METABELIAN p-GROUPS WHICH CONTAIN A SELF-CENTRALIZING ELEMENT[1]

BY

MARC W. KONVISSER

Introduction. An element x of a group G is called self-centralizing in G if the set $c_G(x)$ of all elements commuting with x is just the cyclic group generated by x. The existence of a self-centralizing element has a profound effect on the structure of the group. In this paper we will concern ourselves with groups G which are finite metabelian p-groups, $p \neq 2$, and which contain a self-centralizing element x.

We will analyze the structure of such a group by examining the action of the automorphism induced by a self-centralizing element x on a normal subgroup M of G. We will find a decomposition of M which is analogous to that of a vector space under the action of a linear transformation.

First we define the subsets Y_i of M by

$$Y_0 = 1, \qquad Y_i = \{g \mid g \in M \quad \text{and} \quad [g, x] \in Y_{i-1}\} \quad \text{for } i = 1, 2, \cdots$$

It is clear from the definition that the Y_i's are invariant under the action of x. Since G is nilpotent, it is easily seen that $1 = Y_0 < Y_1 < \cdots < Y_m = M$ for some integer m. In Lemma 4 we show that each Y_i is a subgroup. Thus, the decomposition of M under x is analogous to a block triangular decomposition of a vector space under a linear transformation. In Theorem 1 we show that $Y_i \lhd Y_{i+1}$ and Y_{i+1}/Y_i is cyclic for $i = 0, 1, \cdots$. Thus, the blocks Y_{i+1}/Y_i are one dimensional and the decomposition of M into the subgroups Y_i is triangular under x.

As a simple consequence of Theorem 1, we find that the number of generators of a metabelian p-group, $p \neq 2$, containing a self-centralizing element is less than or equal to its class. Theorem 2 gives a different bound for the number of generators of an arbitrary p-group. It is shown that for the groups discussed in Theorem 1, we can exhibit a system of generators which is economical in the sense that it satisfies the bounds of Theorem 2 and Corollary 2. We conclude with an example which shows that both bounds are best possible.

Our notation will be that of Huppert [3] with the addition of the symbol $[a, {}_ix]$ for the Engel element $[a, x, \cdots, x]$ where x appears i times.

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The following identities will be useful in many of the calculations in this paper.

IDENTITIES. *Let G be a group with elements $a, b, c, \cdots$; then*

1. $[ab, c] = [a, c]^b[b, c] = [a, c][a, c, b][b, c]$
2. $[a, bc] = [a, c][a, b]^c = [a, c][a, b][a, b, c]$
3. $[a, b, c^a][c, a, b^c][b, c, a^b] = 1$,

and if G is a metabelian group, then

4. $[a, b, c, d] = [a, b, d, c]$
5. $[a, b, c][c, a, b][b, c, a] = 1$
6. $[ab, c] = [a, b][a, c]$ *for b in G'*
7. $[b, a^m] = \prod_{i=1}^{m} [b, {}_ia]^{C(m,i)}$

Proof. Identities 1 and 2 are found in [2, p. 150]. Identity 3 can be found in [7, Theorem 5.1]. Identity 4 is from [8, Lemma 34.51]. Identity 5 follows trivially from Identity 3, and Identity 6 is a simple consequence of Identity 2. Identity 7 is proved in [4, Lemma 3].

We will need several technical lemmas.

LEMMA 1. *Let G be a metabelian p-group, $p \neq 2$, and $\langle [c, {}_{j-1}x] \rangle \lhd G$; then*

$$\langle [c^{p^n}, {}_{j-1}x] \rangle = \langle [c, {}_{j-1}x]^{p^n} \rangle.$$

Proof (by induction on n). Since G is metabelian,

$$[[c, x], {}_{j-2}x]^p = [[c, x]^p, {}_{j-2}x]$$

by Identity 6. So for $n = 1$ it will suffice to show that

$$\langle [c^p, {}_{j-1}x] \rangle = \langle [[c, x]^p, {}_{j-2}x] \rangle.$$

Now using Identities 6 and 7 we have

$$[[x, c^p], {}_{j-2}x] = \prod_{k=1}^{p} [[x, {}_kc]^{C(p,k)}, {}_{j-2}x].$$

Letting $\delta_k = C(p, k)/p$ we get

$$[[x, c^p], {}_{j-2}x] = d\prod_{k=2}^{p-1} [[x, {}_kc], {}_{j-2}x]^{p\delta_k}[x, {}_pc, {}_{j-2}x],$$

