DOUBLE CENTRALIZERS OF INJECTIVES AND PROJECTIVES OVER ARTINIAN RINGS

BY

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Introduction and notation. Throughout this paper R denotes a left artinian ring. If M is an R-module, let $C(M)$ denote its set of R-endomorphisms viewed as a ring of operators on the opposite side of M and let $D(M)$ be the *double centralizer* of M. Thus if $M = {}_RM$ is a left R-module $D(M) = C(M_{C(M)})$ and M is a left module over the ring $D(M)$. Moreover $[\lambda(r)](m) = rm$, $r \in R$, $m \in M$, defines a canonical ring homomorphism $\lambda : R \to D(M)$ which is a monomorphism in case M is faithful. (If M is a right R-module the canonical ring homomorphism of R into $D(M)$ will be denoted by ρ.) If $\lambda(R) = D(M)$ M is said to have the *double centralizer property* over R.

Since Nesbitt and Thrall [8] proved that all the faithful modules over a quasi-Frobenius ($= QF$) algebra have the double centralizer property, many authors have studied double centralizers—usually with an eye to determining which rings (in addition to QF rings) possess this property. Here we give characterizations of those left artinian rings over which every left injective (equivalently, right projective) faithful module has the double centralizer property (Theorem 5.). In the process we show that the double centralizers of the left injectives over R are the same as the double centralizers of the direct summands of R_R (Theorem 2.); show how to calculate the double centralizers of all the faithful left injectives and right projectives as subrings of the double centralizer of a certain distinguished direct summand of R_R (Theorom 3.); and give criteria for a given left injective (right projective) to have the double centralizer property (Theorem 4.).

We shall use the following additional notation. Let N be the (Jacobson) radical of R. If M is a left (right) R-module then $S(M)$ denotes the socle of M, $T(M) = M/NM$ ($T(M) = M/MN$) and we write $E(M)$ for the injective hull (see [2]) of M.

Principal results. A module U is said to *generate* a module V over R in case $V = \sum\{\text{Im } \alpha \mid \alpha : U \to V\}$ and U *cogenerates* V if $\bigcap\{\text{Ker } \gamma \mid \gamma : V \to U\} = 0$. Note that U generates (cogenerates) V if and only if V is an epimorph of (can be embedded in) a direct sum (product) of copies of U.

These notions and the following lemma allow us to greatly reduce the size of the modules under consideration.

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1. LEMMA. *Let U and V be left R-modules. Then, letting Res denote restriction to U, we have the commutative diagram of ring homomorphisms*

$$\begin{array}{ccc} & R & \\ {}^{\lambda}\swarrow & & \searrow^{\lambda} \\ D(U \oplus V) & \xrightarrow{\text{Res}} & D(U) \end{array}$$

in which Res *is an isomorphism if U both generates and cogenerates V.*

Proof. Let $W = U \oplus V$ and identify U and V with their natural injections into W. Let $\theta : W \to U$ and $\varphi : W \to V$ be the natural projections and let $C = C(W)$. Then $D(W) = C(W_C)$ and $D(U) = C(U_{\theta C\theta})$. Thus if $a \in D(W)$ we have $a \mid_U \in D(U)$ and Res: $a \to a \mid_U$ is a unital ring homomorphism making the diagram commute. It follows from an argument given by Morita (see the proof of Theorem (1.1), II $\Rightarrow$ I, in [6]) that if U either generates or cogenerates V then Res is a monomorphism. Thus to see that Res is an isomorphism when U both generates and cogenerates V we need only show that under this assumption every $\theta C\theta$-endomorphism of U can be extended to a C-endomorphism of W. Let $b : U \to U$ over $\theta C\theta$. For $c_1, \cdots, c_n \in C$ and $u_1, \cdots, u_n \in U$ let

$$\bar{b}(\sum u_i c_i) = \sum b(u_i)c_i .$$

If $\sum u_i c_i = 0$ then

$$\begin{aligned} \sum b(u_i)c_i &= \sum b(u_i)c_i\theta + \sum b(u_i)c_i\varphi \\ &= b(\sum u_i c_i \theta) + \sum b(u_i)c_i\varphi \\ &= 0 + \sum b(u_i)c_i\varphi \in V. \end{aligned}$$

But for each $\varphi c\theta : V \to U$ we have

$$\sum b(u_i)c_i\varphi c\theta = b(\sum u_i c_i \varphi c\theta) = 0.$$

Thus since U congenerates V we see that $\bar{b} : UC \to W$ is a well defined C-map extending b. Moreover, the assumption that U generates V implies that $UC = W$ and the lemma is proved.

