ON THE ZEROS OF A CLASS OF DIRICHLET SERIES II

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1. Introduction

In [2] we considered the distribution of zeros for a large class of Dirichlet series. One of our theorems concerned the number of zeros on the critical line, but the remainder were concerned with the distribution in a vertical strip. In this paper our theorems pertain to the number of zeros on the critical line.

For Theorem 1 we consider the same class of series studied in [2]. Theorems 2 and 3 are concerned with a slightly different class, but in Theorem 4 we shall relate the two classes. We shall conclude the paper with several examples. For many of these examples, the proof of an infinite number of zeros on the critical line has not been heretofore given.

Class 1. Let $\{\lambda_n\}$ and $\{\mu_n\}$ be two sequences of positive numbers tending to ∞, and $\{a(n)\}$ and $\{b(n)\}$ two sequences of complex numbers not identically zero. Let

$$\Delta(s) = \prod_{k=1}^{N} \Gamma(\alpha_k s + \beta_k),$$

where N is a positive integer, $\alpha_k > 0$, and β_k is an arbitrary complex number. Consider the functions φ and ψ representable as Dirichlet series

$$\varphi(s) = \sum_{n=1}^{\infty} a(n)\lambda_n^{-s}, \quad \psi(s) = \sum_{n=1}^{\infty} b(n)\mu_n^{-s}, \quad s = \sigma + it,$$

with finite abscissae of absolute convergence σ_a and σ_a^*, respectively. If r is real, we say that φ and ψ satisfy the functional equation

$$\Delta(s)\varphi(s) = \Delta(r - s)\psi(r - s),$$

if there exists in the s-plane a domain D which is the exterior of a compact set S, and a function $R(s)$, such that in D

(i) $R(s)$ is holomorphic,

(ii) $R(s) = \Delta(s)\varphi(s), \qquad \sigma > \sigma_a$
$\qquad = \Delta(r - s)\psi(r - s), \quad \sigma < r - \sigma_a^*,$

(iii) there exists a constant $K > 0$ such that

$$R(s) = O(\exp |s|^K),$$

as $|s|$ tends to ∞.

Class 2. Let $\{\lambda_n\}$, $\{a(n)\}$, $\Delta(s)$ and $\varphi(s)$ be given as above. Suppose that $\varphi(s)$ has an analytic continuation to $\sigma \geq h$, where $0 \leq h < \sigma_a$, the singularities of $R(s) = \Delta(s)\varphi(s)$ in $\sigma \geq h$ are confined to a compact set S, and $\varphi(h + it)$ is of finite order.

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2. Notation and results

In the sequel let $A = \sum_{k=1}^{N} \alpha_k$, $\beta_k = \beta_{k1} + i\beta_{k2}$ with β_{k1} and β_{k2} real, and $B = \sum_{k=1}^{N} \beta_{k1}$. $K, K_1, K_2, \cdots$ always denote positive constants, k, k_1 and k_2 real constants, and k' a complex constant, none of which are necessarily the same with each occurrence. The summation sign $\sum$ appearing with no indices will always mean $\sum_{n=1}^{\infty}$. For c, real, the integral sign $\int_{c-i\infty}^{c+i\infty}$ shall be denoted by $\int_{(c)}$.

For Re $z > 0$ and $c > 0$, let

$$E_N(z) = \int_{(c)} \Delta(s) z^{-s}\, ds.$$

Note that if $\Delta(s) = \Gamma(s)$, $E_1(z) = e^{-z}$. Let

$$\Phi(z) = \sum a(n) E_N(\lambda_n z),$$

where Re $z > 0$.

In each of the three main theorems we make the following assumptions:

(2.1) $R(s)$ is holomorphic on $\sigma = h$,
(2.2) $R(s)$ is real on $\sigma = h$,
(2.3) $\cos(\frac{1}{2}\pi A - \varepsilon) > 0$ for $\varepsilon > 0$ small enough.

(2.2) could be replaced by the seemingly more general condition: there exists complex constants a and b such that $ae^{-bs}R(s)$ is real on $\sigma = h$. However, upon letting $b = c + id$, where c and d are real, we find that when $\sigma = h$,

$$e^{-dt} a e^{-bs} R(s) = a_1 B^{-s} R(s),$$

where $a_1 = ae^{-idh}$ and $B = e^{-c}$. Thus, we may replace $a(n)$ by $a_1 a(n)$ and λ_n by $B\lambda_n$ to obtain another series of either Class 1 or 2.

THEOREM 1. *Let φ be of Class* 1. *Suppose that* $\sigma_a = \sigma_a^*$, $h = \frac{1}{2}r$,

$$A\sigma_a + B < \tfrac{1}{2}N + 1, \tag{2.4}$$

and

$$\alpha_k h + \beta_{k1} - \tfrac{1}{2} \geq 0, \quad k = 1, \cdots, N. \tag{2.5}$$

Assume that as ε tends to 0, $\varepsilon > 0$,

$$\Phi(e^{i(\frac{1}{2}\pi A - \varepsilon)}) = o(1/\varepsilon). \tag{2.6}$$

Then $\varphi(s)$ has an infinite number of zeros on $\varphi = h$.

