EXTENSIONS OF COHOMOLOGY THEORIES

BY
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Introduction

The orientability of a sphere bundle relative to a cohomology theory h is essential to some of the basic results of algebraic topology which center around the Thom isomorphism theorem [3], [8], [12]. In the case of integral cohomology corresponding results can be obtained in the non-orientable case by introducing local coefficients. The purpose of this paper is to give an analogous construction for an arbitrary cohomology theory.

Let $\mathcal{P}^*$ be the category of finite CW-pairs and let $\mathcal{P}(B(\mathcal{O}))$ be the category of triples (X, A, f) with $(X, A) \in \mathcal{P}^*$ and $f : X \to B(\mathcal{O})$. We will construct, for a spectrum $\mathbf{F}$, a cohomology theory $H(\ \ ; \mathbf{F})$ on $\mathcal{P}^*(B(\mathcal{O}))$ which is an extension of the cohomology theory on $\mathcal{P}^*$ determined by $\mathbf{F}$. The essential property of this extension is that if $\mathcal{Y} = (Y, X, r)$ is an $(n - 1)$-sphere bundle with classifying map $f : X \to B(\mathcal{O})$ then $(\mathcal{Y}, f)$ has a Thom class. This allows us to establish a Thom isomorphism theorem for arbitrary sphere bundles.

MacAlpine [9] has recently introduced twisted cohomology groups associated with a spectrum $\mathbf{F}$ and bundle $\mathcal{Y}$. By virtue of the Thom isomorphism theorem we can exhibit the relation between his groups and ours (see (12.9)). There is also some overlap between our work and recent work of Israel Berstein on the Thom isomorphism.

A Thom class u for $(\mathcal{Y}, f)$ relative to $\mathbf{F}$ determines an Euler class $\mathfrak{X}$ in the usual way. Many of the properties of the integral Euler class carry over here and, in particular, a necessary condition that $\mathcal{Y}$ admit a cross-section is that $\mathfrak{X} = 0$. On the other hand, if $\mathbf{F}$ is the sphere spectrum $\mathbf{S}$ and X is $(2n - 2)$-connected, this is also a sufficient condition (Theorem (13.23)). An application of this fact to immersions of manifolds in Euclidean space is given in Section 14. Further applications are given in [13].

1. Preliminaries

The unit interval will be denoted by I. The space of maps $X \to B$ with the compact open topology will be denoted by $\mathfrak{M}(X, B)$ and

$$\mathfrak{M}(g) : \mathfrak{M}(X, B) \to \mathfrak{M}(X, B')$$

(respectively, $\mathfrak{M}(g) : \mathfrak{M}(X', B) \to \mathfrak{M}(X, B)$) will denote the map induced by $g : B \to B'$ (respectively, $g : X \to X'$).

Recall that the *reduced join* of pointed spaces F and F', with base-points

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x_0 and x_0' respectively, is the pointed space

$$F \wedge F' = F \times F'/F \times \{x_0'\} \cup \{x_0\} \times F'$$

with base-point $[x_0, x_0']$. We have a canonical mapping

$$\varkappa^{i,j} : \Omega^i(F) \wedge \Omega^j(F') \to \Omega^{i+j}(F \wedge F') \tag{1.1}$$

by $\varkappa^{i,j}[\sigma, \tau](t_1 \cdots t_{i+j}) = [\sigma(t_1 \cdots t_i), \tau(t_{i+1} \cdots t_{i+j})]$.

The boundary operator in the homotopy sequence of the path space fibration

$$\Omega(F) \to P(F) \xrightarrow{q} F,$$

where $q(\sigma) = \sigma(0)$, produces an identification

$$\partial_{\#} : \pi_n(F) \longrightarrow \pi_{n-1}(\Omega(F)), \qquad n \geq 1. \tag{1.2}$$

Explicitly, $\partial_{\#}[\sigma] = [\tilde{\sigma}]$ where

$$\tilde{\sigma}(t_1 \cdots t_{n-1})(t_m) = \sigma(t_1 \cdots t_{n-1}, t_m).$$

We have a pairing

$$\wedge : \pi_s(F) \otimes \pi_t(F') \to \pi_{s+t}(F \wedge F') \tag{1.3}$$

defined by letting $[\sigma] \wedge [\sigma']$ be the class of

$$I^s \times I^t \xrightarrow{\sigma \times \sigma'} F \times F' \xrightarrow{\omega} F \wedge F'$$

where ω is the projection map. Now consider the diagram

$$\begin{tikzcd}
\pi_{s-1}(\Omega(F)) \otimes \pi_t(F') \arrow[r, "\wedge"] & \pi_{s+t-1}(\Omega(F) \wedge F') \arrow[dr, "\varkappa_{\#}^{1,0}"] & \\
\pi_s(F) \otimes \pi_t(F') \arrow[u, "(-1)^t\partial_{\#} \times 1"] \arrow[d, "1 \otimes \partial_{\#}"'] \arrow[r, "\wedge"] & \pi_{s+t}(F \wedge F') \arrow[r, "\partial_{\#}"] & \pi_{s+t-1}(\Omega(F \wedge F')) \\
\pi_s(F) \otimes \pi_{t-1}(\Omega(F')) \arrow[r, "\wedge"] & \pi_{s+t-1}(F \wedge \Omega(F')) \arrow[ur, "\varkappa_{\#}^{0,1}"'] &
\end{tikzcd} \tag{1.4}$$

(1.5) LEMMA. *The above diagram is commutative.*

The proof is left to the reader.

We will take S^n to be the one-point compactification of R^n. The point at infinity will be denoted by z_n and is to be the base-point. We have a natural correspondence

$$\mathfrak{M}(X \wedge S^n, Y) \to \mathfrak{M}(X, \Omega^n(Y)) \tag{1.6}$$

by $f \to \tilde{f}$ where

$$\begin{aligned}\tilde{f}&(x, t_1 \cdots t_m)\\ &= f(x, \tan(\pi/2(2t_1 - 1)), \cdots, \tan(\pi/2(2t_m - 1))) \quad \text{if } 0 < t_i < 1,\\ &= z_0 \quad \text{otherwise.}\end{aligned}$$

In general, this is a continuous, one-one, and onto mapping and if X is compact, it is a homeomorphism. These facts are easily deduced from [4].

We have an identification

$$\zeta_{n,m} : S^n \wedge S^m \to S^{n+m} \tag{1.7}$$

by

$$\begin{aligned}\zeta_{n,m}[x, y] &= (x, y), \quad x \neq z_n, y \neq z_m,\\ &= y_{n+m}, \quad x = z_n \text{ or } y = z_m,\end{aligned}$$

and the following diagrams are commutative:

$$\begin{array}{ccc} S^n \wedge S^m \wedge S^t & \xrightarrow{\zeta_{n,m} \wedge 1} & S^{n+m} \wedge S^t \\ \downarrow{\scriptstyle 1 \wedge \zeta_{m,t}} & & \downarrow{\scriptstyle \zeta_{n+m} \wedge 1} \\ S^n \wedge S^{m+t} & \xrightarrow{\zeta_{n,m+t}} & S^{n+m+t} \end{array} \tag{1.8}$$

$$\begin{array}{ccc} S^n \wedge S^m & \xrightarrow{\zeta_{n,m}} & S^{n+m} \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle C_{n,m}} \\ S^m \wedge S^n & \xrightarrow{\zeta_{m,n}} & S^{n+m} \end{array} \tag{1.9}$$

where $\alpha[x, y] = [y, x]$ and $C_{n,m}$ is the extension of

$$C^*_{n,m} : R^{n+m} \to R^{n+m}$$

defined by

$$C^*_{m,m}(x_1 \cdots x_{n+m}) = (x_{n+1} \cdots x_{n+m}, x_1 \cdots x_n).$$

2. Cohomology theories

For a space B, let $\mathcal{P}(B)$ (respectively, $\mathcal{P}^*(B)$) be the category of objects (X, A, f) such that (X, A) is a CW-pair (respectively, (X, A) is a finite CW-pair) and $f \in \mathfrak{M}(X, B)$. A *mapping*

$$g : (X, A, f) \to (X', A', f')$$

in either category is to be an ordinary map $g : (X, A) \to (X', A')$ such that $f = f'g$. Let $\mathcal{A}$ denote the category of abelian groups.

A *cohomology theory* h on $\mathcal{P}(B)$ consists of a sequence of contravariant functors $h^n : \mathcal{P}(B) \to \mathcal{A}$ and natural transformations

$$d^n : h^n \circ T \to h^{n+1},$$

where $T : \mathcal{P}(B) \to (B)$ is defined by $t(X, A, f) = (A, \varphi, f_{|A})$ and $T(g) = g_{|A}$. They are to have the following properties.

(2.1) *For* $(X \times I, A \times I, F) \in \mathcal{P}(B)$

$$h^n(i_0) : h^n(X \times I, A \times I, F) \to h^n(X, A, f_0)$$

is an isomorphism, where $i_0 : (X, A) \to (X \times I, A \times I)$ *is defined by* $i_0(x) = (x, 0)$ *and* $f_0 = Fi_0$.

