FINITE-DIMENSIONAL SCHAUDER DECOMPOSITIONS IN π_λ AND DUAL π_λ SPACES

BY

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I. Introduction

DEFINITION. Let X be a Banach space and let $\lambda \geq 1$. X is a π_λ space (resp. dual π_λ space) iff there is a net $\{S_d : d \in D; \leq\}$ of linear projections on X such that

(1) each S_d has finite-dimensional range;

(2) $\lim_d S_d(x) = x$, for each $x \in X$;

(3) $\| S_d \| \leq \lambda$, for each $d \in D$;

(4) $S_e S_d = S_d$, for $e \geq d$ (resp. $S_d S_e = S_d$, for $e \geq d$).

$\{S_d : d \in D\}$ is called a π_λ (resp. dual π_λ) decomposition for X.

The concepts of π_λ and dual π_λ spaces are dual in the sense that if $\{S_d\}$ is a π_λ (resp. dual π_λ) decomposition for X, then $\{S_d^*\}$ satisfies the definition of dual π_λ (resp. π_λ) decomposition for X^* except that the convergence in (2) is weak * convergence. In case $\{S_d\}$ is a dual π_λ decomposition for X, it is easy to prove that $\{S_d^*\}$ is a π_λ decomposition for the Banach space $\mathrm{cl}_{X^*}(\bigcup_{d \in D} \text{Range } (S_d^*))$. We note that if $\{S_d\}$ is either a π_λ or dual π_λ decomposition for X, then $\bigcup_{d \in D}$ Range (S_d) is dense in X.

Interesting results concerning π_λ and π_1^∞ spaces (Section III) have been obtained by Lindenstrauss, [2], and Michael and Pełczynski, [3] and [4]. The proof of Lemma 3.1 in [2] can be modified to show that a dual π_λ space is a π_β space for any $\beta > \lambda$, so that many of these results apply to dual π_λ spaces as well.

The main results of Section II relate the concepts of dual π_λ and π_1 decompositions to basis theory:

THEOREM 1. *A separable Banach space, X, has a finite-dimensional Schauder decomposition iff X is a dual π_λ space, for some $\lambda \geq 1$.*

THEOREM 2. *A separable Banach space, X, has a finite-dimensional monotone Schauder decomposition iff X is a π_1 space.*

Recall that $\{P_n, M_n\}_{n=1}^\infty$ is a Schauder decomposition for X iff each P_n is a continuous linear projection of X onto M_n; $P_n P_m = 0$, for $n \neq m$; and for each $x \in X$, $x = \sum_{n=1}^\infty P_n(x)$. If $\{P_n, M_n\}_{n=1}^\infty$ is a Schauder decomposition for X, define the partial sum operators, S_n, by $S_n = \sum_{i=1}^n P_i$. $\{S_n\}_{n=1}^\infty$ is

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pointwise convergent to the identity operator, hence $\{S_n\}_{n=1}^{\infty}$ is uniformly bounded when X is a Banach space. We denote by $G(\{M_n\})$ the number $\sup_{n=1,2,3,\cdots} \| S_n \|$, and call $G(\{M_n\})$ the Grynblum constant of the decomposition. If $G(\{M_n\}) = 1$, the Schauder decomposition is said to be monotone. If each P_n (and hence each S_n) has finite-dimensional range, $\{P_n, M_n\}_{n=1}^{\infty}$ is called a finite-dimensional Schauder decomposition.

It is easy to prove (and essentially known—see [5]) that a sequence $\{S_n\}_{n=1}^{\infty}$ of operators on X is a π_λ-dual π_λ decomposition for X iff $\{S_n\}_{n=1}^{\infty}$ is the sequence of partial sum operators associated with a finite-dimensional Schauder decomposition for X with Grynblum constant no larger than λ. In particular, the "only if" parts of Theorems 1 and 2 are immediate.

In Section III we prove that every $C(K)$ space is a dual π_1^{∞} space. The Michael and Pełczynski result [4] that $C(K)$ is a π_1^{∞} space when K is compact metric is an immediate consequence of Theorem 4.

