REPRESENTATION OF A COMPLEMENTED ALGEBRA ON A LOCALLY COMPACT SPACE

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1. It was shown in [8] that for each simple complemented algebra A there exists a measure space (S, μ) and a real-valued positive function $k(s)$ on S such that A is isomorphic to the set of all measurable functions $f(s, t)$ on $S \times S$ for which the expression $\int\int |f(t, s)|^2 k(s)\, d\mu(t)\, d\mu(s)$ is finite. In this note we intend to show that there is a certain very natural topology τ on S with respect to which the function $k(s)$ is continuous almost everywhere and the measure μ is a Radon measure.

2. Let A be a Banach algebra whose underlying Banach space is a Hilbert space. Then A is called a complemented algebra [6] if the orthogonal complement of every right (left) ideal of A is again a right (left) ideal. To exclude the trivial case, when the product of any two members of the algebra is zero, we assume all algebras in the paper to be semi-simple.

We use the term *Radon measure* to refer to a measure on a locally compact Hausdorff space which corresponds to an integral on the set of all complex-valued continuous functions with a compact support (in the way it does, for example, in §6 of Naimark's book [5]). (An explicit definition of a Radon measure can be found in [4, page 9].) We assume that the reader is familiar with §6 of [5] and we are going to use the terminology of this section of Naimark's book.

Below is an example of a simple complemented algebra (compare with the example in [8]).

Example. Let (S, τ) be a locally compact Hausdorff space and let μ be a Radon measure on S. Let $k(s)$ be a measurable real-valued function on S bounded below by a positive number and finite except on a locally zero set [5, page 131]. (In particular $k(s)$ may be continuous at each point in S at which it is finite.) Let A be the set of all complex-valued measurable functions $x(t, s)$ on $S \times S$ such that $\int\int |x(t, s)|^2 k(s)\, d\mu(t)\, d\mu(s) < \infty$. Then A is a complemented algebra with respect to the multiplication

$$(xy)(t, s) = \int x(t, r) y(r, s)\, d\mu(r)$$

and the scalar product $(x, y) = \int\int x(t, s)\bar{y}(t, s) k(s)\, d\mu(s)\, d\mu(t)$.

If $k(s)$ is essentially bounded then A is a two-sided H^*-algebra.

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3. We intend to show that each simple complemented algebra is of the type described in the example above. The proof is similar to the proof employed in [8] but it has certain modifications due mostly to the fact that we are introducing a topology on the basic space S. Also the author decided to give more details (he feels that the article [8] was too much condensed).

The lemma below plays an essential part in the proof. It may be of an interest by itself.

LEMMA. *For each bounded normal operator T on a Hilbert space H there exists a locally compact Hausdorff space $(S, \mathfrak{T})$, a Radon measure μ on S and a bounded continuous complex-valued function h on S such that H is isomorphic to $L^2(S, \mu)$ and T corresponds to the multiplication of members of $L^2(S, \mu)$ with h. If T is self-adjoint (positive) then h is real (positive); if T is 1-1 then $h(s) \neq 0$ locally almost everywhere on S* [5, page 131].

Proof. Let B be the closed commutative algebra of operators generated by T and the identity operator I. By Theorem 64 of [4] (or the proposition in section 2, §17 of [5]) there exists a family $\{H_\alpha\}_{\alpha \in \Gamma}$ of mutually orthogonal closed subspaces of H, each invariant under members of B, such that for each $\alpha \in \Gamma$ the set B_α of restrictions of members of B to H_α has simple spectrum [4, page 149] (if we consider B_α as a representation of B) and such that $H = \sum_{\alpha \in \Gamma} H_\alpha$. This means that for each $\alpha \in \Gamma$ there exists $\varphi_\alpha \in H$ such that the set $\{P\varphi_\alpha \mid P \in B\}$ is dense in H_α.

For each $\alpha \in \Gamma$ let us apply the Gelfand theory to the algebra B_α (of restrictions of members of B to H_α) (see, for example [2] or 26A in [3]). We obtain a family $\{\mathfrak{M}_\alpha, \mathfrak{T}_\alpha\}_{\alpha \in \Gamma}$ of compact Hausdorff spaces such that each B_α is isomorphic and isometric to the set $L(\mathfrak{M}_\alpha)$ of all continuous complex-valued functions on $\mathfrak{M}_\alpha$. In particular the restriction T_α of T to H_α corresponds to a continuous function h_α such that $|h_\alpha(M)| \leq \|T\|$ for all $M \in \mathfrak{M}_\alpha$. If T is self-adjoint then each h_α is real, also $h_\alpha \geq 0$ in the case when T is positive; if T is one to one then $h_\alpha(M) \neq 0$ almost everywhere on $\mathfrak{M}_\alpha$.