The assumption $\sigma_a = \sigma_a^*$ is not necessary but simplifies the proof. Note that conditions (2.4) and (2.5) imply that $A(\sigma_a - h) < 1$, and thus the theorem is applicable to only series with a fairly narrow critical strip.

COROLLARY 1. *Suppose that* $\Delta(s) = \Gamma(s)$. *For* Re $z > 0$, *let*

$$P(z) = \frac{1}{2\pi i} \int_C R(s) z^{-s}\, ds,$$

where C denotes a curve, or curves, encircling the singularities of $R(s)$. Instead of (2.6) *assume that* $\arg\{a(n)\}$ *is constant for all n and that as x tends to* 0, $x > 0$,

$$|P(x)| = c(\varphi)/x + O(x^{-\rho}),$$

where $c(\varphi)$ is a non-negative constant and $0 \leq \rho < 1$. Suppose there exists a $\lambda \in \{\lambda_n\}$ such that

$$(2.7) \qquad (2\pi)^{\frac{1}{2}}c(\varphi) < \lambda^{-h} \,|\, a(\lambda)\,|.$$

Then, there exists an infinite number of zeros for $\varphi(s)$ on the line $\sigma = h$.

THEOREM 2. *Let φ be of Class 2. Suppose there exists a constant $\rho \geq 0$ such that as ε tends to* 0, $\varepsilon > 0$,

$$(2.8) \qquad \Phi(e^{i(\frac{1}{2}\pi A - \varepsilon)}) = O(\varepsilon^{-\rho}).$$

If

$$(2.9) \qquad \rho < Ah + B - \tfrac{1}{2}N + 1,$$

then $\varphi(s)$ has an infinite number of zeros on $\sigma = h$.

THEOREM 3. *Let φ be of Class 2 and let $R(s)$ be holomorphic for $\sigma \geq h$. Suppose that for* $\operatorname{Re} z > 0$,

$$(2.10) \qquad \Phi(z) = O\{(\operatorname{Re} z)^{-h}\},$$

uniformly with respect to $|\arg z| \leq \frac{1}{2}\pi - \delta$, where $\delta > 0$ is fixed. Let c be chosen so that $a(c) \neq 0$, but $a(n) = 0$, $n < c$. Define $\varphi^*(s) = \lambda_c^s \varphi(s) - a(c)$ and

$$(2.11) \qquad \Psi(t) = \int_t^{t+H} a(c)^{-1}\varphi^*(h + iu)\, du,$$

where H is a positive constant to be chosen later. Assume also that

$$(2.12) \qquad \int_T^{2T} |\Psi(t)|^2\, dt = O(T),$$

as T tends to ∞. If $A \equiv 1 \pmod 4$, assume that

$$(2.13) \qquad N - 1 \leq 2h(A - 1) + 2B;$$

if $A \not\equiv 1 \pmod 4$, assume that

$$(2.14) \qquad N - 1 \leq 2hA + 2B.$$

Then, if $N_0(T)$ denotes the number of zeros of $\varphi(s)$ for $0 < t < T$,

$$N_0(T) > KT.$$

One can state a like result for $N_0(-T)$, $T > 0$. Condition (2.12) is a very restrictive one, and it took considerable work for Lekkerkerker [10] to establish such a condition for a subclass of series when $\Delta(s) = \Gamma(s)$. We remark that the proof and statement of Lemma 7 in his thesis [10] are incorrect, but that the remaining results are still valid with only a few minor modifications in the proofs. Our proof of Theorem 3 uses Lekkerkerker's variation of the Fourier transform method of Hardy and Littlewood [14, p. 222] used by them to obtain the analogous result for $\zeta(s)$.

THEOREM 4. *Let φ be of Class 2 and suppose that β_k is real, $k = 1, \cdots, N$. Then $R(s)$ is real on $\sigma = h$ if and only if there exists a function $\psi(s) = \sum b(n)\mu_n^{-s}$ such that*

$$R(s) = \Delta(2h - s)\psi(2h - s) \tag{2.15}$$

and

$$\overline{a(n)} = b(n); \quad \lambda_n = \mu_n . \tag{2.16}$$

3. Preliminary results

We shall need a form of Stirling's formula:

$$\Gamma(\sigma + it) = t^{\sigma+it-\frac{1}{2}}e^{-\frac{1}{2}\pi t-it+\frac{1}{2}i\pi(\sigma-\frac{1}{2})}(2\pi)^{\frac{1}{2}}(1 + 1/12(\sigma + it) + O(1/t^2)), \tag{3.1}$$

as t tends to ∞, uniformly for $-\infty < \sigma_1 \leq \sigma \leq \sigma_2 < \infty$. A similar formula may be given for $t < 0$ as t tends to $-\infty$ by using the fact that $\Gamma(\bar{s}) = \overline{\Gamma(s)}$.

LEMMA 1. *We have*

$$E_N(z) = O(|z|^k e^{-K|z|^{1/A}}),$$

uniformly for $|\arg z| \leq \frac{1}{2}\pi - \delta$, where $\delta > 0$ is fixed.