(2.2) *For* $(X, A, f) \in \mathcal{P}(B)$, *the sequence*

$$\cdots \xrightarrow{d^{n-1}} h^n(X, A, f) \xrightarrow{h^n(j)} h^n(X, f) \xrightarrow{h^n(i)} h^n(A, f_{|A}) \xrightarrow{d^n} \cdots$$

is exact, where $i : A \to X$ *and* $f : X \to (X, A)$ *are inclusions.*

(2.3) *If* $X = A_1 \cup A_2$ *where* A_1 *and* A_2 *are subcomplexes of* X, *then for* $f \in \mathfrak{M}(X, B)$,

$$h^n(i) : h^n(X, A_2, f) \to h^n(A_1, A_1 \cap A_2, f_{|A_1})$$

is an isomorphism, where $i : (A_1, A_1 \cap A_2) \to (X, A_2)$ *is the inclusion.*

A cohomology theory on $\mathcal{P}^*(B)$ is defined analogously.

Example. When B consists of a single point we may identify $\mathcal{P}(B)$ with the category of CW-pairs. Having done this, axiom (2.1) is easily seen to be equivalent to the homotopy axiom so that a cohomology theory on $\mathcal{P}(B)$ is just a generalized cohomology theory as in [12].

Remark. Hereafter we will revert to the usual convention and write g^* instead of $h^n(g)$ for a mapping $g \in \mathcal{P}(B)$.

Notice that (2.1) implies that i_1^* is also an isomorphism, for we have a commutative diagram

(2.4)
$$\begin{tikzcd}
h^n(X \times I, A \times I, F) \arrow[d, "\alpha^*"'] \arrow[dr, "i_1^*"] & \\
 & h^n(X, A, f_1) \\
h^n(X \times I, A \times I, F^{-1}) \arrow[ur, "i_0^*"'] &
\end{tikzcd}$$

where $\alpha : X \times I \to X \times I$ is the involution $(x, t) \to (x, 1 - t)$.
Now for $F : X \times I \to B$, a homotopy from f_0 to f_1, define

(2.5) $$F_{\#} : h^n(X, A, f_1) \to h^n(X, A, f_0)$$

by $F_{\#} = i_0^* i_1^{*-1}$. The isomorphism $F_{\#}$ has the following easily established properties.

(2.6) *The assignments* $f \to h^n(X, A, f)$, $f \in \mathfrak{M}(X, B)$, *and* $F \to F_{\#}$, $F \in \mathfrak{M}(X, B)^I$, *define a local system of abelian groups on* $\mathfrak{M}(X, B)$.

(2.7) *Commutativity holds in the diagram*

$$\begin{tikzcd} h^n(A, f_{1|A}) \arrow[r, "d^n"] \arrow[d, "(F_{|A\times I})_{\#}"] & h^{n+1}(X, A, f_1) \arrow[d, "F_{\#}"] \\ h^n(A, f_{0|A}) \arrow[r, "d^n"] & h^{n+1}(X, A, f_0). \end{tikzcd}$$

(2.8) *For* $g : (X', A') \to (X, A)$, *commutativity holds in the diagram*

$$\begin{tikzcd} h^n(X, A, f_1) \arrow[r, "g^*"] \arrow[d, "F_{\#}"] & h^n(X', A', f_1 g) \arrow[d, "(F(g \times 1))_{\#}"] \\ h^n(X, A, f_0) \arrow[r, "g^*"] & h^n(X', A', f_0 g). \end{tikzcd}$$

Finally, suppose that a homotopy $G : (X', A') \times I \to (X, A)$ from g_0 to g_1 and a map $f : X \to B$ are given. We have the following "homotopy property".

(2.9) *Commutativity holds in the diagram*

$$\begin{tikzcd} & h^n(X', A', fg_1) \arrow[dd, "(fG)_{\#}"] \\ h^n(X, A, f) \arrow[ur, "g_1^*"] \arrow[dr, "g_0^*"'] & \\ & h^n(X', A', fg_0). \end{tikzcd}$$

Let h and $\tilde{h}$ be cohomology theories on B and $\tilde{B}$ respectively. A *homomorphism* $\tau : h \to \tilde{h}$ *covering* $\tau : B \to \tilde{B}$ is to be a collection of homomorphisms

(2.10) $\lambda : h^n(X, A, f) \to h^n(X, A, \tau f)$, $(X, A, f) \in \mathcal{P}(B)$, $-\infty < n < \infty$,

satisfying the obvious commutativity relations. In the case where τ is an inclusion and each λ in (2.10) is an isomorphism, we will call $\tilde{h}$ an *extension* of h.

3. The spectral sequence of a fibration

In [1] we described the spectral sequence associated with a fibration $\pi : Y \to X$ and map $f : X \to B$. Here, we wish to outline a more general version of this spectral sequence.

Suppose now that $\pi : Y \to X$ is locally trivial, X is a polyhedron and for each pair (K, L) of subcomplexes of X the pair $(\pi^{-1}(K), \pi^{-1}(L))$ is a CW-pair. For $x \in X$ let $F(x) = \pi^{-1}(x)$.

Fix a map $g : Y \to B$. We will describe how the collection of groups $h^n(F(x), g)$, $x \in X$, form a local system over X. Let $\sigma : I \to X$ be a path from x_0 to x_1. By the covering homotopy property, construct

$$S(\sigma) : F(x_0) \times I \to Y$$

such that $S(\sigma)_0 : F(x_0) \to F(x_0)$ is the identity and $\pi S(\sigma)(y, t) = \sigma(t)$, $y \in F(x_0)$, $0 \leq t \leq 1$. We then have $S(\sigma)_1 : F(x_0) \to F(x_1)$. Define

$$\sigma_{\#} : h^n(F(x_1), g) \to h^n(F(x_0), g) \tag{3.1}$$

to be the composition

$$h^n(F(x_1), g) \xrightarrow{S(\sigma)_1^*} h^n(F(x_0), S(\sigma)_1 g) \xrightarrow{(S(\sigma)g)_{\#}} h^n(F(x_0), g). \tag{3.2}$$

It is easy to establish the following.

(3.3) Lemma. *The assignment of $h^n(F(x), g)$ to $x \in X$ and $\sigma_{\#}$ to $\sigma \in X^I$ is a local system on X.*

Denote this local system by $[h^n(F, g)]$.

Let X_p be the p-skeleton of X and let $Y_p = \pi^{-1}(X_p)$. Piecing together the exact sequences

$$\cdots \xrightarrow{d} h^n(Y, Y_p, g) \xrightarrow{j^*} h^n(Y, Y_{p-1}, g) \xrightarrow{i^*} h^n(Y_p, Y_{p-1}, g) \to \cdots$$

leads to an exact couple with

$$E_1^{p,q} = h^{p+q}(Y_p, Y_{p-1}, g), D_1^{p,q} = h^{p+q}(Y, Y_{p-1}, g). \tag{3.4}$$

A natural identification

$$\psi : E_2^{p,q} \to H^p(X; [h^q(F, g)]) \tag{3.5}$$

is constructed just as in the special case treated in [1] where we had assumed that g factored through X. That is, $g = f\pi$ for some map $f : X \to B$. The required modifications will be left to the reader. (They amount essentially to replacing $f\pi$ by g everywhere that it appears in [1, Section 4].)

Now we have a filtration

$$h^n(Y, g) = J^{0,n} \supset \cdots \supset J^{p,n-p} \supset \cdots \tag{3.6}$$

where

$$J^{p,n-p} = \text{Image } (h^n(Y, Y_{p-1}, g) \to h^n(Y, g)).$$

Setting $E_\infty^{p,n-p} = J^{p,n-p}/J^{p+1,n-p-1}$ we have

(3.7) Theorem. *Suppose that either* (a) *X is finitely coconnected or* (b) *Y and $F(x)$, $x \in X$, are finitely coconnected. Then*

(1) *For each pair (p, q) there is an integer $r(p, q)$ such that $E_{r(p,q)}^{p,q} \simeq E_\infty^{p,q}$.*

(2) *The filtration* (3.6) *is finite.*

See Theorem (4.14) of [1].

Now, suppose that h and $\tilde{h}$ are cohomology theories on $\mathcal{P}(B)$ and $\mathcal{P}(\tilde{B})$ respectively and $\lambda : h \to \tilde{h}$ is a homomorphism covering $\tau : B \to \tilde{B}$.

(3.8) THEOREM. *If $\lambda^n : h^n(X, f) \to \tilde{h}^n(X, \tau f)$ is an isomorphism when X is a point, $-\infty < n < +\infty$, then*

$$\lambda^n : h^n(X, A, f) \to \tilde{h}^n(X, A, \tau f)$$

is an isomorphism for all $(X, A, f) \in \mathcal{P}(B)$.

When B and $\tilde{B}$ are points, this is a theorem of Dold [3]. The spectral sequence argument in the general case is exactly the same.

4. Sectioned fibrations

Let $\mathfrak{I}$ denote the category with objects $\mathcal{E} = (E, B, p, \Delta)$ where E, B are spaces, $p : E \to B$, $\Delta : B \to E$ are maps and $p\Delta = 1$. A mapping

$$(4.1) \qquad (\lambda, \tau) : (E, B, p, \Delta) \to (E', B', p', \Delta')$$

in $\mathfrak{I}$ consists of $\lambda : E \to E'$ and $\tau : B \to B'$ such that $p'\lambda = \tau p$ and $\Delta'\tau = \lambda\Delta$. When $B = B'$ and τ is the identity we will abbreviate (λ, τ) to λ.

