map of fibrations

$$\begin{array}{ccccc} BP[2] & \longrightarrow & BW & \to & BBP[n] \\ \downarrow & & \downarrow & & \downarrow \\ BP[2] & \xrightarrow{\iota_1} & BS\bar{X} & \to & BBP \end{array}$$

for we have just shown that ι_1^* is epic.

For $m = 1$ we then deduce the result from the map of fibrations

$$\begin{array}{ccccc} B[2] & \xrightarrow{\iota_1} & BSW & \longrightarrow & BBP[n] \\ \gamma\downarrow & & \downarrow & & \downarrow \\ BP & \xrightarrow{\iota_2} & BW & \longrightarrow & BBP[n] \end{array}$$

using the facts that $w_1 \in \operatorname{Im} \iota_2^*$ and that ι_1^* and γ^* are epic, Q.E.D.

4.3 *Proof of* 1.3. We use the spectral sequence of the Bott periodicity fibration:

$$SU \xrightarrow{\iota} BBSO \xrightarrow{\rho} B\ \mathrm{Spin}$$

In $E^*(\rho)$, we have $\tau e_{2i+1} = w_{i+1}^2$ if $i \neq 2^j$ and $\tau e_3 = 0$ where

$$H^*(SU) = E[e_3, e_5, e_7, \cdots]$$

and

$$H^*(B\ \mathrm{Spin}) = P[w_i \mid i \neq 2^j + 1 \text{ and } i \geqq 4].$$

By [8], for example, we know that $e_{2^j+1} = S_q^I e_3$ where $I = (2^j, 2^{j-1}, \cdots, 2)$. Thus all the e_{2^j+1} trangress to 0. This determines $E^*(\rho)$ completely and we conclude that

$$H^*(BBSO) = E[w_i \mid i \neq 2^j + 1 \text{ and } i \geqq 4] \otimes E[S_q^I e_3 \mid I = (2^j, 2^{j-1}, \cdots, 2)],$$

Q.E.D.

4.4 Lemma

$$H^*(U[5]) = \frac{H^*(K(Z, 5))}{\text{Ideal } [AS_q^3\, b_5]} \otimes P[\sigma\theta_{2i} \mid L(i) > 4]$$

where L and the θ_{2i} *are the objects of Stong* [8] *and* σ *is the suspension of* $E^*(\rho)$, *the spectral sequence of fibration* $U[5] \to * \to BU[6]$. *Furthermore,*

$$\frac{H^*(K(Z, 5))}{\text{Ideal } [AS_q^3\, b_5]} = E[S_q^I\, b_5 \mid I \in Y]$$

where, in the notation of [8], Y *is the set of all admissible sequences of the form*

$$[0, \cdots, 0, 2], \quad [0, \cdots, 4], \quad \text{or} \quad [0, \cdots, 0, 2, 0, \cdots, 0, 2].$$

Proof. Consider first the fibration

$$CP \xrightarrow{\kappa} U[5] \xrightarrow{\pi} U[3] = SU.$$

$E^*(\pi)$ has fiber $P[\alpha]$ and base $E[e_3, e_5, e_7, \cdots]$ with $\tau\alpha^k = e_{k+1}$ when $k = 2^i$. Thus

$$E^\infty(\pi) = E[\alpha^k e_{k+1} \mid k = 2^i \text{ and } i \geqq 1] \otimes E[e_n \mid n \neq 2^i + 1].$$

We can take $e_n = \sigma\theta_{n+1}$ if $L(n + 1) > 4$ and the generators of $E[S_q^I b_5 \mid I \in Y]$

have exactly the right dimensions made up the rest of the elements of the simple system of generators of $H^*(U[5])$ obtained for $E^\infty(\pi)$.

Routine calculations show that, if E is the spectral sequence with fiber

$$E[S_q^I b_5 \mid I \in Y] \otimes E[\sigma\theta_{2i} \mid L(i) > 4],$$

base $H^*(BU[6])$, and b_5 transgressing to b_6, then the base of E is trangressively generated. Hence by possibly throwing out some generators of its fiber, we get $E^*(\pi)$. But by our previous dimension count, we cannot eliminate any generators of the fiber of E, and the map on the fiber is an isomorphism, Q.E.D.

4.5 PROPOSITION.

$$H^*(BB\text{ Spin}) = \frac{H^*(K(Z, 5))}{\text{Ideal }[AS_q^3 b_5]} \otimes E[\rho^*\theta_i \mid L(i) \geqq 4]$$

where ρ is projection in the fibration $U[5] \to BB$ Spin $\to BO[8]$ and where θ_i and L are objects of Strong [8].

The proof is similar to the proof of 1.3 using 4.4 with Stong's calculation of $H^*(BO[8])$.

Propositions 1.3 and 4.5 serve mainly to pin down our calculation of $H^*(BW)$ in 4.2.

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