As in Theorem 61 of [4] we define an integral J_α on $L(\mathfrak{M}_\alpha)$ by setting $J_\alpha(f) = (P_f \varphi_\alpha, \varphi_\alpha)$ where P_f is the member of B_α corresponding to a function $f \in L(\mathfrak{M}_\alpha)$ via (inverse) Gelfand mapping (φ_α is defined above).

Now as in [8] we set $S = \bigcup_{\alpha \in \Gamma} \mathfrak{M}_\alpha$ (we think about spaces $\mathfrak{M}_\alpha$ as being distinct) and then we define a topology $\mathfrak{T}$ on S by the requirement that a subset O of S is open if and only if $O \cap \mathfrak{M}_\alpha$ belongs to $\mathfrak{T}_\alpha$ for each $\alpha \in \Gamma$. Then $(S, \mathfrak{T})$ is a locally compact Hausdorff space such that a complex-valued function f on S is continuous if and only if each restriction f_α of f to $\mathfrak{M}_\alpha$ is continuous. Note also that each $\mathfrak{M}_\alpha$ is both open and closed (in fact each $\mathfrak{M}_\alpha$ is a compact subset of S).

We define an integral J on the set $L(S)$ of all continuous complex-valued functions with compact support in S as follows. For each $f \in L(S)$ we select $\alpha_1, \alpha_2, \cdots, \alpha_n \in \Gamma$ so that f vanishes outside $\bigcup_{k=1}^{n} \mathfrak{M}_{\alpha_k}$ and define

$$J(f) = \sum_{k=1}^{n} J_{\alpha_k}(f_{\alpha_k})$$

(f_{α_k} is the restriction of f to $\mathfrak{M}_{\alpha_k}$).

Now we can apply to J (and $L(S)$) the theory of integration developed by Naimark in §6 of his book [5]. Let μ be the corresponding measure. Let h be the function defined by the condition that the restriction of h to each $\mathfrak{M}_\alpha$ is h_α. Then $(S, \mathfrak{T})$, μ and h has the properties specified in the Lemma. If T is one to one then the product of h with a non-zero member of $L^2(S, \mu)$ is a non-zero member of $L^2(S, \mu)$. In this case the set $\{s \in S \mid h(s) \neq 0\}$ is a locally zero set.

THEOREM. *Each simple complemented algebra A is of the form described in the example above. More specifically for each simple complemented algebra A there exists a locally compact Hausdorff space $(S, \mathfrak{T})$, a Radon measure μ on S and a bounded continuous non-negative-valued function $h(s)$ on S, positive locally almost everywhere, such that A is isomorphic to the algebra of all measurable complex-valued functions $x(t, s)$ on $S \times S$ with the property that*

$$\int \int | x(t, s)|^2 h(s)^{-2} \, d\mu(t) \, d\mu(s)$$

is finite (note that $h(s)^{-2}$ corresponds to $k(s)$ in the above example).

Proof. As in [8] we use Theorem 3 of [7] which states that there exists a positive self-adjoint operator α on a Hilbert space H, such that A is isomorphic to the algebra of all (Hilbert-Schmidt) operators x on H such that $x\alpha$ is of the Hilbert-Schmidt type. The scalar product (x, y) corresponds to $tr(x\alpha^2 y)$ and the operator α has a bounded inverse β, which is also self-adjoint and positive. Applying the above lemma to the operator β we obtain a locally compact space $(S, \mathfrak{T})$, a Radon measure μ on S and a bounded continuous nonnegative function $h(s)$ on S (with $h(s) > 0$ locally almost everywhere) such that H is isomorphic to $L^2(S, \mu)$ and β acts on $L^2(S, \mu)$ as a multiplication with h.

But the set (σc) of all Hilbert-Schmidt operators on $L^2(S, \mu)$ is isomorphic to $L^2(S \times S, \mu \times \mu)$ where the multiplication of members of (σc) corresponds to the operation $(xy)(s, t) = \int x(s, r) y(r, t) \, d\mu(r)$ (see, for example, Theorem 4 on page 35 in [9]). Also it can be easily verified that the products $x\beta$ of a Hilbert-Schmidt operator x and the operator β will correspond to $x(s, t)h(t)$ (if $x(s, t)$ corresponds to x); $x\alpha$ will correspond to $x(s, t)h(t)^{-1}$.

Thus A can be realized as the set of all members $f(s, t)$ of $L^2(S \times S, \mu \times \mu)$ such that $\int \int | f(s, t) |^2 k(s) \, d\mu(s) \, d\mu(t) < \infty$ where $k(s) = 1/h(s)^2$.

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