The *loop space* of $\mathcal{E} = (E, B, p, \Delta)$ in $\mathfrak{I}$ is

$$(4.2) \qquad \Omega(\mathcal{E}; \Delta) = (\Omega(E; \Delta), B, \Omega(p), \Omega(\Delta))$$

where

$$\Omega(\mathcal{E}; \Delta) = \{\sigma : I \to E \mid \sigma(I) \subset p^{-1}(b), \text{ some } b \in B, \sigma(0) = \sigma(1) = \Delta(b)\},$$

and $\Omega(p)(\sigma) = \sigma(I)$, $\Omega(\Delta)(b)(t) = \Delta(b)$, $0 \leq t \leq 1$.

Given $\mathcal{E} = (E, B, p, \Delta)$, $\mathcal{E}' = (E', B', p', \Delta')$ in $\mathfrak{I}$ we have their *product*

$$(4.3) \qquad \mathcal{E} \times \mathcal{E}' = (E \times E', B \times B', p \times p', \Delta \times \Delta')$$

and their *reduced join*

$$(4.4) \qquad \mathcal{E} \blacktriangle \mathcal{E}' = (E \blacktriangle E', B \times B', p \blacktriangle p', \Delta \blacktriangle \Delta')$$

where $E \blacktriangle E'$ is the quotient of $E \times E'$ obtained by identifying

$$p^{-1}(b) \times \Delta'(b') \cup \Delta(b) \times p'^{-1}(b')$$

with a point for each pair $(b, b') \in B \times B'$, and

$$p \blacktriangle p'[e, e'] = (p(e), p'(e')), \quad \Delta \blacktriangle \Delta'(b, b') = [\Delta(b), \Delta'(b')].$$

In the case where $B = B'$ we have the *internal product*

$$(4.5) \qquad \mathcal{E} \oplus \mathcal{E}' = (E \oplus E', B, p \oplus p', \Delta \oplus \Delta')$$

and the *internal reduced join*

$$(4.6) \qquad \mathcal{E} \mathbin{\overline{\blacktriangle}} \mathcal{E}' = (E \mathbin{\overline{\blacktriangle}} E', B, p \mathbin{\overline{\blacktriangle}} p', \Delta \mathbin{\overline{\blacktriangle}} \Delta')$$

where

$$E \oplus E' = \{(e, e') \in E \times E' \mid p(e) = p'(e')\}$$

and $E \mathbin{\bar{\blacktriangle}} E'$ is obtained from $E \oplus E'$ by identifying

$$p^{-1}(b) \times \Delta'(b) \cup \Delta(b) \times p'^{-1}(b)$$

with a point, $b \in B$.

We have a mapping

$$(4.7) \qquad \varkappa^{i,j} : \Omega^i(E) \blacktriangle \Omega^j(E') \to \Omega^{i+j}(E \blacktriangle E')$$

and when $B = B'$ a mapping

$$(4.8) \qquad \varkappa^{i,j} : \Omega^i(E) \mathbin{\bar{\blacktriangle}} \Omega^j(E') \to \Omega^{i+j}(E \mathbin{\bar{\blacktriangle}} E')$$

defined by the rule

$$\varkappa^{i,j}[\sigma, \tau](t_1 \cdots t_{i+j}) = [\sigma(t_1 \cdots t_i), \tau(t_{i+1} \cdots t_{i+j})].$$

Notice that in each case the pair $(\varkappa^{i,j}, 1)$ is a mapping in $\mathfrak{J}$.

We will say that $\mathcal{E} = (E, B, p, \Delta)$ in $\mathfrak{J}$ is a *sectioned fibration* if $p : E \to B$ is a fibre map in the sense of Serre. The following fact is easily established.

(4.9) *The loop space $\Omega(\mathcal{E}; \Delta)$ of a sectioned fibration $\mathcal{E}$ is a sectioned fibration.*

Given $(X, A, f) \in \mathcal{P}(B)$ and $\mathcal{E} = (E, B, p, \Delta) \in \mathfrak{J}$, let

$$(4.10) \qquad \mathcal{L}(X, A, f; \mathcal{E}) = \{g : X \to E \mid pg = f \quad \text{and} \quad g_{|A} = \Delta f_{|A}\},$$

with the compact open topology, and let

$$(4.11) \qquad L(X, A, f; \mathcal{E}) = \pi_0(\mathcal{L}(X, A, f; \mathcal{E})).$$

We regard $\mathcal{L}(X, A, f; \mathcal{E})$ as a pointed space with base-point Δf. Notice that by the exponential law we have an identification

$$(4.12) \qquad \rho : \mathcal{L}(X, A, f; \Omega(\mathcal{E}; \Delta) \to \Omega(\mathcal{L}(X, A, f; \mathcal{E}))$$

by $\rho(g)(t)(x) = g(x)(t), x \in X, 0 \leq t \leq 1$.

Also, for $(X, A, f) \in \mathcal{P}(B)$, $(X, A', f') \in \mathcal{P}(B')$ and $\mathcal{E} = (E, B, p, \Delta)$, $\mathcal{E}' = (E', B', p', \Delta')$ in $\mathfrak{J}$ we have a mapping

$$(4.13) \quad \psi : \mathcal{L}(X, A, f, \mathcal{E}) \wedge \mathcal{L}(X, A', f', \mathcal{E}') \to \mathcal{L}(X, A \cup A', f \times f', \mathcal{E} \blacktriangle \mathcal{E}')$$

by lettin $\psi[g, g']$ be the composition

$$X \xrightarrow{g \times g'} E \times E' \xrightarrow{\omega} E \blacktriangle E'$$

where ω is the projection map.

One more remark. In the sequel we will usually abbreviate $\Omega(\mathcal{E}; \Delta)$ to $\Omega(\mathcal{E})$ and $\Omega(E; \Delta)$ to $\Omega(E)$.

5. Spectra

Recall from [12] that a spectrum $\mathbf{F}$ (here to be called a *point spectrum*) is a collection of pointed spaces F_k and connecting maps $\varepsilon_k : F_k \to \Omega(F_{k+1})$, $-\infty < k < +\infty$. The *n-th* homotopy group of $\mathbf{F}$ is defined to be

$$\pi_n(\mathbf{F}) = \text{DIR LIM}_k\ \pi_{n+k}(F_k), \tag{5.1}$$

the direct limit taken via

$$\pi_{n+k}(F_k) \xrightarrow{\varepsilon_{k\#}} \pi_{n+k}(\Omega(F_{k+1})) \xrightarrow{\partial_{\#}^{-1}} \pi_{n+k+1}(F_{k+1}).$$

Example. The *sphere spectrum* $\mathbf{S}$ is defined as follows. The *k-th* space is S^k, $k \geq 0$, and $\Omega^{-k}(S^0)$, $k < 0$. The connecting maps are

$$\tilde{\zeta}_{k,1} : S^k \to \Omega(S^{k+1}), \quad k \geq 0,$$

and the identity

$$\Omega^{-k}(S^0) \to \Omega(\Omega^{-(k+1)}(S^0)), \quad k < 0.$$

Here $\tilde{\zeta}_{k,1}$ corresponds to $\zeta_{k,1} : S^k \wedge S^1 \to S^{k+1}$ under the correspondence (1.6).

A *B-spectrum* $\mathbf{E}$ is a collection of sectioned fibrations

$$\mathcal{E}_k = (E_k, B, p_k, \Delta_k)$$

and maps

$$\varepsilon_k : (E_k, B, p_k, \Delta_k) \to (\Omega(F_{k+1}), B, \Omega(p_{k+1}), \Omega(\Delta_{k+1})), \quad -\infty < k < \infty. \tag{5.2}$$

Of course when B is a point we have a natural identification between B-spectra and point spectra.

Suppose $\mathbf{E}$ and $\mathbf{E}'$ are B and B' spectra respectively and $\tau : B \to B'$ is a map. A *mapping* $\lambda : \mathbf{E} \to \mathbf{E}'$ covering τ is a collection of maps $\lambda_k : E_k \to E_k'$, $-\infty < k < \infty$, such that

$$(\lambda_k, \tau) : (E_k, B, p_k, \Delta_k) \to (E_k', B', p_k', \Delta_k') \tag{5.3}$$

is a mapping and the diagram

$$\begin{array}{ccc} E_k & \xrightarrow{\lambda_k} & E_k' \\ \downarrow{\scriptstyle \varepsilon_k} & & \downarrow{\scriptstyle \varepsilon_k'} \\ \Omega(E_{k+1}) & \xrightarrow{\Omega(\lambda_{k+1})} & \Omega(E_{k+1}') \end{array} \tag{5.4}$$

is homotopy commutative by a homotopy which is fibre and cross-section preserving.