where $d = [[x, c]^p, {}_{j-2}x]$, and using Identities 4 and 6

$$[[x, c]^p, {}_{j-2}x] = d[x, {}_pc, {}_{j-2}x]\prod_{k=2}^{p-1} [d, {}_{k-1}c]^{\delta_k}.$$

Now $\langle [c, {}_{j-1}x] \rangle \lhd G$ implies $\langle d \rangle = \langle [c, {}_{j-1}x]^p \rangle \lhd G$ implies $\langle d^p \rangle \lhd G$.

We consider the above equation modulo $\langle d^p \rangle$.

$$[[x, c^p], {}_{j-2}x] \equiv d[x, {}_pc, {}_{j-2}x] \mod \langle d^p \rangle$$

Since for $k \geq 2$, $\langle [d, {}_{k-1}c] \rangle < \langle d \rangle$ implies $\langle [d, {}_{k-1}c] \rangle \leq \langle d^p \rangle$. But $[x, {}_pc, {}_{j-2}x, c] \in \langle d \rangle$ implies $[x, c, {}_{j-2}x, c, c] \in \langle d^p \rangle$ so we have

$$[[x, c^p], {}_{j-2}x] \equiv d[x, c, {}_{j-2}x, {}_{p-1}c] \equiv d \mod \langle d^p \rangle,$$

which implies that

$$\langle[[x, c^p], {}_{j-2}x]\rangle = \langle d\rangle = \langle[x, c]^p, {}_{j-2}x]\rangle.$$

Induction Step. Assume $\langle[c^{p^m}, {}_{j-1}x]\rangle = \langle[c, {}_{j-1}x]^{p^m}\rangle$ for all $m < n$.

To apply induction we must first show that $\langle[c, {}_{j-1}x]\rangle \lhd G$ implies $\langle[c^{p^k}, {}_{j-1}x]\rangle \lhd G$ for any integral $k > 0$. Now

$$\begin{aligned}[c^{p^k}, {}_{j-1}x] &= [[x, c^{p^k}]^{-1}, {}_{j-2}x] \\ &= [[x, c^{p^k}], {}_{m-2}x]^{-1} && \text{by Identity 6} \\ &= [\textstyle\prod_{i=1}^{p^k} [x, {}_ic]^{C(p^k,i)}, {}_{j-2}x]^{-1} && \text{by Identity 7} \\ &= \textstyle\prod_{i=1}^{p^k} [x, {}_ic, {}_{j-2}x]^{-C(p^k,i)} && \text{by Identity 6.}\end{aligned}$$

Hence,

$$[c^{p^k}, {}_{j-1}x] \in \langle[c, {}_{j-1}x]\rangle \lhd G.$$

Since all subgroups of a cyclic normal subgroup are normal,

$$\langle[c^{p^k}, {}_{j-1}x]\rangle \lhd G.$$

Now by induction since $n > 1$ we get

$$\begin{aligned}\langle[c^{p^n}, {}_{j-1}x]\rangle &= \langle[c^{p^{n-1}}, {}_{j-1}x]^p\rangle \\ &= \langle([c, {}_{j-1}x]^{p^{n-1}})^p\rangle && \text{applying induction again} \\ &= \langle[c, {}_{j-1}x]^{p^n}\rangle.\end{aligned}$$