A left injective module E and a right projective module F are said to form a *pair* over R in case $E(T(Rf_1)), \cdots, E(T(Rj_n))$ represent all the indecomposable direct summands of E where $f_1, \cdots, f_n$ is an orthogonal set of primitive idempotents such that $f_1 R, \cdots, f_n R$ are (to within isomorphism) the indecomposable direct summands of F. Note that if $f = f_1 + \cdots + f_n$ then E (respectively, F) is both generated and cogenerated by $E(T(Rf))$ (respectively, fR). We shall say that such an f is a *basic idempotent* for E and F.

The notion of a pair is a generalization of the dualty between left injectives and right projectives over a finite dimensional algebra. Just as is the case for dual modules over such an algebra, the double centralizes of the members of a pair are the same. That is

2. THEOREM. *Let ${}_RE$ and F_R form a pair. Then there is a ring isomorphism ψ making the diagram*

$$\begin{array}{ccc} & R & \\ {}^{\lambda}\swarrow & & \searrow^{\rho} \\ D(E) & \xrightarrow{\psi} & D(F) \end{array}$$

commute.

Proof. Let E and F be a pair with basic idempotent f. Then by the lemma we may assume that $E = E(T(Rf))$ and $F = fR$. Let $C = C(E)$. Then $\mathrm{Hom}_C(E, E) = D(E)$ and $\mathrm{Hom}_{fRf}(fR, fR)$ is the opposite ring of $D(F)$. According to [3, Lemma (2.3)]

$$[\lambda'(fr)](x) = frx, \quad fr \in fR, x \in E$$

and

$$[\rho'(x)](fr) = frx, \quad x \in E, fr \in fR$$

define an $fRf - R$ isomorphism

$$\lambda' : fR \to \mathrm{Hom}_C(E, fE)$$

and an $R - C$ isomorphism

$$\rho' : E \to \mathrm{Hom}_{fRf}(fR, fE).$$

Now, because λ' and ρ' are isomorphisms, we can define, for $a \in \mathrm{Hom}_C(E, E)$, $fr \in fR$, $[\psi(a)](fr)$ to be the unique element of fR satisfying

$$[\psi(a)](fr) \cdot x = fr \cdot a(x), \quad \text{for all} \quad x \in E.$$

This gives a ring anti-isomorphism

$$\psi : \mathrm{Hom}_C(E, E) \to \mathrm{Hom}_{fRf}(fR, {}^{f}R)$$

whose inverse γ is defined by the equation

$$fr \cdot [\gamma(b)](x) = b(fr) \cdot x, \quad b \in \mathrm{Hom}_{fRf}(fR, fR), x \in E, fr \in fR.$$

Thus, with the observation that ψ is also a left R-map, the theorem is proved.

Let $f_1, \cdots, f_m$ be orthogonal primitive idempotents such that $T(Rf_1), \cdots, T(Rf_m)$ represent the distinct isomorphism classes of minimal left ideals of R and $f_0 = f_1 + \cdots + f_m$. Then $E(T(Rf_0))$ must be a faithful injective that appears as a direct summand of every faithful left injective R-module. On the other hand, no minimal left ideal annihilates f_0 on the right; and if P_R is a projective such that $T(P)$ does not contain a copy of some $T(f_i R)$ and $T(Rf_i) \cong S \leq {}_R R$ then $f_i S \neq 0$ and $Pf_i S \subseteq PNf_i S = 0$. So a projective right R-module is faithful if and only if it contains a direct summand isomorphic to $f_0 R$. Thus we call $E_0 = E(T(Rf_0))$ the *minimal faithful left injective* for R and $f_0 R$ the *minimal faithful right projective* for R. These two distinguished modules form a pair. So by 2. we may identify their double centralizers. Henceforth we shall write $D_0 = D(f_0 R) = D(E_0)$. The next theorem shows, given a faithful pair ${}_R E$ and F_R, how to calculate their double centralizer D as a subring of D_0 with $R \subseteq D \subseteq D_0$.