The construction of a cohomology theory $h(\ \ ;\mathbf{E})$ on $\mathcal{P}(B)$ from a B-spectrum $\mathbf{E}$ is carried out as follows. Given $(X, A, f) \in \mathcal{P}(B)$ form the point spectrum $\mathcal{L}(X, A, f; \mathbf{E})$ whose *k-th* space is $\mathcal{L}(X, A, f, \mathcal{E}_k)$ and whose connecting maps are

$$\mathcal{L}(X, A, f, \mathcal{E}_k) \xrightarrow{\mathcal{L}(\varepsilon_k)} \mathcal{L}(X, A, f, \Omega(\mathcal{E}_{k+1})) \xrightarrow{\rho} \Omega(\mathcal{L}(X, A, f, \mathcal{E}_{k+1})).$$

Then let

$$(5.5) \qquad h^n(X, A, f; \mathbf{E}) = \pi_{-n}(\mathcal{L}(X, A, f; \mathbf{E})).$$

For a map $g : (X, A, f) \to (X', A', f')$ in $\mathcal{P}(B)$, the collection of maps $\mathcal{L}(g) : \mathcal{L}(X', A', f', \mathcal{E}_k) \to \mathcal{L}(X, A, f, \mathcal{E}_k)$, $-\infty < k < \infty$, constitute a map of spectra

$$\mathcal{L}(g) : \mathcal{L}(X', A', f'; \mathbf{E}) \to \mathcal{L}(X, A, f, \mathbf{E}).$$

Let

$$(5.6) \qquad h^n(g, \mathbf{E}) = \mathcal{L}(g)_{\#} : h^n(X', A', f'; \mathbf{E}) \to h^n(X, A, f; \mathbf{E}).$$

The boundary operator

$$(5.7) \quad d^n(X, A, f; \quad) : h^n(A, f; \quad) \to h^{n+1}(X, A, f; \quad), \quad (X, A, f) \in \mathcal{P}(B),$$

is defined to be the direct limit of the homomorphisms

$$(-1)^k \partial_{\#} : \pi_k(\mathcal{L}(A, f, \mathcal{E}_{n+k})) \to \pi_{k-1}(\mathcal{L}(X, A, f, \mathcal{E}_{n+k}))$$

where $\partial_{\#}$ is the boundary operator in the homotopy sequence of the fibration

$$\mathcal{L}(X, A, f, \mathcal{E}_{n+k}) \xrightarrow{\mathcal{L}(j)} \mathcal{L}(X, f, \mathcal{E}_{n+k}) \xrightarrow{\mathcal{L}(i)} \mathcal{L}(A, f, \mathcal{E}_{n+k}),$$

and

$$A \xrightarrow{i} X \xrightarrow{j} (X, A)$$

are the inclusions. That the collection of functors $h^n(\quad; \mathbf{E})$ and natural transformations d^n define a cohomology theory on $\mathcal{P}(B)$ is established in [1].

As expected, a mapping $\lambda : \mathbf{E} \to \mathbf{E}'$ covering $\tau : B \to B'$ determines a homomorphism of cohomology theories

$$(5.8) \qquad \lambda_{\#} : h(\quad; \mathbf{E}) \to \mathrm{h}(\quad; \mathbf{E})$$

covering τ.

The *loop spectrum* $\Omega(\mathbf{E})$ of $\mathbf{E}$ is the spectrum whose *k-th* space is $\Omega(\mathcal{E}_{k+1})$ and whose connecting maps are $\Omega(\varepsilon_{k+1})$. The maps

$$\varepsilon_k : \mathcal{E}_k \to \Omega(\mathcal{E}_{k+1})$$

define a map of spectra $\varepsilon : \mathbf{E} \to \Omega(\mathbf{E})$ and it is clear from the definitions that

$$(5.9) \qquad \varepsilon_{\#} : h(\quad; \mathbf{E}) \to h(\quad; \Omega(\mathbf{E}))$$

is an equivalence.

6. Pairings of spectra

Whitehead [12] has defined the notion of a pairing of point spectra and has shown how such a pairing yields a pairing of the associated cohomology theories. The purpose of this section is to describe the generalization of these ideas to B-spectra.

For us, a pairing $\eta : (\mathbf{F}^1, \mathbf{F}^2) \to \mathbf{F}^3$ of point spectra is to consist of maps

$$\eta_{s,t} : F_s^1 \wedge F_t^2 \to F_{s+t}^3, \qquad -\infty < s, t < +\infty, \tag{6.1}$$

having the following property. Consider the diagram

$$\begin{tikzcd}
\Omega(F_{s+1}^1) \wedge F_t^2 \arrow[r, "\kappa^{1,0}"] \arrow[d, "\varepsilon_s^1 \wedge 1"] & \Omega(F_{s+1}^1 \wedge F_t^2) \arrow[drr, "\Omega(\eta_{s+1,t})"] & & \\
F_s^1 \wedge F_t^2 \arrow[r, "\eta_{s,t}"] \arrow[d, "1 \wedge \varepsilon_t^2"] & F_{s+t}^3 \arrow[rr, "\varepsilon_{s+t}^3"] & & \Omega(F_{s+t+1}^3) \\
F_s^1 \wedge \Omega(F_{t+1}^2) \arrow[r, "\kappa^{0,1}"] & \Omega(F_s^1 \wedge F_{t+1}^2) \arrow[urr, "\Omega(\eta_{s,t+1})"'] & &
\end{tikzcd} \tag{6.2}$$

We are to have

$$(-1)^t[\Omega(\eta_{s+1,t})\kappa^{1,0}(\varepsilon_s^1 \wedge 1)] = [\varepsilon_{s+t}^3, \eta_{s,t}] = [\Omega(\eta_{s,t+1})\kappa^{0,1}(1 \wedge \varepsilon_t^2)], \tag{6.3}$$

in the group $[F_s^1 \wedge F_t^2, \Omega(F_{s+t+1}^2)]$.

(6.4) *Example.* The *natural pairing* $\lambda^1 : (\mathbf{F}, \mathbf{S}) \to \mathbf{F}$ is defined by letting $\lambda_{s,t}^1 : F_s \wedge S^t \to F_{s+t}$, $t \geq 0$, be the mapping which corresponds to

$$F_s \xrightarrow{\varepsilon_s} \Omega(F_{s+1}) \xrightarrow{\Omega(\varepsilon_{s+1})} \cdots \xrightarrow{\Omega^{t-1}(\varepsilon_{s+t-1})} \Omega^t(F_{s+t}),$$

under the correspondence (1.6). For $t < 0$, $\lambda_{s,t}^1$ is to be the constant map. The *natural pairing* $\lambda^2 : (\mathbf{S}, \mathbf{F}) \to \mathbf{F}$ is defined by letting $\lambda_{s,t}^2$ be $(-1)^{st}$ times the composition

$$S^t \wedge F_s \xrightarrow{\alpha} F_s \wedge S^t \xrightarrow{\lambda_{s,t}^1} F_{s+t}$$

where $\alpha[y, x] = [x, y]$.

Given a pairing $\eta : (\mathbf{F}^1, \mathbf{F}^2) \to \mathbf{F}^3$, let

$$\eta_{k,l*} : \pi_{s+k}(F_k^1) \otimes \pi_{t+l}(F_l^2) \to \pi_{s+t+k+l}(F_{k+l}^3) \tag{6.5}$$

be the composition

$$\pi_{s+k}(F_k^1) \otimes \pi_{t+l}(F_l^2) \xrightarrow{\wedge} \pi_{s+t+k+l}(F_k^1 \wedge F_l^2) \xrightarrow{\eta_{k,l\#}} \pi_{s+t+k+l}(F_{k+l}^3).$$

Using the commutativity of the diagram (1.4) we see that the homomorphisms $(-1)^{tk}\eta_{k,l_*}$ commute with the direct limit maps and so define a pairing

$$\eta_{\#} : \pi_s(\mathbf{F}^1) \otimes \pi_t(\mathbf{F}^2) \to \pi_{s+t}(\mathbf{F}^3). \tag{6.6}$$

Now suppose that B^i-spectra

$$\mathbf{E}^i = \pi\mathcal{E}_k^i = \{(E_k^i, B^i, p_k^i, \Delta_k^i); \mathcal{E}_k^i\}, \qquad i = 1, 2, 3,$$

and a map

$$\mu : B^1 \times B^2 \to B^3$$

are given. A *μ-pairing* $\eta : (\mathbf{E}^1, \mathbf{E}^2) \to \mathbf{E}^3$ is a collection of maps

$$\eta_{s,t} : E_s^1 \blacktriangle E_t^2 \to E_{s+t}^3$$

such that

$$(6.7) \qquad (\eta_{s,t}, \mu) : (E_s^1 \blacktriangle E_t^2, B^1 \times B^2, p_s^1 \blacktriangle p_t^2, \Delta_s^1 \blacktriangle \Delta_t^2) \to (E_{s+t}^3, B^3, p_{s+t}^3, \Delta_{s+t}^3)$$

is a mapping in $\mathfrak{I}$, and in the diagram

$$(6.8) \qquad \begin{tikzcd} \Omega(E_{s+1}^1) \blacktriangle E_t^2 \arrow[r, "\kappa^{1,0}"] & \Omega(E_{s+1}^1 \blacktriangle E_t^2) \arrow[drr, "\Omega(\eta_{s+1,t})"] & & \\ E_s^1 \blacktriangle E_t^2 \arrow[u, "\varepsilon_s^1 \blacktriangle 1"] \arrow[d, "1 \blacktriangle \varepsilon_t^2"'] \arrow[r, "\eta_{s,t}"] & E_{s+t}^3 \arrow[rr, "\varepsilon_{s+t}^3"] & & \Omega(E_{s+t+1}^3) \\ E_s^1 \blacktriangle \Omega(E_{t+1}^2) \arrow[r, "\kappa^{0,1}"] & \Omega(E_s^1 \blacktriangle E_{t+1}^2) \arrow[urr, "\Omega(\eta_{s,t+1})"'] & & \end{tikzcd}$$

we have

$$(6.9) \quad (-1)^t[\Omega(\eta_{s+1,t})\kappa^{1,0}(\varepsilon_s^1 \blacktriangle 1)] = [\varepsilon_{s+1}^3 \eta_{s,t}] = [\Omega(\eta_{s,t+1})\kappa^{0,1}(1 \blacktriangle \varepsilon_t^2)]$$

in the group

$$L(E_s^1 \blacktriangle E_t^2, \Delta_t^1 \blacktriangle \Delta_t^2(B^1 \times B^2), \mu(p_s^1 \blacktriangle p_t^2); \Omega(\mathcal{E}_{s+t+1}^3)).$$

