NUMERICAL INVARIANTS OF KNOTS

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In the following, we will define two numerical invariants of knots and show that they are equivalent. The first, the bridge number of a knot, describes the minimum number of overcrossing arcs of any representation of a given knot, and was first described by Schubert [1]. The second, the local maximum number of a knot, is the minimum number of local maxima of any representation of a given knot.

In the following, a *knot* is a piecewise linear embedding of S^1 into E^3. Let C denote the cube

$$\{(x, y, z) \mid 0 \leqq x \leqq 1, 0 \leqq y \leqq 1, 0 \leqq z \leqq 1\}$$

and D_t, $0 \leqq t \leqq 1$, the disk

$$\{(x, y, z) \mid 0 \leqq x \leqq 1, 0 \leqq y \leqq 1, z = t\}.$$

If a knot K is represented as n arcs $A_1, A_2, \cdots, A_n$ where

$$A_i = \{(x, y, z) \mid x = i/(n+1), y = \tfrac{1}{2}, 0 \leqq z \leqq 1\},$$

through C and n connecting arcs on the boundary of C, then K is said to be in a *standard n bridge position*. The *bridge number* of a knot K, denoted by $B(K)$ is the minimum number of A_i among all standard n bridge positions representing K. A knot K is said to be in *standard position with n local maxima* if K is represented as being contained in C so that

(1) $K \cap D_1$ is n points,

(2) $K \cap D_t$, $0 \leqq t \leqq 1$ is a finite number of points,

(3) if $p \in (K \cap (\text{interior } C))$ and N is a neighborhood of p, then there exists a point $p' \in N$ such that the z coordinate of p' is greater than the z coordinate of p.

The *local maximum number* of a knot K, denoted $M(K)$, is the least number of local maxima among all standard positions with n local maxima representing K.

It is easy to show that a knot can be put in a standard n bridge position or in a standard position with n local maxima. Hence, given K, $M(K)$ and $B(K)$ are well defined.

THEOREM 1. *If K is a knot, then $M(K) = B(K)$.*

We first prove that $B(K) \geqq M(K)$ by showing that if K is a knot in standard n bridge position, then K can be moved by a space homeomorphism to a position with n local maxima.

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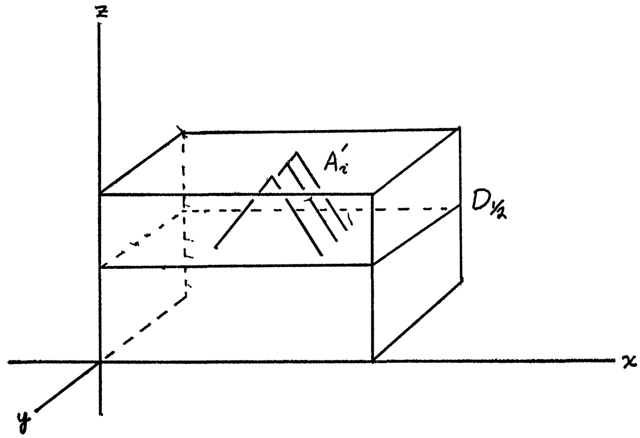

FIGURE 1

Assume that K is in a standard n bridge position and that D is an open disk on the boundary of C such that (closure of D) $\cap K = \Phi$. Thus K lies in the interior of a closed disk (boundary of C) $-$ D, except for n linear segments. Let

$$A_i' = \{(x, y, z) \mid x = z/2 \text{ or } x = 1 - z/2, y = i/(n+1), 1/2 \leqq z \leqq 1\}$$

(Fig. 1). Define h_1 to be a space homeomorphism taking (boundary of C)—D onto $D_{1/2}$ and A_i onto A_i'. Hence $h_1(K - \bigcup_{i=1}^{n} A_i)$ is n arcs, $B_1, \cdots, B_n$, lying in the interior of $D_{1/2}$. Let a_i, b_i denote the endpoints of B_i and x_i the midpoint of B_i. Construct the arc B_i' by moving $\alpha = (x, y, 1/2) \in B_i$ to

$$\left(x, y, \frac{d(\alpha, x_i)}{2d(a_i, x_i)}\right)$$

where $d(\alpha, x_i)$ denotes the arc length along B_i of the point α to x_i. Let h_2 be a space homeomorphism that moves B_i to B_i', $i = 1, 2, \cdots, n$, and leaves $\bigcup_{i=1}^{n} A_i'$ fixed. Hence $h_2 h_1(K)$ is in a standard position with n local maxima.