The constants k and K could be determined exactly, but we have no need to do so. Bochner [4] has incorrectly estimated $E_N(z)$; equations (145)–(147) in his paper are incorrect. For the special case $\Delta(s) = \Gamma^N(s)$, the estimate for $E_N(z)$ has been given in [1].

Proof of Lemma 1. We induct on N. For $N = 1$ we have

$$E_1(z) = (z^{\beta_1/\alpha_1}/\alpha_1)\exp(-z^{1/\alpha_1}).$$

Hence,

$$\begin{aligned} E_1(z) &= O(|z|^{\beta_{11}} \exp\{-|z|^{1/\alpha_1}\cos(\arg z/\alpha_1)\}) \\ &= O(|z|^k \exp(-K|z|^{1/\alpha_1})). \end{aligned}$$

From the theory of multiple Mellin transforms [13, pp. 53, 60] we find that

$$\begin{aligned} E_N(z) = &\int_0^\infty \frac{u_{N-1}^{\beta_N/\alpha_N}}{\alpha_N}\exp(-u_{N-1}^{1/\alpha_N})\,du_{N-1} \\ &\cdot\int_0^\infty \frac{u_{N-2}^{\beta_{N-1}/\alpha_{N-1}}}{\alpha_{N-1}}\exp(-u_{N-2}^{1/\alpha_{N-1}})\,du_{N-2}\cdots\int_0^\infty \frac{u_1^{\beta_2/\alpha_2}}{\alpha_1\alpha_2} \\ &\cdot\left(\frac{z}{u_1\cdots u_{N-1}}\right)^{\beta_1/\alpha_1}\exp\{-u_1^{-1/\alpha_2} - (z/u_1\cdots u_{N-1})^{1/\alpha_1}\}\,du_1 \\ = &\int_0^\infty \frac{u_{N-1}^{\beta_N/\alpha_N}}{\alpha_N}\exp(-u_{N-1}^{1/\alpha_N})E_{N-1}(z/u_{N-1})\,du_{N-1}. \end{aligned}$$

Letting $A_{N-1} = \sum_{k=1}^{N-1} \alpha_k$, we find by the induction hypothesis that

$$E_N(z) = O\left(|z|^{k_1} \int_0^\infty u^{k_2} \exp\{-u^{1/\alpha_N} - K(|z|/u)^{1/A_{N-1}}\}\, du\right)$$
$$= O\left(|z|^{k_1} \left\{\int_0^1 + \int_1^\infty\right\} u^{k_2} \exp\{-|z|^{1/A}(u + Ku^{-\alpha_N/A_{N-1}})\}\, du\right)$$
$$= O(|z|^{k_1}\{I_1 + I_2\}),$$

say. Let $f(u) = u^{k_2} \exp\{-|z|^{1/A} K u^{-\alpha_N/A_{N-1}}\}$. If $k_2 \geq 0$, $f(u)$ is increasing on $(0, \infty)$; if $k_2 < 0$, $f(u)$ has a relative maximum at

$$u = (-k_2 A_{N-1}/\alpha_N K)^{-A_{N-1}/\alpha_N} |z|^{A_{N-1}/\alpha_N A}.$$

In both cases

$$I_1 = O\left(|z|^k e^{-K|z|^{1/A}} \int_0^1 \exp(-|z|^{1/A} u)\, du\right)$$
$$= O(|z|^k e^{-K|z|^{1/A}}).$$

For $k_2 \geq 0$,

$$I_2 = O\left(|z|^{k_1} \int_1^\infty u^{k_2} \exp(-|z|^{1/A} u)\, du\right)$$
$$= O(|z|^k e^{-K|z|^{1/A}}),$$

upon successive integrations by parts. For $k_2 < 0$,

$$I_2 = O\left(|z|^k \exp\{-K|z|^{1/A} |z|^{-1/A}\} \int_1^\infty \exp(-|z|^{1/A} u)\, du\right)$$
$$= O(|z|^k e^{-K|z|^{1/A}}).$$

Combining our estimates for I_1 and I_2, we are finished.

We now see that $E_N(z)$ behaves exponentially for $|z|$ large in any sector in the right-half plane bounded away from the imaginary axis. In particular, we note that if $A \not\equiv 1 \pmod 4$, (2.6) and (2.8) are trivially satisfied.

LEMMA 2. *Let φ be of Class 1 with $\Delta(s) = \Gamma(s)$ and let $P(z)$ be defined as in Corollary 1. Then, for* $\operatorname{Re} z > 0$,

$$\Phi(z) = z^{-r} \sum b(n) e^{-\mu_n/z} + P(z).$$

A proof of Lemma 2 has been given by Bochner [4].

4. Proof of Theorem 1

Choose $\varepsilon > 0$ so that (2.3) holds, $\delta > 0$, and T_0 such that

$$\sup_{s \in S} |t| < T_0.$$

From the functional equation, (3.1), and the Phragmén-Lindelöf principle we easily deduce that for $r - \sigma_a - \delta \leq \sigma \leq \sigma_a + \delta$,

$$\varphi(s) = O(|t|^{A(\sigma_a+\delta-\sigma)}), \tag{4.1}$$

as $|t|$ tends to ∞.