(6.10) *Example.* From $\eta : (\mathbf{E}^1, \mathbf{E}^2) \to \mathbf{E}^3$ we have an induced pairing $\eta^{1,1} : (\Omega(\mathbf{E}^1), \Omega(\mathbf{E}^2)) \to \Omega^2(\mathbf{E}^3)$ defined by letting

$$\eta_{s,t}^{1,1} : \Omega(E_{s+1}^1) \blacktriangle \Omega(E_{t+1}^2) \to \Omega^2(E_{s+t+2}^3)$$

be the composition

$$\Omega(E_{s+t}^1) \wedge \Omega(E_{t+1}^2) \xrightarrow{\kappa^{1,1}} \Omega^2(E_{s+1}^1 \wedge E_{t+1}^2)$$
$$\cdot \xrightarrow{\Omega^2(\eta_{s+1,t+1})} \Omega^2(E_{s+t+2}^3) \xrightarrow{(-1)^t} \Omega^2(E_{s+t+2}^3).$$

A straightforward calculation shows that $\eta^{1,1}$ satisfies (6.9). Furthermore, the diagram

$$\begin{tikzcd} E_s^1 \blacktriangle E_t^2 \arrow[r, "\eta_{s,t}"] \arrow[d, "\varepsilon_s^1 \blacktriangle \varepsilon_t^2"] & E_{s+t}^3 \arrow[r, "\varepsilon_{s+t}^3"] & \Omega(E_{s+t+1}^3) \arrow[d, "\Omega(\varepsilon_{s+t+1}^3)"] \\ \Omega(E_{s+1}^1) \blacktriangle \Omega(E_{t+1}^2) \arrow[rr, "\eta_{s,t}^{1,1}"] & & \Omega^2(E_{s+t+2}^3) \end{tikzcd}$$

is commutative.

Now suppose that cohomology theories h_i on $\mathcal{P}(B^i)$, $i = 1, 2, 3$, and

$\mu : B^1 \times B^2 \to B^3$ are given. A *μ-pairing* $\eta : h_1 \otimes h_2 \to h_3$ consists of homomorphisms

(6.12) $\quad \eta : h_1^s(X, A_1, f_1) \otimes h_2^t(X, A_2, f_2) \to h_3^{s+t}(X, A_1 \cup A_2, \mu(f_1 \times f_2)),$

$(X, A_1, f_1) \in \mathcal{P}(B^1)$, $(X, A_2, f_2) \in \mathcal{P}(B^2)$, having the following properties. (as usual, denote $\eta(u \otimes v)$ by $u \cup v$.)

(6.13) *If $g : X \to X'$ is such that*

$$g : (X, A_1, f_1) \to (X', A_1', f_1') \quad \textit{and} \quad g : (X, A_2, f_2) \to (X', A_2', f_2')$$

are maps in $\mathcal{P}(B^1)$ and $\mathcal{P}(B^2)$ respectively and

$$u \in h_1^s(X', A_1', f_1'), \qquad v \in h_2^t(X', A_2', f_2')$$

then

$$g^*(u \cup v) = g^*(u) \cup g^*(v)$$

in $h_3^{s+t}(X, A_1 \cup A_2, \mu(f_1 \times f_2))$.

For the next two properties let $(X, A, f_1) \in \mathcal{P}(B^1)$ and $(X, A, f_2) \in \mathcal{P}(B^2)$ be given and let $i : A \to X$ denote the inclusion.

(6.14) *If $u \in h_1^{s-1}(A, f_1)$ and $v \in h_2^t(X, f_2)$ then*

$$d(u \cup i^*(v)) = d(u) \cup v$$

in $h_3^{s+t}(X, A, \mu(f_1 \times f_2))$.

(6.15) *If $u \in h_1^s(X, f_1)$ and $v \in h_2^{t-1}(A, f_2)$ then*

$$d(i^*(u) \cup v) = (-1)^s u \cup d(v)$$

in $h_3^{s+t}(X, A, \mu(f_1 \times f_2))$.

These are the analogue of the axioms given by Steenrod [11] for a pairing of two ordinary cohomology theories to a third.

Suppose that $M : X \times I \to B^1$, $N : X \times I \to B^2$ are homotopies. We then have a homotopy

$$X \times I \xrightarrow{M \times N} B^1 \times B^2 \xrightarrow{\mu} B^3$$

and a diagram

$$(6.16) \qquad \begin{array}{ccc} h_1^s(X, A_1, M_1) \otimes h_2^t(X, A_2, N_1) & \xrightarrow{\eta} & h_3^{s+t}(X, A_1 \cup A_2, \mu(M_1 \times N_1)) \\ \big\downarrow M_{\#} \otimes N_{\#} & & \big\downarrow \mu(M \times N)_{\#} \\ h_1^s(X, A_1, M_0) \otimes h_2^t(X, A_2, N_0) & \xrightarrow{\eta} & h_3^{s+t}(X, A_1 \cup A_2, \mu(M_1 \times N_1)) \end{array}$$

(6.17) LEMMA. *The above diagram is commutative.*

This is immediate from (6.13) and the definition of the operator $\#$.

We will now describe how a μ-pairing $\eta : (\mathbf{E}^1, \mathbf{E}^2) \to \mathbf{E}^3$ of spectra determines a μ-pairing

$$\eta_{\#} : h(\ ; \mathbf{E}^1) \otimes h(\ ; \mathbf{E}^2) \to h(\ ; \mathbf{E}^3)$$

of cohomology theories. For $(X, A_1, f_1) \in \mathcal{P}(B^1)$ and $(X, A_2, f_2) \in \mathcal{P}(B^2)$ let

$$(6.18) \quad \begin{aligned} \mathcal{L}^\wedge(\eta_{s,t}) : \mathcal{L}(X, A_1, f_1, \mathcal{E}_s^1) \wedge \mathcal{L}(X, A_2, f_2, \mathcal{E}_t^2) \\ \to \mathcal{L}(X, A_1 \cup A_2, \mu(f_1 \times f_2), \mathcal{E}_{s+t}^3) \end{aligned}$$

be the composition

$$\begin{aligned} \mathcal{L}(X, A_1, f_1, \mathcal{E}_s^1) \wedge \mathcal{L}(X, A_2, f_2, \mathcal{E}_t^2) \xrightarrow{\psi} \mathcal{L}(X, A_1 \cup A_2, f_1 \times f_2, \mathcal{E}_s^1 \blacktriangle \mathcal{E}_t^2) \\ \xrightarrow{\mathcal{L}(\eta_{s,t})} \mathcal{L}(X, A_1 \cup A_2, \mu(f_1 \times f_2), \mathcal{E}_{s+t}^3), \end{aligned}$$

where ψ is given in (4.13).

(6.19) LEMMA. *The collection of maps $\mathcal{L}^\wedge(\eta_{s,t})$ is a pairing of point spectra*

$$\mathcal{L}^\wedge(\eta) : (\mathcal{L}(X, A_1, f_1; \mathbf{E}^1), \mathcal{L}(X, A_2, f_2; \mathbf{E}^2)) \to \mathcal{L}(X, A_1 \cup A_2, \mu(f_1 \times f_2); \mathbf{E}^3).$$

This is a straightforward calculation (compare [12, Lemma (6.4)]). Now we may define

$$(6.20) \quad \begin{aligned} \eta_\# : h^s(X, A_1, f_1; \mathbf{E}^1) \otimes h^t(X, A_2, f_2; \mathbf{E}^2) \\ \to h^{s+t}(X, A_1 \cup A_2, \mu(f_1 \times f_2); \mathbf{E}^3) \end{aligned}$$

to be

$$\begin{aligned} \mathcal{L}^\wedge(\eta)_\# : \pi_{-s}(\mathcal{L}(X, A_1, f_1; \mathbf{E}^1)) \otimes \pi_{-t}(\mathcal{L}(X, A_2, f_2; \mathbf{E}^2)) \\ \to \pi_{-(s+t)}(\mathcal{L}(X, A_1 \cup A_2)\mu(f_1 \times f_2); \mathbf{E}^3)) \end{aligned}$$

as in (6.6).

(6.21) THEOREM. *The collection of pairings $\eta_\#$ in* (6.20) *defines a μ-pairing of cohomology theories*

$$\eta_\# : h(\quad; \mathbf{E}^1) \otimes h(\quad; \mathbf{E}^2) \to (\quad; \mathbf{E}^3).$$

The proof is straightforward and will be omitted (compare [12]).

7. Construction of μ

We will use Milnor's construction [10] of a universal principal G-bundle $(E(G), B(G), r(G))$ for a countable CW-group G. Recall that $E(G) = \bigcup_{n=0}^\infty E_n(G)$ where $E_n(G)$ is the $(n+1)$-fold join of G with itself. There is the natural action of G on the right of $E(G)$ and $B(G)$ is the orbit space. We have $G \subset E(G)$ as the first factor in the infinite join.