If P is a linear operator, we denote by $R(P)$ the range of P, and by $\ker(P)$, the null space of P. For $a > 0$, let $B(a) = \{x : \| x \| \leq a\}$. I denotes the identity operator. $C(K)$ is the Banach space of scalar (i.e., real or complex) valued continuous functions on the compact Hausdorff space K, endowed with the sup norm. If A is a subset of a linear space, sp A denotes the linear span of A.

II. The basis theorems

LEMMA 1. *Let X be a normed space and Y a separable subspace of X. Suppose $\{S_d : d \in D; \leq\}$ is an equicontinuous net of linear operators of finite range on X which converges pointwise to I. Let M and a be positive numbers. Then there is $\{d_1 \leq d_2 \leq d_3 \leq \cdots\} \subset D$ such that $\lim_{n\to\infty} S_{d_n}(x) = x$, for each $x \in Y$, and $S_{d_{n+1}}$ moves each point of $B(M) \cap \text{sp} \bigcup_{i=1}^{n} R(S_{d_i})$ a distance less than $a/2^n$.*

Proof. Let $\{x_i\}_{i=1}^{\infty}$ be dense in Y. Choose $d_1 \in D$ such that $\| x_1 - S_{d_1}(x_1) \| < a$. Suppose that $d_1 \leq d_2 \leq \cdots \leq d_n$ have been chosen. Choose $d_{n+1} \in D$ such that $d_n \leq d_{n+1}$ and for each

$$x \in A = \{x_i\}_{i=1}^{n+1} \cup [B(M) \cap \text{sp} \bigcup_{i=1}^{n} R(S_{d_i})],$$

$$\| x - S_{d_{n+1}}(x) \| < a/2^n.$$

This choice is possible because $\{S_d : d \in D\}$ converges pointwise to I and is equicontinuous, so that the convergence is uniform on compact sets. A is closed, bounded, and finite dimensional, hence is compact. Now for each i, $\lim_{n\to\infty} S_{d_n}(x_i) = x_i$. Since $\{x_i\}_{i=1}^{\infty}$ is dense in Y and $\{S_{d_n}\}_{n=1}^{\infty}$ is equicontinuous, $\{S_{d_n}\}_{n=1}^{\infty}$ converges pointwise on Y to I.

Proof of Theorem 1. Suppose that X is a dual π_λ space. Let $M > \lambda$. We show that X has a finite-dimensional Schauder decomposition with Grynblum constant no larger than M.

By Lemma 1, we can assume that X has a dual π_λ decomposition $\{S_n\}_{n=1}^{\infty}$ such that for each n and each $x \in B(M) \cap \text{sp} \bigcup_{i=1}^{n} R(S_i)$,

$$\| x - S_{n+1}(x) \| < (M - \lambda)/2^n. \tag{1}$$

For $j \geq n$, let $T_n^j = S_j S_{j-1} \cdots S_n$. Now if $j > n$,

$$\| T_n^j \| \leq [\textstyle\sum_{i=n}^{j-1} (M - \lambda)/2^i] + \lambda < M. \tag{2}$$

If $j = n + 1$, (2) follows from the fact that $\| S_n \| \leq \lambda$, (1), and the inequality

$$\| T_n^{n+1}(x) \| \leq \| S_{n+1} S_n(x) - S_n(x) \| + \| S_n(x) \|.$$

In general, if (2) holds for j, then for $x \in B(1)$

$$\begin{aligned} \| T_n^{j+1}(x) \| &\leq \| S_{j+1} T_n^j(x) - T_n^j(x) \| + \| T_n^j(x) \| \\ &\leq (M - \lambda)/2^j + [\textstyle\sum_{i=n}^{j-1} (M - \lambda)/2^i] + \lambda, \end{aligned}$$

so that (2) also holds if $j + 1$ is substituted for j. Note that this argument also shows that for $x \in B(1)$ and $j > i \geq n$,

$$\| T_n^j(x) - T_n^i(x) \| \leq \textstyle\sum_{k=i}^{j-1} (M - \lambda)/2^k. \tag{3}$$

Thus the Cauchy criterion guarantees that $\lim_{j\to\infty} T_n^j(x)$ exists for each $x \in X$ and $n = 1, 2, 3, \cdots$. Let $T_n = \lim_{j\to\infty} T_n^j$. Clearly each T_n is linear and $\| T_n \| \leq M$.