Next, we assume that a knot K' is in a standard position with n local maxima. Let $p_1, p_2, \cdots, p_n$ be the n points of $K' \cap D_1$ and $x_1, x_2, \cdots, x_n$ vertices of K' such that $p_i x_i$ is a linear segment of K'. Define $y_1, \cdots, y_n$ such that $y_i \in D_1$ and the triangles $y_i x_i p_i$ and $y_j x_j p_j$, $i \neq j$, have at most x_i in common (Fig. 2). A space homeomorphism taking $x_i p_i$ to $x_i y_i p_i$ leaving $K' - \bigcup_{i=1}^{n} x_i p_i$ fixed will be denoted by h_3. Let h_4 be a space homeomorphism that moves $h_3(K')$, without introducing local maxima, such that

(1) no three points of $D_t \cap h_3(K')$, $0 \leqq t \leqq 1$, have the same x coordinate,

(2) only a finite number of $D_t \cap h_3(K')$ have two points with the same x coordinate

(3) if $D_t \cap h_3(K')$ has two points with the same x coordinate then $D_t \cap h_3(K')$ contains no vertices of K'.

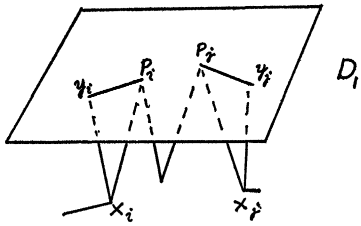

FIGURE 2

To construct h_4, first order the vertices $h_3(K')$ and put a small spherical neighborhood about each vertex. Then, using the ordering, move the i^{th} vertex to another point in its neighborhood so that the 1-simplexes joining this new point are in general position with respect to the y axis and satisfy condition (3) with respect to the previously moved l-simplexes.

Without loss of generality, we can assume that $h_4 h_3(K') - \bigcup_{i=1}^n y_i p_i$ is contained in the interior of C. Hence

$$h_4 h_3(K') - \bigcup_{i=1}^n (\text{interior of } y_i p_i)$$

consists of n arcs $B_1', B_2', \cdots, B_n'$ having n vertices $b_1', \cdots, b_n'$ which are local minima. Let $L_1 = \{(x, y, z) \mid (x, y, z) \in C \text{ and } y = 1\}$, $L_2 = \{(x, y, z) \mid (x, y, z) \in C$ and there exists a point $p \in \bigcup_{i=1}^n B_i'$, $p = (x', y', z')$ such that $x' = x$, $z' = z$, and $y \geqq y'\}$, and $L_3 = \{(x, y, z) \mid (x, y, z) \in C$ and for some $b_i' = (x', y', z')$, $x = x', z = z'$ and $y \geqq y'\}$.

We now show that

$$L_1 \cup L_2 \cup L_3 \cup (\bigcup_{i=1}^n B_i')$$

simplicially collapses to

$$L_1 \cup L_3 \cup (\bigcup_{i=1}^n B_i') \qquad [2].$$

By property (2) of h_4 we can define $D_{t_1}, D_{t_2}, \cdots, D_{t_K}$, such that $1 = t_1 > t_2 > \cdots > t_K > 0$, all crossings of $h_4 h_3(K')$ in the y direction lie between D_{t_2} and D_{t_K}, and no more than one crossing lies between D_{t_i} and $D_{t_{i+1}}$, $i = 2, 3, \cdots, K-1$. Let $C_i = \{(x, y, z) \mid (x, y, z) \in C \text{ and } t_i \geqq z \geqq t_{i+1}\}$. As $L_2 \cap C_1$ consists of a disjoint collection of 2-cells, we can collapse $L_2 \cap C_1$ to

$$L_2 \cap C_1 \cap (L_1 \cup L_3 \cup (\bigcup_{i=1}^n B_i') \cup D_{t_2}).$$

Assume that $L_2 \cap (\bigcup_{i=1}^j C_i)$ has been collapsed to

$$L_2 \cap (\bigcup_{i=1}^j C_i) \cap (L_1 \cup L_3 \cup (\bigcup_{i=1}^j B_i') \cup D_{t_{j+1}}).$$

There is at most one $B_i' \cap C_{j+1}$ a connected component of which is an overcrossing arc. Collapse the part of $L_2 \cap C_{j+1}$ lying below this arc, then collapse the remaining part of $L_2 \cap C_{j+1}$. Hence we have $L_2 \cap (\bigcup_{i=1}^{j+1} C_i)$ collapsed to

$$L_2 \cap (\bigcup_{i=1}^{j+1} C_i) \cap (L_1 \cup L_3 \cup (\bigcup_{i=1}^n B_i') \cup D_{t_{j+2}}).$$