Let $\mathcal{O}_n$ denote the n-dimensional orthogonal group. The usual inclusion $i_n : \mathcal{O}_n \to \mathcal{O}_{n+1}$ induces inclusions

$$E(i_n) : E(\mathcal{O}_n) \to E(\mathcal{O}_{n+1}) \quad \text{and} \quad B(i_n) : B(\mathcal{O}_n) \to B(\mathcal{O}_{n+1}).$$

In this way we regard $E(\mathcal{O}_n)$ and $B(\mathcal{O}_n)$ as subspaces of $E(\mathcal{O}_{n+1})$ and $B(\mathcal{O}_{n+1})$ respectively. Let $\lambda_{n,m} : \mathcal{O}_n \times \mathcal{O}_m \to \mathcal{O}_{n+m}$ denote the usual inclusion and let $A^n \in \mathcal{O}_{2n+2}$ be defined by the formula

$$(7.1) \quad A^n(x_1 \cdots x_{2n+2}) = (x_1 \cdots x_n, x_{n+2} \cdots x_{2n+1}, x_{n+1}, x_{2n+2}).$$

Next, define $B^n \in \mathcal{O}_{2n}$ by setting $B^0 = 1$ and $B^n = A^0 \cdots A^{n-1}$, $n > 0$.

We will construct $\lambda_{n,n}$-equivariant maps

$$\nu_n : E(\mathcal{O}_n) \times E(\mathcal{O}_n) \to E(\mathcal{O}_{2n}) \tag{7.2}$$

having the following properties.

(7.3) $\nu_n : \mathrm{E}(\mathcal{O}_n) \times \{1\} \to \mathrm{E}(\mathcal{O}_{2n})$ *is given by* $\nu_n(x, 1) = x \cdot B^n$.

(7.4) *Commutativity holds in the diagram*

$$\begin{tikzcd} E(\mathcal{O}_n) \times E(\mathcal{O}_n) \arrow[r, "\nu_n"] \arrow[d, "E(i_n) \times E(i_n)"] & E(\mathcal{O}_{2n}) \arrow[r, "E(i_{2n+1})E(i_{2n})"] & E(\mathcal{O}_{2n+2}) \arrow[d, "A^n"] \\ E(\mathcal{O}_{n+1}) \times E(\mathcal{O}_{n+1}) \arrow[rr, "\nu_{n+1}"] & & E(\mathcal{O}_{2n+2}). \end{tikzcd}$$

Let $\nu_0 : E(\mathcal{O}_0) \times \mathrm{E}(\mathcal{O}_0) \to E(\mathcal{O}_0)$ be the projection onto the first factor and assume now that ν_n has been constructed. Let M^* denote the orbit of $E(\mathcal{O}_n) \times \mathrm{E}(\mathcal{O}_n)$ in $E(\mathcal{O}_{n+1}) \times \mathrm{E}(\mathcal{O}_{n+1})$.

$$M^* = \{(x_1 S_1, x_2 S_2) \mid x_i \in E(\mathcal{O}_n), S_i \in \mathcal{O}_{n+1}, i = 1, 2\}.$$

Let $M = (E(\mathcal{O}_{n+1}) \times \mathcal{O}_{n+1}) \cup M^*$ and define

$$\begin{aligned} &\bar{\nu}_{n+1}(x, S) \\ &\quad = xB^{n+1}\lambda_{n+1,n+1}(1, S), && x \in \mathrm{E}(\mathcal{O}_{n+1}), S \in \mathcal{O}_{n+1}. \\ &\bar{\nu}_{n+1}(x_1 S_1, x_2 S_2) \\ &\quad = \nu_n(x_1, x_2)A^n\lambda_{n+1,n+1}(S_1, S_2), && x_i \in E(\mathcal{O}_n), S_i \in \mathcal{O}_{n+1}, i = 1, 2. \end{aligned} \tag{7.5}$$

Using the relations

(a) $A^j\lambda_{n+1,n+1}(S, 1) = \lambda_{n+1,n+1}(S, i)A^j, 0 \leq j \leq n + 1, S \in \mathcal{O}_{n+1}$,

(b) $A^n\lambda_{n+1,n+1}(S, T) = \lambda_{n,n}(S, T)A^n, S, T \in \mathcal{O}_n$,

we see that $\bar{\nu}_{n+1}$ is well defined and is $\lambda_{n+1,n+1}$-equivariant. Now apply Theorem 7.1 of [6] to find a $\lambda_{n+1,n+1}$-equivariant extension

$$\nu_{n+1} : E(\mathcal{O}_{n+1}) \times E(\mathcal{O}_{n+1}) \to E(\mathcal{O}_{2n+2}).$$

Properties (7.3) and (7.4) are easily checked.

Let $B(\mathcal{O}) = \bigcup_{n=0}^{\infty} \mathrm{B}(\mathcal{O}_n)$ and take $1 \in \mathcal{O}_0 \subset E(\mathcal{O}_0) = B(\mathcal{O}_0)$ as the base-point b_0 of $\mathrm{B}(\mathcal{O})$. Let

$$\mu_n : B(\mathcal{O}_n) \times \mathrm{B}(\mathcal{O}_n) \to \mathrm{B}(\mathcal{O}_{2n}) \tag{7.6}$$

be the orbit map of ν_n. From property (7.5) we have a commutative diagram

$$\begin{tikzcd} B(\mathcal{O}_n) \times B(\mathcal{O}_n) \arrow[r, "\mu_n"] \arrow[d] & B(\mathcal{O}_{2n}) \arrow[d] \\ B(\mathcal{O}_{n+1}) \times B(\mathcal{O}_{n+1}) \arrow[r, "\mu_{n+1}"] & B(\mathcal{O}_{2n+2}). \end{tikzcd} \tag{7.7}$$

Thus we may define

$$(7.8)\qquad \mu : B(\mathcal{O}) \times B(\mathcal{O}) \to B(\mathcal{O})$$

to be the direct limit of the μ_n. Property (7.4) implies that

$$(7.9)\qquad \mu : B(\mathcal{O}) \times \{b_0\} \to \mathrm{B}(\mathcal{O})$$

is given by $\mu(b, b_0) = b$.

Remark. Note that the mapping μ_n is a classifying map for Whitney sum. From this fact it follows that μ endows $B(\mathcal{O})$ with an H-space structure which is the canonical one. That is, the natural mapping $\tilde{K}\mathcal{O}(X) \to [X, \mathrm{B}(\mathcal{O})]$, $X \in \mathcal{P}^*$, is an isomorphism.

8. Construction of $H(\ \ ; F)$

The usual left action of $\mathcal{O}_n$ on R^n extends to an action of $\mathcal{O}_n$ on S^n by setting $Wz_n = z_n$, $W \in \mathcal{O}_n$. Now let $\mathbf{F} = \{F_k, \varepsilon_k\}$ be a point spectrum and let $\mathcal{O}_n$ act on $F_k \wedge S^n$ by $W[x, y] = [x, Wy]$, $W \in \mathcal{O}_n$. Form the sectioned fibration

$$(8.1)\qquad \mathfrak{F}_k^n = (F_k^n, \mathrm{B}(\mathcal{O}_n), p_k^n, \Delta_k^n)$$

where $F_k^n = \mathrm{E}(\mathcal{O}_n) \times (F_{k-n} \wedge S^n)/\mathcal{O}_n$, the action of $\mathcal{O}_n$ defined by

$$W\cdot[e, [x, y]] = [eW^{-1}, W[x, y]], \quad W \in \mathcal{O}_n,$$

and $p_k^n[e, [x, y]] = [e]$, $\Delta_k^n[e] = [e, [x_{k-n}, z_n]]$, where x_{k-n} is the base-point of F_{k-n}.

The mapping

$$F_{k-n} \wedge S^n \xrightarrow{\varepsilon_{k-n} \wedge 1} \Omega(F_{k+1-n}) \wedge S^n \xrightarrow{\varkappa^{1,0}} \Omega(F_{k+1-n} \wedge S^n)$$

is equivariant so that we may define

$$(8.2)\qquad \varepsilon_k^n : F_k^n \to \Omega(F_{k+1}^n)$$

to be

$$E(\mathcal{O}_n) \times (F_{k-n} \wedge S^n)/\mathcal{O}_n$$
$$\cdot \xrightarrow{[1 \times \varkappa^{1,0}(\varepsilon_{k-n} \wedge 1)]} \mathrm{E}(\mathcal{O}_n) \times \Omega(F_{k-n+1} \wedge S^n)/\mathcal{O}_n = \Omega(F_{k+1}^n).$$

We now have a $\mathrm{B}(\mathcal{O}_n)$-spectrum

$$(8.3)\qquad \mathbf{F}^n = \{\mathfrak{F}_k^n = (F_k^n, B(\mathcal{O}_n), p_k^n, \Delta_k^n); \varepsilon_k^n\}.$$

Next, we define a mapping of spectra

$$(8.4)\qquad \theta^n : \mathbf{F}^n \to \Omega(\mathbf{F}^{n+1})$$

covering the inclusion $B(i_n) : \mathrm{B}(\mathcal{O}_n) \to \mathrm{B}(\mathcal{O}_{n+1})$ as follows. Let

$$\tilde{\zeta}_{n,1} : S^n \to \Omega(S^{n+1})$$

correspond to $\zeta_{n,1} : S^n \wedge S^1 \to S^{n+1}$ under the correspondence (1.6). There is the inherited action of $\mathcal{O}_{n+1}$ on $\Omega(S^{n+1})$ and $\tilde{\zeta}_{n,1}$ is i_n-equivarient. Consequently,

$$F_{k-n} \wedge S^n \xrightarrow{1 \wedge \tilde{\zeta}_{n,1}} F_{k-n} \wedge \Omega(S^{n+1}) \xrightarrow{\varkappa^{0,1}} \Omega(F_{k-n} \wedge S^{n+1})$$

is i_n-equivariant. Let

$$\theta_k^n : F_k^n \to \Omega(F_k^{n+1}) \tag{8.5}$$

be the composition

$$F_k^n \xrightarrow{[E(i_n) \times (1 \wedge \tilde{\zeta}_{n,1})]} \Omega(F_k^{n+1}) \xrightarrow{(-1)^k} \Omega(F_k^{n+1}).$$

It is easily checked that θ^n is a mapping of spectra.