Now for $n \geq m$ and $j \geq m$,

$$\begin{aligned} T_m^j T_n &= S_j \cdots S_m \lim_{i\to\infty} S_i \cdots S_n \\ &= \lim_{i\to\infty} S_j \cdots S_m S_i \cdots S_n \\ &= \lim_{i\to\infty} S_j \cdots S_m = T_m^j. \end{aligned}$$

Thus for $n \geq m$, $T_m T_n = T_m$. Similarly, for $j \geq m \geq n$,

$$\begin{aligned} T_m^j T_n &= S_j \cdots\cdot S_m \lim_{i\to\infty} S_i \cdots\cdot S_n \\ &= \lim_{i\to\infty} S_j \cdots\cdot S_m S_i \cdots\cdot S_n \\ &= S_j \cdots\cdot S_m S_{m-1} \cdots\cdot S_n = T_n^j. \end{aligned}$$

Thus for $m \geq n$, $T_m T_n = T_n$. That is, $T_m T_n = T_{\min(n,m)}$.

We next show that $\{T_n\}_{n=1}^\infty$ pointwise converges to I. Since $\bigcup_{n=1}^\infty R(S_n)$ is dense in X and $\{T_n\}_{n=1}^\infty$ is equicontinuous, it is sufficient to show that for each $x \in \bigcup_{n=1}^\infty R(S_n)$, $\lim_{n\to\infty} T_n(x) = x$. Let $x \in \bigcup_{n=1}^\infty R(S_n)$, say $x \in R(S_i)$, and without loss of generality assume that $x \in B(1)$. If $j > n > i$, we have from (3) and (1) that

$$\begin{aligned} \| T_n^j(x) - x \| &\leq \| T_n^j(x) - S_n(x) \| + \| S_n(x) - x \| \\ &\leq [\textstyle\sum_{k=n}^{j-1} (M - \lambda)/2^k] + (M - \lambda)/2^{n-1}. \end{aligned}$$

Passing to the limit on j, we get that for $n > i$,

$$\| T_n(x) - x \| \leq \textstyle\sum_{k=n-1}^{\infty} (M - \lambda)/2^k.$$

Passing to the limit on n, we have that $\lim_{n\to\infty} \| T_n(x) - x \| = 0$.

Now for each n,

$$(4) \qquad S_n T_n = S_n \quad \text{and} \quad T_n S_n = T_n,$$

so that $\ker(T_n) = \ker(S_n)$, and thus $R(T_n)$ and $R(S_n)$ have the same dimension. Therefore $\{T_n\}_{n=1}^{\infty}$ is a π_M-dual π_M decomposition for X, and the remarks in the introduction complete the proof.

Remark 1. Using the notation of Theorem 1, we have from (4) that T_n is an isomorphism from $R(S_n)$ onto $R(T_n)$ with inverse S_n. Thus for each n, $d(R(T_n), R(S_n)) \leq M\lambda$, where

$$d(A, B) = \inf\{\|T\| \cdot \|T^{-1}\| : T \text{ is an isomorphism from } A \text{ onto } B\}.$$

If each S_n is of norm 1, then each T_n is of norm 1, so that $R(T_n)$ and $R(S_n)$ are isometric. Of course, in this case the generated Schauder decomposition is monotone.