It follows that

$$L_1 \cup L_2 \cup L_3 \cup (\bigcup_{i=1}^{n} B_i')$$

collapses to

$$L_1 \cup L_3 \cup (\bigcup_{i=1}^{n} B_i).$$

Let $N(L_1 \cup L_2 \cup L_3 \cup (\bigcup_{i=1}^{n} B_i'))$ denote the second derived neighborhood of $L_1 \cup L_2 \cup L_3 \cup (\bigcup_{i=1}^{n} B_i')$ in C. Define h_5 to be a space homeomorphism taking C onto $N(L_1 \cup L_2 \cup L_3 \cup (\bigcup_{i=1}^{n} B_i'))$ leaving $\bigcup_{i=1}^{n} B_i'$ fixed.

As $(L_1 \cup L_2 \cup L_3 \cup (\bigcup_{i=1}^{n} B_i'))$ collapses to $(L_1 \cup L_3 \cup (\bigcup_{i=1}^{n} B_i'))$, there is a space homeomorphism h_6 taking

$$N(L_1 \cup L_2 \cup L_3 \cup (\bigcup_{i=1}^{n} B_i')) \quad \text{onto} \quad N(L_1 \cup L_3 \cup (\bigcup_{i=1}^{n} B_i'))$$

leaving $\bigcup_{i=1}^{n} B_i'$ fixed [2]. Let

$$L_3' = \{(x, y, z) \mid z = 1/2, 1 \geqq y \geqq 1/2, \text{and } x = i/(n+1)\}.$$

There exists a space homeomorphism h_7' taking L_1 onto L_1, B_i' onto A_i, and L_3 onto L_3'. Let h_7 be a space homeomorphism of

$$N(L_1 \cup L_3 \cup (\bigcup_{i=1}^{n} B_i')) \quad \text{onto} \quad N(L_1 \cup L_3 \cup (\bigcup_{i=1}^{n} A_i))$$

such that

$$h_7(L_1 \cup L_3 \cup (\bigcup_{i=1}^{n} B_i')) = h_7'(L_1 \cup L_3 \cup (\bigcup_{i=1}^{n} B_i')).$$

Finally, let h_8 be a homeomorphism taking $N(L_1 \cup L_3' \cup (\bigcup_{i=1}^{n} A_i))$ onto C leaving $\bigcup_{i=1}^{n} A_i$ fixed (Fig. 3). Hence $h_8 h_7 h_6 h_5 h_4 h_3(K')$ is a standard n bridge position.

A *link* is a piecewise linear embedding of $\bigcup_{i=1}^{n} S_i'$ onto E^3. If K is a link,

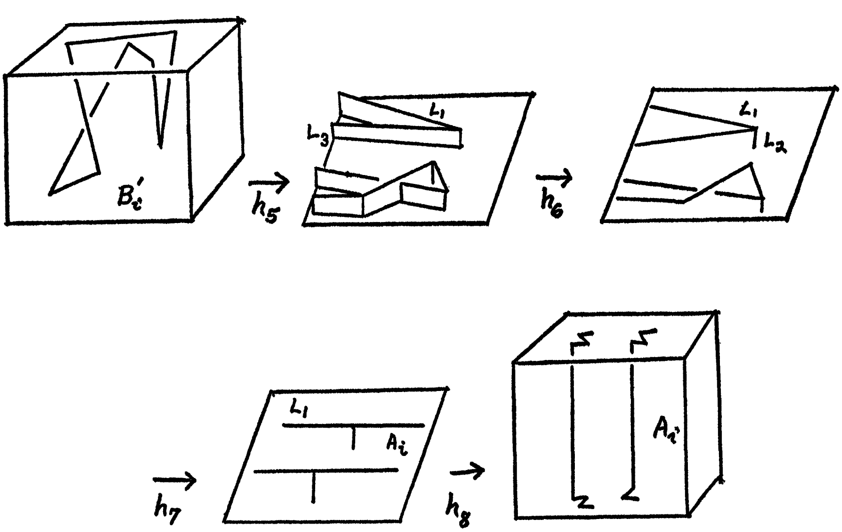

FIGURE 3

then we can derive $B(K)$ and $M(K)$ in a similar manner as in knots. Hence, using the same proof as in Theorem 1 we obtain the following.

THEOREM 2. *If K is a link, then $B(K) = M(K)$.*

REFERENCES

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