Now for $(X, A, f) \in \mathcal{P}^*(\mathrm{B}(\mathcal{O}))$ let $n(X)$ be the smallest integer n such that $f(X) \subset \mathrm{B}(\mathcal{O}_n)$ and let

$$H^k(X, A, f; \mathbf{F}) = \text{DIR LIM}_{n \geq n(X)}\, h^k(X, A, f; \mathbf{F}^n) \tag{8.6}$$

be the direct limit taken via the homomorphisms

$$h^k(X, A, f; \mathbf{F}^n) \xrightarrow{\theta_{\#}^n} h^k(X, A, f; \Omega(\mathbf{F}^{n+1})) \xrightarrow{(\varepsilon_{\#}^{n+1})^{-1}} h^k(X, A, f; \mathbf{F}^{n+1}).$$

(See (5.9).)

For $g : (X, A, f) \to (X', A', f')$ in $\mathcal{P}^*(B(\mathcal{O}))$ let

$$H^k(g; \mathbf{F}) : H^k(X', A', f'; \mathbf{F}) \to H^k(X, A, f; \mathbf{F}) \tag{8.7}$$

be the direct limit of the

$$h^k(g; \mathbf{F}^n) : h^k(X', A', f'; \mathbf{F}^n) \to h^k(X, A, f; \mathbf{F}^n), \qquad n \geq \max\,(n(X), n(X')).$$

The boundary operator

$$d^k : H^k(A, f_{|A}; \mathbf{F}) \to H^{k+1}(X, A, f; \mathbf{F}), \quad (X, A, f) \in \mathcal{P}^*(\mathrm{B}(\mathcal{O})), \tag{8.8}$$

is defined to be the direct limit of the maps

$$d^k : h^k(A, f_{|A}; \mathbf{F}^n) \to h^{k+1}(X, A, f; \mathbf{F}^n), \qquad n \geq n(X).$$

(8.9) THEOREM. *The collection of functors* $H^k(\quad; \mathbf{F})$ *and natural transformations* d^k, $-\infty < k < \infty$, *constitute a cohomology theory on* $\mathcal{P}^*(\mathrm{B}(\mathcal{O}))$.

This is easily established. Now suppose that $\lambda : \mathbf{F} \to \mathbf{G}$ is a map of point spectra. Define

$$\lambda^n : \mathbf{F}^n \to \mathbf{G}^n \tag{8.10}$$

to be the collection $\lambda_k^n : F_k^n \to G_k^n$ where

$$\lambda_k^n[e, [x, y]] = [e[\lambda_{k-n}(x), y]] \qquad e \in E(\mathcal{O}_n),\ y \in F_{k-n},\ y \in S^n.$$

The induced homomorphisms $\lambda_{\#}^n : \mathrm{h}(\quad; \mathbf{F}^n) \to \mathrm{h}(\quad; \mathbf{G}^n)$ commute with the direct maps and so define

$$\lambda_{\#} : H(\ ; \mathbf{F}) \to \mathrm{H}(\ ; \mathbf{G}). \tag{8.11}$$

Evidently, the assignments $\mathbf{F} \to \mathrm{H}(\ ;\mathbf{F})$ and $\lambda \to \lambda_\#$ define a covariant functor from the category of point spectra to the category of cohomology theories on $\mathcal{P}^*(B(\mathcal{O}))$.

9. The associated μ-pairing

Suppose that $\mathbf{F} = \{F_k, \varepsilon_k\}, \mathbf{G} = \{G_k, \gamma_k\}$ and $\mathbf{D} = \{D_k, \delta_k\}$ are point spectra with F_k and G_k countable CW-complexes, $-\infty < k < \infty$. We will associate with each pairing $\eta : (\mathbf{F}, \mathbf{G}) \to \mathbf{D}$ a μ-pairing

$$\eta_\# : \mathrm{H}(\ ;\mathbf{F}) \otimes H(\ ;\mathbf{G}) \to H(\ ;\mathbf{D}) \tag{9.1}$$

where $\mu : B(\mathcal{O}) \times B(\mathcal{O}) \to B(\mathcal{O})$ is the mapping (7.9).

Consider the mapping

$$\alpha : (F_{s-n} \wedge S^n) \times (G_{t-n} \wedge S^n) \to (F_{s-n} \wedge G_{t-n}) \wedge (S^n \wedge S^n) \tag{9.2}$$

by $\alpha([x, y], [w, y']) = [[x, w], [y, y']]$. Since F_{s-n} and G_{t-n} are countable CW-complexes both

$$(F_{s-n} \wedge S^n) \times (G_{t-n} \wedge S^n) \quad \text{and} \quad (F_{s-n} \wedge G_{t-n}) \wedge (S^n \wedge S^n)$$

inherit a CW-structure. Now we may check that α is continuous by checking that its restriction to each cell is continuous.

We have the product action of $\mathcal{O}_n \times \mathcal{O}_n$ on $S^n \wedge S^n$ and $\zeta_{n,n} : S^n \wedge S^n \to S^{2n}$ in (1.7) is $\lambda_{n,n}$-equivariant. The composition

$$\begin{aligned}
&(\mathrm{E}(\mathcal{O}_n) \times (F_{s-n} \wedge S^n) \times (\mathrm{E}(\mathcal{O}_n) \times (G_{t-n} \wedge S^n))\\
&\xrightarrow{\alpha'} E(\mathcal{O}_n) \times E(\mathcal{O}_n) \times ((F_{s-n} \wedge G_{t-n}) \wedge (S^n \wedge S^n))\\
&\xrightarrow{\nu_n \times (\eta_{s-n,t-n} \wedge \zeta_{n,n})} E(\mathcal{O}_{2n}) \times (D_{s+t-2n} \wedge S^{2n}),
\end{aligned}$$

where $\alpha'(e, [x, y], e', [w, y']) = (e, e', [x, w], [y, y'])$, is continuous since α in (9.2) is continuous and is $\lambda_{n,n}$-equivariant. Let

$$''\eta^n_{s,t} : F^n_s \times G^n_t \to D^{2n}_{s+t} \tag{9.3}$$

denote the orbit mapping. This in turn yields

$$'\eta^n_{s,t} : F^n_s \blacktriangle G^n_t \to D^{2n}_{s+t} \tag{9.4}$$

and we have a mapping of sectioned fibrations

$$('\eta^n_{s,t}, \mu_n) : \mathfrak{F}^n_s \blacktriangle \mathfrak{G}^n_t \to \mathfrak{D}^{2n}_{s+t}. \tag{9.5}$$

Now define

$$\tilde{\eta}^n_{s,t} : F^n_s \blacktriangle G^n_t \to \Omega(D^{2n}_{s+t+1}) \tag{9.6}$$

to be the composition

$$F^n_s \wedge G^n_t \xrightarrow{'\eta^n_{s,t}} D^{2n}_{s+t} \xrightarrow{\delta^{2n}_{s+t}} \Omega(D^{2n}_{s+t+1}) \xrightarrow{(-1)^{sn}} \Omega(D^{2n}_{s+t+1}).$$

By a direct calculation we have

(9.7) LEMMA. *The collection of mappings* $\tilde{\eta}^n_{s,t}$ *constitute a* μ_n*-pairing* $\tilde{\eta}^n : (\mathbf{F}^n, \mathbf{G}^n) \to \Omega(\mathbf{G}^{2n})$.