Let $\lambda \in \{\lambda_n\}$ be chosen so that $a(\lambda) \neq 0$. By Cauchy's theorem, for $T > T_0$,

$$\begin{aligned} i\int_{T_0}^{T} e^{-\varepsilon t}\lambda^{h+it}\varphi(h+it)\,dt &= \int_{h+iT_0}^{h+iT} e^{\varepsilon i(s-h)}\lambda^{s}\varphi(s)\,ds \\ &= \left\{\int_{h+iT_0}^{\sigma_a+\delta+iT_0} + \int_{\sigma_a+\delta+iT_0}^{\sigma_a+\delta+iT} + \int_{\sigma_a+\delta+iT}^{h+iT}\right\} e^{\varepsilon i(s-h)}\lambda^{s}\varphi(s)\,ds \\ &= I_1 + I_2 + I_3, \end{aligned} \tag{4.2}$$

say. Clearly, $I_1 = O(1)$. Using (4.1), we find that

$$I_3 = O(e^{-\varepsilon T}T^{A(\sigma_a+\delta-h)}).$$

Now, $f(T) = e^{-\varepsilon T}T^{A(\sigma_a+\delta-h)}$ has a maximum at $T = A(\sigma_a+\delta-h)/\varepsilon$. Hence, for all $T > T_0$ we conclude that

$$I_3 = O(\varepsilon^{-A(\sigma_a+\delta-h)}).$$

Lastly,

$$\begin{aligned} I_2 &= e^{-i\epsilon h}\int_{\sigma_a+\delta+iT_0}^{\sigma_a+\delta+iT}\{e^{\varepsilon is}a(\lambda) + e^{\varepsilon is}\sum_{\lambda_n\neq\lambda} a(n)(\lambda/\lambda_n)^{s}\}\,ds \\ &= (ia(\lambda)/\varepsilon)(1+O(\varepsilon)) + O(1). \end{aligned}$$

Thus, by (4.2) we have shown that upon letting T tend to ∞,

$$\int_{T_0}^{\infty} e^{-\varepsilon t}\lambda^{h+it}\varphi(h+it)\,dt = a(\lambda)/\varepsilon + O(\varepsilon^{-A(\sigma_a+\delta-h)}),$$

since $A(\sigma_a - h) < 1$. It follows that

$$\begin{aligned} \int_{T_0}^{\infty} e^{-\varepsilon t}|\varphi(h+it)|\,dt &= \lambda^{-h}\int_{T_0}^{\infty} e^{-\varepsilon t}|\lambda^{h+it}\varphi(h+it)|\,dt \\ &\geq \lambda^{-h}\left|\int_{T_0}^{\infty} e^{-\varepsilon t}\lambda^{h+it}\varphi(h+it)\,dt\right| \\ &= \lambda^{-h}|a(\lambda)|/\varepsilon + O(\varepsilon^{-A(\sigma_a+\delta-h)}). \end{aligned} \tag{4.3}$$

For Re $z > 0$ and $c > \sigma_a$ we have

$$\begin{aligned} \Phi(z) &= (1/2\pi i)\int_{(c)} R(s)z^{-s}\,ds \\ &= (1/2\pi i)\left\{\int_{C_1} + \int_{C_2}\right\} R(s)z^{-s}\,ds, \end{aligned} \tag{4.4}$$

where

$$C_1 = (h - i\infty, h - iT_0) \cup (h + iT_0, h + i\infty)$$

and

$$C_2 = (h - iT_0, c - iT_0) \cup (c - iT_0, c + iT_0) \cup (c + iT_0, h + iT_0).$$

Letting $z = \exp\{i(\frac{1}{2}\pi A - \varepsilon)\}$, we have shown that

$$\Phi(e^{i(\frac{1}{2}\pi A - \varepsilon)}) = (1/2\pi i)\int_{C_1} R(s)e^{-i(\frac{1}{2}\pi A-\varepsilon)s}\,ds + O(1).$$

Now, from (3.1) as t tends to ∞,

$$(4.5)\quad |\Delta(s)| = De^{-\frac{1}{2}\pi A|t|}\{\textstyle\prod_{k=1}^{N} |\alpha_k t + \beta_{k2}^{\alpha_k\sigma+\beta_{k1}-\frac{1}{2}}\}\cdot\{1 + O(1/t^2)\},$$

where $D = (2\pi)^{\frac{1}{2}N}\exp(-\frac{1}{2}\pi\sum_{k=1}^{N}\beta_{k2})$. A similar formula holds as t tends to $-\infty$. In particular, as t tends to $-\infty$,

$$R(h + it) = O(|t|^K e^{-\frac{1}{2}\pi A|t|}),$$

and so as ε tends to 0,

$$\int_{-\infty}^{-T_0} R(h + it)e^{(\frac{1}{2}\pi A-\varepsilon)t}\,dt = O(1).$$

We have thus shown that

$$(4.6)\quad e^{i(\frac{1}{2}\pi A-\varepsilon)h}\Phi(e^{i(\frac{1}{2}\pi A-\varepsilon)}) = (1/2\pi)\int_{T_0}^{\infty} R(h + it)e^{(\frac{1}{2}\pi A-\varepsilon)t}\,dt + O(1).$$