3. THEOREM. *Let ${}_RE$ and F_R be faithful and form a pair with basic idempotent f. Write $D = D(E) = D(F)$ and embed R in D and D_0 via either ρ or λ. Then there is a ring monomorphism Γ such that the diagram*

$$\begin{array}{ccc} & R & \\ \swarrow & & \searrow \\ D & \xrightarrow{\Gamma} & D_0 \end{array}$$

is commutative. Moreover, identifying $R = \rho(R) \subseteq D_0$,

$$\Gamma(D) = \{b \in D_0 \mid fRb \subseteq R\}.$$

Proof. Using 1. and 2. we may assume that $f = f_0 + g$ where g is an idempotent in R orthogonal to f_0, $D = D(fR)$, and $R = \rho(R) \subseteq D_0$. Then $B = \{b \in D_0 \mid fRb \subseteq R\}$ is a subring of D_0 and

$$(f_0 r)[\Gamma(a)] = (f_0 r)a, \quad a \in D, f_0 r \in f_0 R$$

defines s ring homomorphism $\Gamma: D \to D_0$. To see that $\Gamma(D) \subseteq B$, let $f_0 s \in f_0 R$, $fr \in fR$. Then if $a \in D$

$$(f_0 s)[fr\Gamma(a)] = ((f_0 s)\rho(fr))a = (f_0 sfr)a$$
$$= f_0 sf(fr)a = (f_0 s)\rho((fr)a)$$

so that $fR\Gamma(a) \subseteq \rho(R) = R$ and $\Gamma: D \to B$. Moreover,

$$(fr)\Omega(b) = frb, \quad b \in B, fr \in fR$$

defines a ring homomorphism $\Omega: B \to D$ with

$$(f_0 r)[\Gamma(\Omega(b))] = (f_0 r)\Omega(b) = (ff_0 r)\Omega(b) = f_0 rb = (f_0 r)b,$$

for $f_0 r \in f_0 R$, $b \in B$. Therefore $\Gamma(\Omega(b)) = b$ for all $b \in B$. Also if $fr \in fR$, $a \in D$ then $(fr)[\Omega(\Gamma(a))] = fr\Gamma(a)$ and as we saw above $(f_0 s)[fr\Gamma(a)] = (f_0 s)[(fr)a]$, for all $f_0 s \in f_0 R$, so that $\Omega(\Gamma(a)) = a$ for all a in D. Thus Γ is the desired isomorphism from D onto B.

Our next theorem shows precisely which left injectives and right projectives have the double centralizer property. Observe that as a consequence there is a unique smallest pair with the double centralizer property—namely, $E(T(Rf_*))$ and $f_* R$, where $f_* = f_0 + f_{m+1} + \cdots + f_{m+k}$ with $T(Rf_{m+j})$, $j = 1, \cdots, k$, representing those simple modules S that are not minimal left ideals and satisfy an exact sequence $0 \to R \to S' \to S \to 0$ where R is essential in S'.

4. THEOREM. *If ${}_RE$ and F_R are faithful and form a pair with basic idempotent f then the following are equivalent:*

(a) *E has the double centralizer property.*

(b) *F has the double centralizer property.*

(c) *Every simple submodule of the left module* D_0/R *appears in* $T(Rf)$.

(d) $\operatorname{Ext}^1_R(S, R) = 0$ *for every simple left module that does not appear in* $T(Rf)$.

Proof. (a) $\Leftrightarrow$ (b). This is immediate from 2.

(c) $\Rightarrow$ (b). If F does not have the double centralizer property then by 3. there is an R-module S' with $R < S' \leq D \leq D_0$ and S'/R simple. But, again by 3., $fS' \subseteq fD \subseteq R$ so S'/R is a simple submodule of D_0/R that does not appear in $T(Rf)$.

(b) $\Rightarrow$ (d). As Mueller [7, Lemma 7] pointed out, a simple module S has $\operatorname{Ext}^1_R(S, R) \neq 0$ if and only if there is an essential extension S' of R with $S'/R \cong S$. If S' is such an over-module of ${}_RR$ and S does not appear in $T(Rf)$ then $fS' \subseteq R$ and

$$fr\bar{\rho}(s') = frs', \quad fr \in fR, s' \in S'$$

defines a monomorphism $\bar{\rho} : S' \to D(fR)$ with $\rho(R) < \bar{\rho}(S') \leq D(fR)$ so that fR and hence F does not have the double centralizer property.