LEMMA 2. *If G is a metabelian p-group, $p \neq 2$, and $c \in N_G\langle[x, c]\rangle$, then*

$$\langle[x, c^{p^n}]\rangle = \langle[x, c]^{p^n}\rangle.$$

Proof (by induction on n). $n = 1$. By Identity 7

$$\begin{aligned}[x, c^p] &= [x, c]^p \textstyle\prod_{k=2}^{p-1} [x, c, {}_{k-1}c]^{C(p,k)}[x, {}_pc] \\ &= [x, c]^p \textstyle\prod_{k=2}^{p-1} [[x, c, c]^p, {}_{k-2}c]^{\delta_k}[x, {}_pc],\end{aligned}$$

where $\delta_k = C(p, k)/p$. Now since $c \in N_G\langle[x, c]\rangle$ implies that $c \in N_G\langle[x, c, c]\rangle$, we see that $\langle[x, c, c, c]\rangle \leq \langle[x, c, c]^p\rangle$, and if we consider the above equation modulo $\langle[x, c, c]^p\rangle$, we have

$$[x, c^p] \equiv [x, c]^p \mod \langle[x, c, c]^p\rangle.$$

By Identity 6 we have $[x, c, c]^p = [[x, c]^p, c]$. Since $c \in N_G\langle[x, c]\rangle$

$$\langle[x, c, c]^p\rangle = \langle[[x, c]^p, c]\rangle < \langle[x, c]\rangle.$$

Thus, $\langle[x, c^p]\rangle = \langle[x, c]^p\rangle$.

Now assume we have shown $\langle[x, c^{p^m}]\rangle = \langle[x, c]^{p^m}\rangle$ for all $m < n$. Since $c \in N_G\langle[x, c]\rangle$, $c^{p^{n-1}} \in N_G\langle[x, c^{p^{n-1}}]\rangle$. So by induction we see that

$$\langle[x, (c^{p^{n-1}})^p]\rangle = \langle[x, c^{p^{n-1}}]^p\rangle,$$

i.e.,

$$\langle[x, c^{p^n}]\rangle = \langle[x, c^{p^{n-1}}]^p\rangle,$$

and applying induction again,

$$\langle [x, c^{p^n}] \rangle = \langle [x, c]^{p^n} \rangle.$$

LEMMA 3. *Let G be a metabelian p-group, $p \neq 2$, and let c normalize $\langle [x, c] \rangle$; then*

$$[x, c^i] = [x, c^j] \quad \text{implies} \quad i \equiv j \mod |\,[x, c]\,|\,.$$

Proof (by induction on the order of $[x, c]$). Assume $|\,[x, c]\,| = p$. Suppose $[x, c^i] = [x, c^j]$. Then using Identity 4 we have

$$[x, c^i] = [x, c]^i \prod_{k=2}^{i} [x, c, {}_{k-1}c]^{C(i,k)} = [x, c]^i$$

since $\langle [x, c, c] \rangle \leq \langle [x, c]^p \rangle = 1$. Doing the same thing for j we get

$$[x, c^i] = [x, c]^i = [x, c]^j = [x, c^j],$$

which implies that $i \equiv j \mod |\,[x, c]\,|$.

Assume the lemma is true for commutators of order $\leq p^n$, and let $|\,[x, c]\,| = p^{n+1}$. Let $H = \langle c, [x, c] \rangle$ and $b = [x, c]$. Then $\langle b^{p^n} \rangle \lhd H$. Let $\sigma : H \rightarrow H/\langle b^{p^n} \rangle$ be the natural homomorphism. By induction

$$[x, c^i] \equiv [x, c^j] \mod \langle b^{p^n} \rangle$$

implies

$$i \equiv j \mod p^n \qquad (\text{since } |\,[x, c]^\sigma\,| = p^n)$$

implies

$$i = j + \delta p^n.$$

We will show that p divides δ which will imply that $i \equiv j \mod p^{n+1}$.

$$\begin{aligned} [x, c^j] &= [x, c^i] = [x, c^{j+\delta p^n}] \\ &= [x, c^j][x, c^{\delta p^n}][x, c^j, c^{\delta p^n}] \\ &= [x, c^j][x, c^{\delta p^n}], \end{aligned}$$

since the order of $[x, c]$ is p^{n+1} and the p-part of the order of the automorphism group of a cyclic p-group is less than the order of the group, we see that the automorphism induced by c on $\langle [x, c] \rangle$ has order less than p^{n+1}. So applying Lemma 2 we see that

$$1 = [x, c^{\delta p^n}] \quad \text{implies} \quad p \text{ divides } \delta.$$

Now we will discuss the structure of finite metabelian p-groups, $p \neq 2$, which contain a self-centralizing element.