Suppose now that there is not an infinite number of zeros on the line $\sigma = h$. Then, there exists a $T^* \geq T_0$ such that $R(h + it)$ is of constant sign for $t \geq T^*$. From (4.5) we see that

$$(4.7)\quad \begin{aligned} &\int_{T^*}^{\infty} |R(h + it)|\, e^{(\frac{1}{2}\pi A-\varepsilon)t}\,dt \\ &= D\int_{T^*}^{\infty} |\varphi(h + it)|\, e^{-\varepsilon t}\left\{\prod_{k=1}^{N} |\alpha_k t + \beta_{k2}|^{\alpha_k h+\beta_{k1}-\frac{1}{2}}\right\}\cdot\{1 + O(1/t^2)\}\,dt. \end{aligned}$$

By (4.1) the contribution of the O-term is

$$O\left(\int_{T^*}^{\infty} e^{-\varepsilon t} t^{A(\sigma_a+\delta)+B-\frac{1}{2}N-2}\,dt\right) = O(1),$$

by (2.4). Also, by (2.5) for T^* large enough,

$$\int_{T^*}^{\infty} |\varphi(h + it)|\, e^{-\varepsilon t}\prod_{k=1}^{N} |\alpha_k t + \beta_{k2}|^{\alpha_k h+\beta_{k1}-\frac{1}{2}}\,dt \geq \int_{T^*}^{\infty} |\varphi(h + it)|\, e^{-\varepsilon t}\,dt.$$

Combining the above with (4.6) and (4.7), we conclude that

$$(4.8)\quad \int_{T^*}^{\infty} |\varphi(h + it)|\, e^{-\varepsilon t}\,dt \leq 2\pi\,|\Phi(e^{i(\frac{1}{2}\pi A-\varepsilon)})|/D + O(1).$$

For ε small enough, (4.3) and (4.8) now yield

$$\lambda^{-h} \mid a(\lambda) \mid /\varepsilon + O(\varepsilon^{-A(\sigma_a+\delta-h)}) \leq 2\pi \mid \Phi(e^{i(\frac{1}{2}\pi A-\varepsilon)}) \mid /D + O(1).$$

Now, from (2.6) we easily have a contradiction, and the theorem is proved.

Proof of Corollary 1. By Lemma 2,

$$\begin{aligned} \mid \Phi(e^{i(\frac{1}{2}\pi A-\varepsilon)}) \mid &\leq \sum \mid a(n) \mid e^{-\lambda_n \sin \varepsilon} \\ &= \mid (\sin \varepsilon)^{-r} \sum b(n) e^{-\mu_n/\sin \varepsilon} + P(\sin \varepsilon) \mid \\ &= \mid P(\sin \varepsilon) \mid + O(1) \\ &= c(\varphi)/\sin \varepsilon + O(\varepsilon^{-\rho}). \end{aligned}$$

(4.9) now yields

$$\lambda^{-h} \mid a(\lambda) \mid /\varepsilon + O(\varepsilon^{-(\sigma_a+\delta-h)}) \leq (2\pi)^{\frac{1}{2}} c(\varphi)/\sin \varepsilon + O(\varepsilon^{-\rho}).$$

In view of (2.7) we arrive at another contradiction, and the corollary is proved.

5. Proof of Theorem 2

Let C denote a curve, or curves, enclosing the singularities of $R(s)$ in the half plane $\sigma > h$. By (4.4), (3.1), (2.1) and the residue theorem, we have for $\operatorname{Re} z > 0$,

$$\Phi(z) = (1/2\pi i) \int_{(h)} R(s) z^{-s}\, ds + (1/2\pi i) \int_C R(s) z^{-s}\, ds.$$

Again, let $z = \exp\{i(\frac{1}{2}\pi A - \varepsilon)\}$ where $\varepsilon > 0$ is small enough so that (2.3) holds. Thus, as ε tends to 0,

$$(5.1) \quad \Phi(e^{i(\frac{1}{2}\pi A-\varepsilon)}) = (1/2\pi) e^{-i(\frac{1}{2}\pi A-\varepsilon)h} \int_{-\infty}^{\infty} R(h+it) e^{t(\frac{1}{2}\pi A-\varepsilon)}\, dt + O(1).$$

Now, suppose there exists only a finite number of zeros for $\varphi(s)$ on $\sigma = h$. The integral on the right-hand side of (5.1) then converges absolutely, and so by (2.8)

$$\int_{-\infty}^{\infty} \mid R(h+it) \mid e^{t(\frac{1}{2}\pi A-\varepsilon)}\, dt = O(\varepsilon^{-\rho})$$

as ε tends to 0. Choose

$$T_0 > \sup(\sup\{\mid t \mid : t \in S\}, \sup_{1 \leq k \leq N}\{\mid -\beta_{k2}\, \alpha_k \mid\})$$

and use (3.1) to obtain

$$\int_{T_0}^{\infty} e^{-\varepsilon t} \mid \varphi(h+it) \mid \prod_{k=1}^{N} \mid \alpha_k t + \beta_{k2} \mid^{\alpha_k h + \beta_{k1} - \frac{1}{2}}\, dt = O(\varepsilon^{-\rho}).$$