COROLLARY 1. *Let X be a dual π_1 space and let Y be a separable subspace of X. Then there is a separable subspace Z of X such that $Y \subset Z$ and Z has a π_1-dual π_1 decomposition.*

Proof. Let $\{S_d : d \in D\}$ be a dual π_1 decomposition for X. Using Lemma 1, we can find $\{d_1 \leq d_2 \leq d_3 \leq \cdots\} \subset D$ such that $\lim_{n\to\infty} S_{d_n}(x) = x$, for each $x \in Y$, and $S_{d_{n+1}}$ moves each point of sp $\bigcup_{i=1}^{n} R(S_{d_i}) \cap B(1)$ a distance less than $\frac{1}{2}^n$. Let $Z = \{x \in X : \lim_{n\to\infty} S_{d_n}(x) = x\}$. Clearly Z is a separable (closed) subspace of X and $Y \subset Z$. Now $\{S_{d_n}\}_{n=1}^{\infty}$ is a dual π_1 decomposition for Z because each $R(S_{d_n})$ is a subset of Z. Thus by Theorem 1 and Remark 1, Z has a π_1-dual π_1 decomposition.

The referee has noted that the proof of Proposition 6.1 in [4] can be generalized to give an easy proof of Theorem 2. Alternatively, Theorem 2 follows immediately from Lemma 1 and the following:

THEOREM 3. *Let X be a Banach space and let $\{S_n\}_{n=1}^{\infty}$ be a π_λ decomposition for X. Suppose that there is a sequence $\{P_n\}_{n=1}^{\infty}$ such that for each n, P_n is a linear projection from $R(S_{n+1})$ onto $R(S_n)$, and that $\prod_{n=1}^{\infty} \|P_n\| = k < \infty$. Then X has a finite-dimensional Schauder decomposition with Grynblum constant no larger than λk.*

Sketch of proof. For $n > j$, let $T_j^n = P_j P_{j+1} \cdots P_{n-1} S_n$. For each j, let $T_j = \lim_{n\to\infty} T_j^n$. (This pointwise limit exists because $\|T_j^n\| \leq \lambda k$, and for each m, $\{T_j^n\}_{n=j+1}^{\infty}$ is eventually constant on $R(S_m)$.) It follows by an argument similar to that used in Theorem 1 that $\{T_n\}_{n=1}^{\infty}$ is a $\pi_{\lambda k}$-dual $\pi_{\lambda k}$ decomposition for X. The remarks in the introduction then complete the proof.

We conclude this section with an unsolved problem:

Problem 1. Does every separable π_λ space have a finite-dimensional Schauder decomposition?

III. Dual π_1^∞ decompositions in $C(K)$ spaces

A π_1 (resp. dual π_1) decomposition $\{S_d : d \in D\}$ is a π_1^∞ (resp. dual π_1^∞) decomposition iff each $R(S_d)$ is isometric to an $l_{n(d)}$ space. It is known [2] that every $C(K)$ space "almost" has a π_1^∞ decomposition, $\{S_d : d \in D\}$, in the sense that each $R(S_d)$ is almost isometric to $l_{n(d)}^\infty$, and that if K is compact metric, $C(K)$ is a π_1^∞ space [3]. It is not known whether every $C(K)$ space is a π_1^∞ space. However, Theorem 4 shows that every $C(K)$ space is a dual π_1^∞ space.

Recall that $\{f_i\}_{i=1}^n \subset C(K)$ is a peaked partition of unity iff each f_i is non-negatively real-valued, $\sum_{i=1}^n f_i$ is the constant 1 function, and $\| f_i \| = 1$. $\mathrm{Sp}(\{f_i\}_{i=1}^n)$ is then called a peaked partition subspace, and is isometric to l_n^∞ (cf., e.g., [3]).