LEMMA 4. *Let G be a finite metabelian p-group with $x \in G$ and $M \lhd G$. Let $Y_0 = 1$ and*

$$Y_i = \{m \mid m \in M \text{ and } [m, {}_ix \mid = 1\} \quad \text{for } i = 1, 2, \cdots.$$

Then Y_i is a group for $i = 1, 2, \cdots$.

Proof. Let $m, n \in Y_i$. Then using Identities 1 and 6 we get

$$[mn^{-1}, {}_ix] = [m, {}_ix][m, x, n^{-1}, {}_{i-1}x][n^{-1}, {}_ix].$$

Now $m \in Y_i$, so $[m, {}_ix] = 1$. We apply Identity 4 to the second term and Identity 1 to the third term to get

$$[mn^{-1}, {}_ix] = [m, {}_ix, n^{-1}][[n, x]^{-1}, {}_{i-1}x][[n, x, n^{-1}]^{-1}, {}_{i-1}x].$$

Applying Identities 4 and 6 we have

$$[mn^{-1}, {}_ix] = [n, {}_ix]^{-1}[n, {}_ix, n^{-1}]^{-1}$$

and since $n \in Y_i$,

$$[mn^{-1}, {}_ix] = 1.$$

So mn^{-1} is in Y_i.

LEMMA 5. *Let G be a finite metabelian p-group, $p \neq 2$, and x be self-centralizing in G. Then if $\langle x \rangle \neq G$,*

$$1 \neq \langle x \rangle \cap G' \lhd G.$$

Proof. $1 \neq Z(G) \cap G' \leq \langle x \rangle \cap G'$. Let $w \in \langle x \rangle \cap G'$ and $g \in G$. Then

$$[w^g, x] = [w, x^{g^{-1}}]^g = [w, x[x, g^{-1}]]^g = [w, x]^g = 1.$$

Hence, $w^g \in G' \cap \langle x \rangle$ since $c_G\langle x \rangle = \langle x \rangle$. So $\langle x \rangle \cap G' \lhd G$. We now prove the main theorem.

THEOREM 1. *Let G be a finite metabelian p-group, $p \neq 2$, and x self-centralizing in G. Let M be a normal subgroup of G. If subgroups Y_i are defined as in Lemma 4, then*

$$Y_i \lhd Y_{i+1} \quad \textit{and} \quad Y_{i+1}/Y_i \textit{ is cyclic for } i = 1, 2, \cdots$$

Proof. Since Y_1 is cyclic, it will suffice to show that

(a) $Y_{i-1} \lhd Y_i$ implies Y_i/Y_{i-1} cyclic
(b) Y_i/Y_{i-1} cyclic implies $Y_i \lhd Y_{i+1}$.

We first show $Y_{i-1} \lhd Y_i$ implies Y_i/Y_{i-1} is cyclic. We use the fact that a p-group, $p \neq 2$, is cyclic if and only if it has exactly one subgroup of order p [2, Theorem 12.5.2]. Let $c, d \in Y_i \backslash Y_{i-1}$ and $c^p, d^p \in Y_{i-1}$. Now $1 \neq \langle x \rangle \cap G' \lhd G$, so we apply Lemma 1 as follows:

$$1 = [c^p, {}_{i-1}x] \quad \text{implies} \quad [c, {}_{i-1}x]^p = 1 \quad \text{and} \quad [c, {}_{i-1}x] \in \langle x \rangle \cap G'$$

$$1 = [d^p, {}_{i-1}x] \quad \text{implies} \quad [d, {}_{i-1}x]^p = 1 \quad \text{and} \quad [d, {}_{i-1}x] \in \langle x \rangle \cap G'.$$