If $\varepsilon = 1/T$, then the above implies that as T tends to ∞,

$$\int_{T_0}^{T} t^{Ah+B-\frac{1}{2}N} \mid \varphi(h+it) \mid dt = O(T^{\rho}),$$

which in turn implies that

$$\int_{h+iT_0}^{h+iT} | s^{Ah+B-\frac{1}{2}N}\varphi(s) | \, | ds | = O(T^\rho).$$

Now, choose $\lambda \in \{\lambda_n\}$ so that $a(\lambda) \neq 0$. Thus,

$$\int_{h+iT_0}^{h+iT} | s^{Ah+B-\frac{1}{2}N}\lambda^s\varphi(s) | \, | ds | = O(T^\rho). \tag{5.2}$$

In the quarter plane $\operatorname{Re} z \geq h$, $\operatorname{Im} z \geq T_0$, the function

$$I(z) = \int_{h+iT_0}^{z} s^{Ah+B-\frac{1}{2}N} \lambda^s\varphi(s)\, ds$$

is holomorphic. If $c > \sigma_a$,

$$\begin{aligned} I(c + iy) &= \sum a(n) \int_{c+iT_0}^{c+iy} s^{Ah+B-\frac{1}{2}N}(\lambda/\lambda_n)^s\, ds + O(1) \\ &= a(\lambda) \int_{c+iT_0}^{c+iy} s^{Ah+B-\frac{1}{2}N}\, ds + \sum_{\lambda_n \neq \lambda} a(n) \int_{c+iT_0}^{c+iy} s^{Ah+B-\frac{1}{2}N}(\lambda/\lambda_n)^s\, ds + O(1) \\ &= k'(c + iy)^{Ah+B-\frac{1}{2}N+1} + O(y^{Ah+B-\frac{1}{2}N}) + O(1), \end{aligned} \tag{5.3}$$

upon an integration by parts for the integrals in the infinite series.

Consider now the function $\mu(x)$ for $I(x + iy)$ i.e. for fixed x,

$$\mu(x) = \inf \{\xi : I(z) = O(| y |^\xi)\}.$$

By (5.3) we have shown that

$$\mu(c) = Ah + B - \tfrac{1}{2}N + 1.$$

From the fact that $\mu(x)$ is convex downward, we must also have

$$\mu(h) \geq Ah + B - \tfrac{1}{2}N + 1.$$

However, from (5.2) and (2.9) this is a contradiction. Hence, $R(h + it)$, and therefore $\varphi(h + it)$, has an infinite number of zeros.

6. Proof of Theorem 3

We consider again (4.4). Since $R(s)$ is holomorphic for $\sigma \geq h$, we have by Cauchy's theorem,

$$\Phi(z) = (1/2\pi i) \int_{(h)} R(s)z^{-s}\, ds. \tag{6.1}$$

For ξ real and $\varepsilon > 0$ small enough so that (2.3) holds, we put $z = \exp\{\xi + i(\frac{1}{2}\pi A - \varepsilon)\}$. If we let

$$F(\xi) = (2\pi)^{\frac{1}{2}} e^{\xi h + ih(\frac{1}{2}\pi A - \varepsilon)}\Phi(e^{\xi + i(\frac{1}{2}\pi A - \varepsilon)}) \quad \text{and} \quad G(t) = R(h + it)e^{(\frac{1}{2}\pi A - \varepsilon)t},$$

we see that (6.1) can be written as

$$F(\xi) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-i\xi t} G(t)\, dt,$$

i.e. $F(\xi)$ and $G(t)$ are Fourier transforms. Since $\varphi(h + it)$ is of finite order, we have from (3.1),

$$G(t) = O(|t|^k \exp\{(\tfrac{1}{2}\pi A - \varepsilon)t - \tfrac{1}{2}\pi A |t|\}),$$

as $|t|$ tends to ∞. It follows that

$$I(t) = \int_t^{t+H} G(u)\, du,$$

as well as $|I(t)|$ and $|I(t)|^2$, are integrable on $(-\infty, \infty)$. A direct calculation shows that the Fourier transform of $I(t)$ is $(e^{i\xi H} - 1)F(\xi)/i\xi$. Thus, by Parseval's formula,

$$\int_{-\infty}^{\infty} |I(t)|^2\, dt = 4 \int_{-\infty}^{\infty} \xi^{-2} \sin^2 \frac{1}{2} \xi H\, |F(\xi)|^2\, d\xi.$$

Now, if $A \equiv 1 \pmod 4$, from (2.10), $F(\xi) = O(\varepsilon^{-h})$ as ε tends to 0. Hence,

$$(6.2)\quad 4 \int_{-\infty}^{\infty} \xi^{-2} \sin^2 \tfrac{1}{2} \xi H\, |F(\xi)|^2\, d\xi < K\varepsilon^{-2h} \int_{-\infty}^{\infty} \xi^{-2} \sin^2 \tfrac{1}{2} \xi H\, d\xi < K_1 \varepsilon^{-2h} H.$$

If $A \not\equiv 1 \pmod 4$, $F(\xi) = O(1)$ as ε tends to 0. Hence,

$$(6.3)\qquad 4 \int_{-\infty}^{\infty} \xi^{-2} \sin^2 \tfrac{1}{2} \xi H\, |F(\xi)|^2\, d\xi < K_2 H.$$