(d) $\Rightarrow$ (c) This follows from the facts that ${}_RR$ is essential in $D(E(R))$ (see [4]) and that $D(E(R)) \cong D_0$ (by 1.).

Putting the preceding results together we see that

5. Theorem. *The following statements about a left artinian ring* R *are equivalent*:

(a) *Every faithful left injective and every faithful right projective R-module has the double centralizer property.*

(b) *The minimal faithful left injective R-module has the double centralizer property.*

(c) *The minimal faithful right projective R-module has the double centralizer property.*

(d) $\operatorname{Ext}^1_R(S, R) = 0$ *for every simple left R-module* S *that is not isomorphic to a minimal left ideal.*

Remarks. (a) In [4] Lambek showed that Utumi's (Johnson's in case of zero singular ideal) left ring of quotients Q for a ring with identity is the double centralizer of the injective hull of the ring viewed as a left module over itself. If R is a left artinian ring then it follows from 1., 2. and Lambek's theorem that Q can be constructed directly from R. That is, $D_0 = \operatorname{Hom}_{f_0Rf_0}(f_0R, f_0R)^0$ is Utumi's ring of quotients for R. We also note that R has zero singular ideal (equivalently, D_0 is semi-simple) if and only if f_0Rf_0 is a semi-simple ring.

(b) Also in [4] Lambek raised the question of whether or not all the rings Q' with $R \leq Q' \leq Q$ (i.e., the rational ring extensions of R) are double centralizers of faithful injective modules. The answer is no. Because, for the ring R of 3×3 matrices of the form

$$\begin{bmatrix} a & x & y \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

with entries in a field K, f_0 is the matrix unit e_{11}, D_0 is the full ring of 3×3 matrices over K and, using 3. (or direct calculation), one can show that neither $D((e_{11} + e_{22})R)$ nor $D((e_{11} + e_{33})R)$ is isomorphic to the ring of upper triangular matrices over K.

(c) If R is a QF-3 ring (i.e., $E({}_R R)$ is projective) then according to [7, Lemma 7] dom dim $R > 1$ if and only if $\operatorname{Ext}_R^1(S, R) = 0$ for every simple left R-module S that is not isomorphic to a minimal left ideal. Thus 5. generalizes the fact that the unique minimal faithful left module over a QF-3 ring R has the double centralizer property if and only if dom dim $R > 1$, a result that was first proved for finite-dimensional algebras by Tachikawa [9] and Mochizuki [5].

(d) When Thrall [10] defined QF-3 and QF-1 (= every faithful left module has the double centralizer property) rings he wondered whether there is a containment relationship between the two classes. In [6] Morita showed that there is not. Wu, Mochizuki and Jans [11] proved that R is QF-3 if and only if the class $\mathfrak{T}$ (for torsion) of modules ${}_R T$ with $\operatorname{Hom}_R(T, R) = 0$ is closed under taking submodules and the class $\mathfrak{L}$ (for torsionless) of left R-modules that are cogenerated by R is closed under taking extensions. It is interesting to note that QF-3 and QF-1 rings share the former property. In fact we shall show that $\mathfrak{T}$ is closed under taking submodules if the faithful left injectives over R have the double centralizer property. To this end suppose $D_0 = R$, $T \in \mathfrak{T}$ and T/M is semi-simple. Then, since $\mathfrak{T}$ is always closed under taking factors we may write $T/M \cong \oplus \sum S_\alpha$ where each S_α is simple and not isomorphic to a minimal left ideal. Thus by [1, VI, 1.2] and 5.,

$$\operatorname{Ext}_R^1(T/M, R) \cong \prod \operatorname{Ext}_R^1(S_\alpha, R) = 0$$

So from the exact sequence

$$0 \to \operatorname{Hom}_R(T/M, R) \to \operatorname{Hom}_R(T, R) \to \operatorname{Hom}_R(M, R) \to \operatorname{Ext}_R^1(T/M, R)$$

we see that $M \leq T$ is torsion whenever $T \in \mathfrak{T}$ and T/M is semi-simple. Now the proof is completed by observing that if M is any submodule of T we can write

$$M = M_0 \leq M_1 \leq \cdots \leq M_n = T$$

where M_k/M_{k-1} is semi-simple for $k = 1, \cdots, n$.

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