Thus,

$$\langle [c, {}_{i-1}x] \rangle = \langle [d, {}_{i-1}x] \rangle.$$

Now we note that the commutators $[d^j, {}_{i-1}x]$ for $j = 1, 2, \cdots, p-1$ all lie in the group $\langle [d, {}_{i-1}x] \rangle$ which has order p, and by Lemma 3 we see that these commutators are all different so there exists an integer δ, $1 \leq \delta \leq p-1$, so that

$$[d^{-\delta}, {}_{i-1}x] = [c, {}_{i-1}x]^{-1}.$$

Now by Identities 1 and 6,

$$[d^{-\delta}c, {}_{i-1}x] = [d^{-\delta}, {}_{i-1}x][d^{-\delta}, x, c, {}_{i-2}x][c, {}_{i-1}x] = [d^{-\delta}, x, c, {}_{i-2}x]$$

and by Identity 4,

$$[d^{-\delta}c, {}_{i-1}x] = [d^{-\delta}, {}_{i-1}x, c] = 1$$

since $|[d^{-\delta}, {}_{i-1}x]| = p$ and $\langle[d^{-\delta}, {}_{i-1}x]\rangle \lhd G$. Thus, $cd^{-\delta} \in Y_{i-1}$, which means that $\langle c\rangle \equiv \langle d\rangle \mod Y_{i-1}$, i.e., Y_i/Y_{i-1} has only one subgroup of order p. Thus, since $p \neq 2$, Y_i/Y_{i-1} is cyclic.

We now show Y_i/Y_{i-1} cyclic implies $Y_i \lhd Y_{i+1}$. Let $y \in Y_i$ and $g \in Y_{i+1}$. Then $y^g \in Y_i$ if and only if $[y, g] \in Y_i$ if and only if $[y, g, {}_ix] = 1$. Now

$$\begin{aligned}[y, g, {}_ix] &= [y, g, x, {}_{i-1}x] \\ &= [x, y, g, {}_{i-1}x]^{-1}[g, x, y, {}_{i-1}x]^{-1} && \text{by Identity 5} \\ &= [[x, y, {}_{i-1}x], g]^{-1}[[g, x], y, {}_{i-1}x]^{-1} && \text{by Identity 4} \\ &= [[g, x, y], {}_{i-1}x]^{-1} && \text{since } y \in Y_i\end{aligned}$$

and since $[g, x] \in Y_i$ and Y_i/Y_{i-1} cyclic imply $[[x, g], y] \in Y_{i-1}$

$$[y, g, {}_ix] = 1.$$

COROLLARY 1. *Let G, x, M, Y_i be as in Theorem I. Then*

(1) $|Y_{i+1}/Y_i| \leq |Y_i/Y_{i-1}|$ *for* $i = 1, 2, \cdots$;
(2) *if* $|Y_{i+1}/Y_i| = |Y_i/Y_{i-1}|$ *and if* $Y_{i+1} = \langle b_{i+1}, Y_i\rangle$, *then*

$$\langle[b_{i+1}, x], Y_{i-1}\rangle = Y_i.$$

Proof. Let $\delta(i+1)$ be chosen so that $b_{i+1}^{p^{\delta(i+1)}} \in Y_i$, but $b_{i+1}^{p^{\delta(i+1)-1}} \notin Y_i$. Since $\langle[b_{i+1}, {}_ix]\rangle \leq \langle x\rangle \cap G'$, $\langle[b_{i+1}, {}_ix]\rangle \lhd G$ so we can apply Lemma 1 to get

$$1 = \langle[b_{i+1}^{p^{\delta(i+1)}}, {}_ix]\rangle = \langle[b_{i+1}, {}_ix]^{p^{\delta(i+1)}}\rangle.$$