From (3.1) it follows that for $|t| \geq T_1$, say,

$$(6.4)\qquad |\Delta(h + it)| > K\, |t|^{Ah+B-\frac{1}{2}N} e^{-\frac{1}{2}\pi A |t|}.$$

Now let

$$(6.5)\qquad T \geq T_0 = \sup(1, 6H, T_1).$$

If $\varepsilon = 1/T$ and $T \leq u \leq 2T + H$, then from (6.5) $\varepsilon u < 3$ and $u^{Ah+B-\frac{1}{2}N} > 3^{-\frac{1}{2}N} T^{Ah+B-\frac{1}{2}N}$. Hence, by (6.4)

$$J(t) = \int_t^{t+H} |G(u)|\, du > KT^{Ah+B-\frac{1}{2}N} \int_t^{t+H} |\varphi(h + iu)|\, du.$$

Using the definition of $\varphi^*(s)$, we have

$$|\varphi(h + iu)| = |a(c)|\, \lambda_c^{-h}\, |1 + a(c)^{-1} \varphi^*(h + iu)|$$
$$\geq |a(c)|\, \lambda_c^{-h}\, \mathrm{Re}\, \{1 + a(c)^{-1} \varphi^*(h + iu)\}.$$

Thus,

$$(6.6)\qquad J(t) > K_3\, T^{Ah+B-\frac{1}{2}N} \{H + \mathrm{Re}\, \Psi(t)\},$$

where $\Psi(t)$ is given by (2.11).

Let S be the subset of $(T, 2T)$ where $|I(t)| = J(t)$. Note that

$$|I(t)| = J(t)$$

if and only if $R(h + it)$ does not have a zero of odd multiplicity on $(t, t + H)$. Letting $\varepsilon = 1/T$, we have by (6.2) and (6.3),

$$\begin{aligned}\int_S J(t)\,dt = \int_S |I(t)|\,dt &\le \int_T^{2T} |I(t)|\,dt \\ &\le \left\{\int_T^{2T} dt \int_T^{2T} |I(t)|^2\,dt\right\}^{\frac{1}{2}} \qquad (6.7)\\ &\le K_1^{\frac{1}{2}} H^{\frac{1}{2}} T^{h+\frac{1}{2}}, \quad A \equiv 1 \pmod 4 \\ &\le K_2^{\frac{1}{2}} H^{\frac{1}{2}} T^{\frac{1}{2}}, \quad A \not\equiv 1 \pmod 4.\end{aligned}$$

Denoting the measure of S by $m(S)$, we have from (6.6) and (2.12),

$$\begin{aligned}T^{-Ah-B+\frac{1}{2}N} \int_S J(t)\,dt &> K_3\left\{Hm(S) + \int_S \operatorname{Re}\Psi(t)\,dt\right\} \\ &> K_3 Hm(S) - K_3\left\{\int_T^{2T} dt \int_T^{2T} |\Psi(t)|^2\,dt\right\}^{\frac{1}{2}} \qquad (6.8)\\ &> K_3 Hm(S) - K_4 T.\end{aligned}$$

Combining (6.7) and (6.8) and employing (2.13) and (2.14), we have in both cases that

$$K_3\, Hm(S) < K_4\, T + K_5\, H^{\frac{1}{2}} T.$$

Now, choose H large enough so that $(K_4/H + K_5/H^{\frac{1}{2}})/K_3 < 1/12$. Then

$$m(S) < T/12. \tag{6.9}$$

Now, subdivide $(T, 2T)$ into $[T/2H]$ pairs of abutting subintervals j_1, j_2, each of length H except for possibly the last j_2. Suppose that ν j_1-intervals contain only points of S. Then, $\nu H \le m(S)$. Of the remaining $[T/2H] - \nu$ pairs of intervals, either j_1 or j_2 contains a zero of $R(h + it)$. By (6.5) and (6.9),

$$[T/2H] - \nu > T/3H - T/12H = T/4H.$$

Hence, $R(h + it)$, and therefore $\varphi(h + it)$, has at least $T/4H$ zeros of odd multiplicity in $(T, 2T)$, and the proof is complete.

7. Proof of Theorem 4

Suppose (2.15) and (2.16) hold. Then, for $\sigma > \sigma_a$,

$$\sum b(n)\mu_n^{-s} = \sum \overline{a(n)}\lambda_n^{-s} = \overline{\sum a(n)\lambda_n^{-\bar{s}}}.$$

By analytic continuation, for all s, $\psi(s) = \overline{\varphi(\bar{s})}$. Thus,

$$\overline{R(\bar{s})} = \Delta(s)\psi(s) = R(2h - s).$$

Upon letting $\sigma = h$, we find that $R(h + it)$ is real.

Conversely, if $R(s)$ is real on $\sigma = h$, it follows from the reflection principle that $R(s)$ assumes conjugate values at the points s and $2h - \bar{s}$. Thus,

$$R(s) = \overline{\Delta(2h - \bar{s})\varphi(2h - \bar{s})} = \Delta(2h - s)\overline{\varphi(2h - \bar{s})}.$$

Hence, (2.15) and (2.16) hold if we take $\psi(s) = \overline{\varphi(\bar{s})}$.