THEOREM 4. *Let K be compact Hausdorff. Then $C(K)$ has a dual π_1^∞ decomposition $\{S_d : d \in D\}$ such that each $R(S_d)$ is a peaked partition subspace.*

Proof. Let D be the collection of all ordered pairs $(\{U_i\}_{i=1}^n, \{x_i\}_{i=1}^n)$ such that $\{U_i\}_{i=1}^n$ is a minimal open cover of K and $x_i \in U_i - \bigcup_{j \neq i} U_j$. Partially order D by

$$(\{U_i\}_{i=1}^n, \{x_i\}_{i=1}^n) \leq (\{V_j\}_{j=1}^m, \{y_j\}_{j=1}^m)$$

iff

$$\{V_j\}_{j=1}^m \text{ refines } \{U_i\}_{i=1}^n \text{ and } \{x_i\}_{i=1}^n \subset \{y_j\}_{j=1}^m.$$

It is straightforward to verify that D is directed by $\leq$. For each $(\{U_i\}_{i=1}^n, \{x_i\}_{i=1}^n) \in D$, pick a peaked partition of unity $\{f_i\}_{i=1}^n$ such that f_i vanishes outside U_i (hence $f_i(x_j) = \delta_{ij}$). For each $d = (\{U_i\}_{i=1}^n, \{x_i\}_{i=1}^n)$ in D, define the projection S_d by $S_d(f) = \sum_{i=1}^n f(x_i) f_i$, where $\{f_i\}_{i=1}^n$ is the peaked partition of unity associated with d. If $d = (\{U_i\}_{i=1}^n, \{x_i\}_{i=1}^n)$ is in D, then clearly

$$\ker (S_d) = \{f \in C(K) : f(x_1) = f(x_2) = \cdots = f(x_n) = 0\}.$$

Thus if $d \leq e$, $\ker (S_e) \subset \ker (S_d)$, and hence $S_d S_e = S_d$. Obviously $\| S_d \| = 1$, for all $d \in D$. To complete the proof we must show that the net $\{S_d : d \in D; \leq\}$ pointwise converges to I. Let $f \in C(K)$ and let $\varepsilon > 0$. Choose a minimal open cover $\{V_j\}_{j=1}^n$ of K such that if $\{x, y\} \subset V_j$, then $| f(x) - f(y) | < \varepsilon$. Suppose $d = (\{U_i\}_{i=1}^m, \{x_i\}_{i=1}^m)$ is in D such that $\{U_i\}_{i=1}^m$ refines $\{V_j\}_{j=1}^n$. Then for all $x \in K$,

$$\begin{aligned} | f(x) - S_d(f)(x) | &= | f(x) - \textstyle\sum_{i=1}^m f(x_i) f_i(x) | \\ &= | \textstyle\sum_{i=1}^m f_i(x)(f(x) - f(x_i)) | \\ &\leq \textstyle\sum_{i=1}^m f_i(x) \, | f(x) - f(x_i) | = k, \end{aligned}$$

where $\{f_i\}_{i=1}^m$ is the peaked partition of unity associated with d. Now if $x \in U_i$, $| f(x) - f(x_i) | < \varepsilon$, since $\{U_i\}_{i=1}^m$ refines $\{V_j\}_{j=1}^n$. If $x \notin U_i$, then $f_i(x) = 0$. Hence $k < \sum_{i=1}^m f_i(x) \varepsilon = \varepsilon$. This completes the proof.

Remark 2. The proof of Corollary 1 shows that a separable subspace of a dual π_1^∞ space, X, is contained in a separable π_1^∞-dual π_1^∞ subspace of X. Thus by Theorem 4, every separable subspace of $C(K)$ is contained in a separable π_1^∞ subspace of $C(K)$. In particular, when K is compact metric, we have the result of Michael and Pełczynski [4], that $C(K)$ is a π_1^∞ space.

Recall that a Hausdorff space K is a Boolean space iff the compact-open subsets of K form a base for the topology. In [1], Dyer notes that Theorem 4 can be improved for Boolean spaces:

THEOREM 5. *If K is a compact Boolean space, then $C(K)$ has a π_1^∞-dual π_1^∞ decomposition $\{S_d : d \in D\}$ such that for each $d \in D$, $R(S_d)$ is spanned the characteristic functions of the elements of a pairwise disjoint compact-open cover of K.*

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