By applying Identity 6 we get

$$1 = \langle[[b_{i+1}, x]^{p^{\delta(i+1)}}, {}_{i-1}x]\rangle,$$

i.e.,

$$[b_{i+1}, x]^{p^{\delta(i+1)}} \in Y_{i-1}.$$

Thus, we see that

$$p^{\delta(i+1)} = |Y_{i+1}/Y_i| \leq |Y_i/Y_{i-1}|$$

and if $|Y_{i+1}/Y_i| = |Y_i/Y_{i-1}|$, then $|[b_{i+1}, x] \mod Y_{i-1}| = |Y_i/Y_{i-1}|$. Hence,

$$|Y_{i+1}/Y_i| = |Y_i/Y_{i-1}| \quad \text{implies} \quad Y_i = \langle[b_{i+1}, x], Y_{i-1}\rangle.$$

COROLLARY 2. *Let G be a finite metabelian p-group of class n containing a self-centralizing element. Then $d(G) \leq n$.*

Proof. Let M be a normal supplement for $\langle x\rangle$. Since the class of G is n, for all $a \in M$, $[a, {}_nx] = 1$. Thus, $M \leq Y_n$.

The groups Y_i/Y_{i-1} are cyclic, so we can find elements b_i of M so that $Y_i = \langle b_i, Y_{i-1}\rangle$. Now

$$G = \langle x, M\rangle = \langle x, b_1, \cdots, b_n\rangle = \langle x, b_2, \cdots, b_n\rangle$$

since $b_1 \in \langle x\rangle$. Hence, $d(G) \leq n$.

Application. We wish to find an economical generating system for a metabelian p-group, G, $p \neq 2$, which contains a self-centralizing element. Applying Theorem 1 to G with $M = G$ we have $1 = Y_0 < Y_1 < \cdots < Y_m = G$. Using Corollary 1 we choose elements b_i of G so that

(a) $x = b_1$,
(b) $Y_i = \langle b_i, Y_{i-1}\rangle$ for $i = 2, 3, \cdots$,
(c) if $| Y_{i+1}/Y_i | = | Y_i/Y_{i-1} |$, then $b_i = [b_{i+1}, x]$ for $i = 2, 3, \cdots$.

Now $G = \langle x, b_2, b_3, \cdots \rangle$, but we may eliminate each b_i for which $| Y_{i+1}/Y_i | = | Y_i/Y_{i-1} |$ from this system of generators. This leaves only those b_i's for which $| Y_{i+1}/Y_i | < | Y_i/Y_{i-1} |$. Call these b_i's $b_{i_1} = x$, $b_{i_2}, \cdots, b_{i_v}$.

We will show that $v \leq w + 1$ where $p^w = | G' \cap \langle x \rangle |$. Now by Lemma 2 $| Y_2/Y_1 | = | [b_2, x] | \leq | G' \cap \langle x \rangle | = p^w$. Thus, for $i \geq 2$, $| Y_{i+1}/Y_i | < | Y_i/Y_{i-1} |$ can happen at most w times. Hence, $v \leq w + 1$, and so $d(G) \leq w + 1$.

This result is a specific case of the following theorem which was pointed out to me by the referee.

THEOREM 2. *Suppose G is a p-group, x an element of G, $C = c_G(x)$, and $| G' \cap C | = p^w$. Then $d(G) \leq w + d(C)$.*

Proof. Since G is a p-group,

$$d(G) = d(G/G') \leq d(G/CG') + d(CG'/G').$$

Using $C/C \cap G' \cong CG'/G'$ and $C' \leq C \cap G'$ we have

$$d(G) \leq d(G/CG') + d(C).$$

Now $| G/CG' | = | G | \, | G' \cap C | / | C | \, | G' |$ and since

$$| G : C | = \text{number of conjugates of } x = \text{number of commutators } [x, a],$$

we have

$$| G | / | C | = | G : C | \leq | G' |.$$

So

$$| G/CG' | \leq | C \cap G' | = p^w.$$

Hence,

$$d(G/CG') \leq w.$$

Thus,

$$d(G) \leq w + d(C).$$

The following example will show that the results of Theorem 2 and Corollary 2 are best possible in the sense that we can find groups for which the bounds are attained.