8. Examples

The conditions of Theorem 1 are satisfied by a class of Epstein zeta-functions associated with quadratic forms of two variables [9]. A still larger class of Epstein zeta-functions satisfy the conditions of Corollary 1 [9]. More precise results for certain Epstein zeta-functions have been given by Potter and Titchmarsh [11]. Siegel [12] has obtained very sharp results for Epstein zeta-functions associated with quadratic forms of a higher number of variables, as well as precise results for Dirichlet L-functions.

If K is an imaginary quadratic field, the hypotheses of Corollary 1 are satisfied by some Dedekind zeta-functions,

$$\zeta_K(s) = \sum F(n)n^{-s},$$

where $F(n)$ denotes the number of non-zero integral ideals of norm n in K. The condition (2.7) here is

$$(2\pi)^{3/2}Rh/(w\,|\,d\,|^{\frac{1}{2}}) < |\,a(\lambda)\,|/\lambda^{\frac{1}{2}},$$

where R denotes the regulator, h the class number, w the number of roots of unity, and d the discriminant of K. However, it has recently been shown ([5], [3]) that the Dedekind zeta-function for any quadratic field possesses an infinite number of zeros on $\sigma = \frac{1}{2}$.

For examples illustrating Theorem 2 when $\Delta(s) = \Gamma(s)$, we refer the reader to Hecke's beautiful paper [8] where a large variety of illustrations are given.

The following examples for a more complicated $\Delta(s)$ appear to be new.

Example 1. Let K denote an algebraic number field of degree

$$n = r_1 + 2r_2,$$

where r_1 denotes the number of real conjugates in K and $2r_2$ the number of imaginary conjugates. Then,

$$\Delta(s) = \Gamma(\tfrac{1}{2}s)^{r_1}\Gamma(s)^{r_2}.$$

If $r_1 \leq 3$, condition (2.9) is satisfied. If also $A = \frac{1}{2}r_1 + r_2 \equiv 3\frac{1}{2}$, 4, $4\frac{1}{2}$ (mod 4), i.e. $n \equiv 7, 8, 9 \pmod 8$, then the Dedekind zeta-function of an ideal class $\mathfrak{C}$ in K,

$$\zeta(s, \mathfrak{C}) = \sum a_m m^{-s}, \quad \sigma > 1,$$

where a_m denotes the number of non-zero integral ideals of norm m in $\mathfrak{C}$, has an infinite number of zeros on the line $\sigma = \frac{1}{2}$.

Example 2. Let $\mathfrak{f}$ denote an ideal in the algebraic number field K and let χ denote a nonprincipal character mod $\mathfrak{f}$. Then the Dirichlet L-function for K and the character χ is defined by [6]

$$L(s, \chi) = \sum\nolimits_{\mathfrak{r}} \chi(\mathfrak{r}) N(\mathfrak{r})^{-s}, \quad \sigma > 1,$$

where the sum is over all non-zero integral ideals $\mathfrak{r}$ of K, and $N(\mathfrak{r})$ denotes the norm of $\mathfrak{r}$. $L(s, \chi)$ is an entire function. If we put

$$D = (dN(\mathfrak{f})\pi^{-n_2} 2^{-2r_2})^{\frac{1}{2}},$$

then for each real character mod $\mathfrak{f} \neq (1)$ and for each character mod (1), where (1) denotes the unit ideal, the function

$$\xi(s, \chi) = D^s \Gamma(\tfrac{1}{2}s)^{r_1} \Gamma(s)^{r_2} L(s, \chi)$$

satisfies the functional equation

$$\xi(s, \chi) = W(\chi)\xi(1 - s, \bar{\chi}),$$

where $W(\chi)W(\bar{\chi}) = 1$ and $|W(\chi)| = 1$. Putting $W(\chi) = e^{i\alpha}$, we have

$$\overline{e^{-\frac{1}{2}i\alpha}\xi(\tfrac{1}{2}+it, \chi)} = \overline{e^{\frac{1}{2}i\alpha}\xi(\tfrac{1}{2}-it, \bar{\chi})} = e^{-\frac{1}{2}i\alpha}\xi(\tfrac{1}{2} + it, \chi).$$

Hence, $e^{-\frac{1}{2}i\alpha}\xi(s, \chi)$ is real on $\sigma = \frac{1}{2}$. We conclude that under the same conditions on K as in Example 1, $L(s, \chi)$ has infinitely many zeros on $\sigma = \frac{1}{2}$.

Example 3. Lastly, we mention that certain zeta-functions with "Grössencharakteren" [7] have an infinite number of zeros on $\sigma = \frac{1}{2}$.

Functions which satisfy the conditions of Theorem 3 include the Dirichlet series of signature $(\lambda, \kappa, \gamma)$, $0 < \lambda < 2$, which are entire and have real coefficients [8]. In particular, if $\tau(n)$ denotes Ramanujan's arithmetical function, the conclusion of Theorem 3 is valid for $\sum \tau(n)/n^s$ with $h = 6$. See [10] for details and other examples.

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