A NONEXISTENCE THEOREM FOR RELATIVE DIFFERENCE SETS

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In [1], Elliott and Butson introduced a generalization $R(m, n, k, d)$ of the concept of group difference set. In this note we give an extension of the Bruck-Ryser nonexistence theorem which is suggested by [2]. The proof is a straightforward generalization of that of the theorem announced in [3], and depends only on the fact that a certain rational (incidence) matrix A for $R(m, n, k, d)$ satisfies the relation

$$AA' = A'A = kI + dJ_{mn} - d(I_m \otimes J_n),$$

J being the matrix of 1's (c.f. [1]).

THEOREM. *The existence of an* $R(m, n, k, d)$ *implies the following*:

(i) *If* $m \equiv 2 \pmod 4$ *and if* n *is even* (*so* $k - nd$ *is a perfect square*), *then the Hilbert symbol* $(k, -1)p = +1$ *for all primes* p.

(ii) *If* m *is odd* (*so* k *is a square*), *then the Hilbert symbol*

$$(k - nd, nm(-1)^{\frac{1}{2}(m-1)m})p = +1$$

for all primes p.

Proof. From the equation involving A above, we see that the minimal polynomial for $A'A$ is $(x - k^2)(x - k)(x - (k - nd))$. For part (i), let W be the $(n - 1)m$-dimensional, A-invariant space of characteristic vectors of $A'A$ associated with the value k. Then with respect to the usual Euclidean inner product A induces a similarity transformation of norm k on W (c.f. [2]). Suppose $m \equiv 2 \pmod 4$, and let α_{kj} be the column vector of length mn with $+1$ in position $km + 1$ and -1 in position $km + j$, with zeros elsewhere. Here $0 \leq k < m$, $2 \leq j \leq n$. Then $\{\alpha_{kj}\}$ is a basis for W with a discriminant which is an m^{th} power, i.e. a square. By the first theorem in [2], the Hilbert symbol

$$(k, (-1)^r)p = +1,$$

where $r = \frac{1}{2}(n - 1)m[(n - 1)m + 1]$.

Similarly, for (ii) let m be odd and let W be the $(m - 1)$-dimensional, A-invariant subspace of characteristic vectors of $A'A$ associated with the value $k - nd$. Then A induces a similarity transformation of norm $k - nd$ on W. Let α_j be the vector with $+1$ in positions $1, 2, \cdots, n$, and -1 positions $jn + 1, \cdots, jn + n$, with zeros elsewhere, for $1 \leq j < n$. Then $\{\alpha_j\}$ is a basis for W with discriminant mn. So again the theorem follows by [2].

REFERENCES

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Received July 29, 1968.

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