Example 1. We construct the group as follows. Let

$$M = \langle a_1 \rangle \times \langle a_2 \rangle \times \cdots \times \langle a_n \rangle,$$

where $|a_i| = p^{n-i+1}$ for $i = 1, 2, \cdots, n$. Let τ be the automorphism of M so that

$$\tau : a_1 \rightarrow a_1, \qquad \tau : a_i \rightarrow a_i a_{i-1}^p \quad \text{for } i = 2, 3, \cdots, n.$$

τ is an automorphism of M since τ preserves the defining relations of M (since M is abelian, this means τ preserves orders of elements) and τ is onto.

Let $G = \langle x, M\rangle$ where $M \lhd G$, and $a^\tau = a^x$ for all $a \in M$, and $x^{|\tau|} = a_1$.

We now show x is self-centralizing in G. It suffices to show $c_M\langle x\rangle = \langle x\rangle \cap M$. Let us define $A_i = \langle a_1, \cdots, a_i\rangle$ for $i = 1, \cdots, n$. We will show $c_M\langle x\rangle = \langle a_1\rangle$. Suppose $g \in M\backslash A_1$ and $1 = [g, x]$. Since $g \in M$, there is an integer j so that $g \in A_j\backslash A_{j-1}$. Thus, g can be written as $g = a_j^{\eta p^\lambda} m$ where $m \in A_{j-1}$ and $(\eta, p) = 1$. Also $p^\lambda < |a_j| = p^{n-j+1}$. Now $1 = [g, x]$ implies

$$\begin{aligned} 1 &\equiv [a_j^{\eta p^\lambda} m, x] \quad \text{mod } A_{j-2} \\ &\equiv [a_j^{\eta p^\lambda}, x] \quad \text{mod } A_{j-2}. \end{aligned}$$

Hence,

$$1 \equiv a_{j-1}^{\eta p^{\lambda+1}} \quad \text{mod } A_{j-2}.$$

But this means that $a_{j-1}^{p^{\lambda+1}} = 1$ so $p^{\lambda+1} \mid |a_{j-1}|$, i.e., $\lambda + 1 \geq n - j$, a contradiction since $p^\lambda < p^{n-j+1}$. Thus, $c_m\langle x\rangle = \langle a_1\rangle = M \cap \langle x\rangle$.

Now $G' = \langle a_1^p, a_2^p, \cdots, a_{n-1}^p\rangle$ so $\langle x\rangle \cap G' = \langle a_1^p\rangle$, and $|\langle x\rangle \cap G'| = p^{n-1}$. Since $d(G) = d(G/G')$ we see that $\{x, a_2, \cdots, a_n\}$ is a minimal generating system for G. Thus, $d(G) = n$.

We also see that since $[a_i, x] = a_{i-1}^p$, $[a_{n,n-1}x] \neq 1$, but $[a_{n,n}x] = 1$. Hence, $\text{cl}(G) \geq n$. Corollary 2 gives $\text{cl}(G) \leq n$ so $\text{cl}(G) = n$, and the bound of Corollary 2 is attained.

References

1. J. Dombek, *Anticenters of several classes of groups*, Amer. Math. Monthly, vol. 69 (1962), pp. 738–741.
2. M. Hall, Jr., *Theory of groups*, Macmillan, New York, 1959.
3. B. Huppert, *Endliche Gruppen*, Springer-Verlag, Heidelberg, 1967.
4. W. Kappe, *On the anticenter of nilpotent groups*, Illinois J. Math., vol. 12 (1968), pp. 603–609.
5. ———, *Self-centralizing elements in regular p-groups*, to appear
6. N. Levine, *The anticenter of a group*, Amer. Math. Monthly, vol. 70 (1960), pp. 61–63.
7. W. Magnus, A. Karrass, and D. Solitar, *Combinatorial group theory*, Wiley, New York, 1966.
8. H. Neumann, *Varieties of groups*, Springer-Verlag, New York, 1967.
9. K. Seksenbaev, *On the anticenter of bundles of groups*, Izv. Akad. Nauk Kazah. SSR Ser. Fiz.-Mat. Nauk, No. 1 (1966), pp. 20–24.

Battelle Memorial Institute
Columbus, Ohio