Now the composition

$$h(\ \ ;\mathbf{F}^n) \otimes h(\ \ ;\mathbf{G}^n) \xrightarrow{\tilde{\eta}^n_\#} h(\ \ ;\Omega(\mathbf{D}^{2n})) \xrightarrow{(\delta^{2n}_\#)^{-1}} h(\ \ ;\mathbf{D}^{2n})$$

is a η_n-pairing of cohomology theories

$$\eta^n_\# : h(\ \ ;\mathbf{F}^n) \otimes h(\ \ ;\mathbf{G}^n) \to h(\ \ ;\mathbf{D}^{2n}). \tag{9.8}$$

By a tedious but entirely straightforward calculation we see that the pairings $(-1)^{n(n+1)/2}\eta^n_\#$ commute with the direct limit maps. Now let $\eta_\#$ in (9.1) be the direct limit of these pairings

10. The equivalence ψ

Let $\mathbf{F} = \{F_k;\, \varepsilon_k\}$ be a point spectrum and form the specturm $\mathbf{F} \wedge S^1$ whose *k-th* space is $F_{k-1} \wedge S^1$ and whose connecting maps

$$\varepsilon^1_k : F_{k-1} \wedge S^1 \to \Omega(F_k \wedge S^1)$$

are the composition

$$F_{k-1} \wedge S^1 \xrightarrow{\varepsilon_{k-1} \wedge 1} \Omega(F_k) \wedge S^1 \xrightarrow{\varkappa^{1,0}} \Omega(F_k \wedge S^1).$$

Define $\theta : \mathbf{F} \to \Omega(\mathbf{F} \wedge S^1)$ by letting θ_k be

$$F_k \xrightarrow{\tau} \Omega(F_k \wedge S^1) \xrightarrow{(-1)^k} \Omega(F_k \wedge S^1),$$

where τ corresponds to the identity mapping $1 : F_k \wedge S^1 \to F_k \wedge S^1$. We have

$$\theta_\# : \pi_m(\mathbf{F}) \to \pi_m(\Omega(\mathbf{F} \wedge S^1)). \tag{10.1}$$

(10.2) LEMMA. $\theta_\#$ *is an isomorphism.*

Proof. Define $\varepsilon^\wedge : \mathbf{F} \wedge S^1 \to \mathbf{F}$ by letting $\varepsilon^\wedge_k : F_{k-1} \wedge S^{-1} \to F_k$ correspond to $(-1)^{k-1}\varepsilon_k$. We have a homotopy commutative diagram of spectra

$$\begin{tikzcd} \mathbf{F} \arrow[d, "\varepsilon"'] \arrow[dr, "\theta"] & \mathbf{F} \wedge S^1 \arrow[l, "\varepsilon^\wedge"'] \arrow[d, "\varepsilon^1"] \\ \Omega(\mathbf{F}) & \Omega(\mathbf{F} \wedge S^1). \arrow[l, "\Omega(\varepsilon^\wedge)"'] \end{tikzcd}$$

Since $\varepsilon_\#$ and $\varepsilon^1_\#$ are isomorphisms, so is $\theta_\#$.

Now, for $(X, A, f) \in \mathcal{P}^*(B(\mathcal{O}_n))$, consider

$$\theta^n_\# : h^m(X, A, f; \mathbf{F}^n) \to h^m(X, A, f; \Omega(\mathbf{F}^{n+1})) \tag{10.3}$$

as in Section 8.

(10.4) LEMMA. $\theta^n_\#$ *is an isomorphism.*

Proof. The preceding lemma implies that

$$\theta_{\#}^{n} : h^{m}(X, f; \mathbf{F}^{n}) \to h^{m}(X, f; \Omega(\mathbf{F}^{n+1}))$$

is an isomorphism when X is a point. Now apply Theorem (3.8).

Let $\mathcal{P}^*$ denote the category of finite CW-pairs. The spectrum $\mathbf{F}$ determines a cohomology theory $j(\ \ ; \mathbf{F})$ on $\mathcal{P}^*$ as in [12]. We have

$$(10.5) \qquad j^{m}(X, A; \mathbf{F}) = \pi_{-n}(X, A, \mathbf{F})),$$

where $\mathfrak{M}(X, A, \mathbf{F})$ is the spectrum whose *k-th* space is $\mathfrak{M}(X, A, F_k)$ and whose connecting maps are

$$(10.6) \quad \mathfrak{M}(X, A, F_k) \xrightarrow{\mathfrak{M}(\varepsilon_k)} \mathfrak{M}(X, A, \Omega(F_{k+1})) \xrightarrow{\rho} \Omega(\mathfrak{M}(X, A, F_{k+1})).$$

The spectrum $\mathbf{F}$ also determines a cohomology theory $H(\ \ ; \mathbf{F})$ on $\mathcal{P}^*(B(\mathcal{O}))$ as in Section 8, and we have an embedding

$$i_0 : \mathcal{P}^* \to \mathcal{P}^*(B(\mathcal{O}))$$

by $(X, A) \to (X, A, \bar{b}_0)$, $\bar{b}_0$ the constant mapping of X to b_0. Let $\bar{H}(\ \ ; \mathbf{F})$ denote the cohomology theory on $\mathcal{P}^*$ induced by i_0 from $H(\ \ ; \mathbf{F})$. We will define an equivalence

$$(10.7) \qquad \psi : j(\ \ ; \mathbf{F}) \to \bar{H}(\ \ ; \mathbf{F}).$$

Recall that $F_k^0 = E(\mathcal{O}_0) \times (F_k \wedge S^0) = B(\mathcal{O}_0) \times F_k$ so that

$$\mathfrak{F}_k^0 = (\mathrm{B}(\mathcal{O}_0) \times F_k, \mathrm{B}(\mathcal{O}_0), p_k^0, \Delta_k^0)$$

where p_k^0 is projection onto the first factor and $\Delta_k^0(b) = (b, \mathrm{x}_k)$, $b \in B(\mathcal{O}_0)$. We have a homeomorphism of function spaces

$$(10.8) \qquad \bar{\psi}_k : \mathfrak{M}(X, A, F_k) \to \mathfrak{L}(X, A, \bar{b}_0, \mathfrak{F}_k^0)$$

by $\bar{\psi}_k(g)(x) = (b_0, g(x))$, $x \in X$. The collection of these $\bar{\psi}_k$ is a map of point spectra

$$(10.9) \qquad \bar{\psi} : \mathfrak{M}(X, A, \mathbf{F}) \to \mathfrak{L}(X, A, \bar{b}_0; \mathbf{F}^0)$$

which yields an isomorphism

$$(10.10) \qquad \bar{\psi}_{\#} : j^{n}(X, A; \mathbf{F}) \to h^{n}(X, A, \bar{b}_0; \mathbf{F}^0).$$

Now define

$$(10.11) \qquad \psi^{n} : j^{n}(X, A; \mathbf{F}) \to H^{n}(X, A, \bar{b}_0; \mathbf{F})$$

to be the composition

$$j^{n}(X, A; \mathbf{F}) \xrightarrow{\bar{\psi}_{\#}} h^{n}(X, A, \bar{b}_0; \mathbf{F}^0) \to H^{n}(X, A, \bar{b}_0; \mathbf{F}),$$

the latter being inclusion into the direct limit which by (10.4) is an isomorphism. Now let ψ in (10.7) be the collection $\{\psi^n\}$.

Note. Hereafter, when we speak of a point spectrum $\mathbf{F}$, it will be understood that each F_k is a countable CW-complex. This is not a severe restriction for, by a theorem of E. H. Brown [2], every cohomology theory on $\mathcal{P}^*$ with countable coefficient group is equivalent to one of the form $j(\;;\mathbf{F})$ where $\mathbf{F}$ is a spectrum of this type.

11. Commutativity

A point spectrum $\mathbf{F}$ is a *ring spectrum* if there is a map $\omega : \mathbf{S} \to \mathbf{F}$ and a pairing $\eta : (\mathbf{F}, \mathbf{F}) \to \mathbf{F}$ such that the diagrams

(11.1)
$$\begin{tikzcd} S^s \wedge F_t \arrow[r, "\omega_s \wedge 1"] \arrow[dr, "\lambda^1_{s,t}"'] & F_s \wedge F_t \arrow[d, "\eta_{s,t}"] & F_s \wedge F_t \arrow[l, "1 \wedge \omega_t"'] \arrow[dl, "\lambda^2_{s,t}"] \\ & F_{s+t} & \end{tikzcd}$$

$$\begin{tikzcd} F_s \wedge F_t \arrow[r, "\eta_{s,t}"] \arrow[d, "\alpha"] & F_{s+t} \arrow[r, "\varepsilon_{s+t}"] & \Omega(F_{s+t+1}) \arrow[d, "(-1)^{st}"] \\ F_t \wedge F_s \arrow[r, "\eta_{t,s}"] & F_{s+t} \arrow[r, "\varepsilon_{s+t}"] & \Omega(F_{s+t+1}) \end{tikzcd}$$

are commutative, $-\infty < s, t < \infty$. Here $\alpha[x, x'] = [x', x]$ and $\lambda^1 : (\mathbf{F}, \mathbf{S}) \to \mathbf{F}$, $\lambda^2 : (\mathbf{S}, \mathbf{F}) \to \mathbf{F}$ are the natural pairings. The commutativity of the second diagram implies that the induced pairing

(11.2) $$\eta_\# : j(\;;\mathbf{F}) \otimes j(\;;\mathbf{F}) \to j(\;;\mathbf{F})$$

is commutative in the graded sense. That is,

$$u \cup v = (-1)^{st} v \cup u, \quad u \in j^s(X, A; \mathbf{F}), \quad v \in j^t(X, A; \mathbf{F}).$$

The purpose of this section is to discuss the commutativity properties of the induced μ-pairing

(11.3) $$\eta_\# : H(\;;\mathbf{F}) \otimes H(\;;\mathbf{F}) \to H(\;;\mathbf{F}).$$

Firse we define an involution

(11.4) $$T : H(\;;\mathbf{F}) \to H(\;;\mathbf{F}),$$

(a natural transformation such that $T^2 = 1$) as follows. Let

$$T^n_k : F^n_k \to F^n_k$$

be the orbit map of

$$E(\mathcal{O}_n) \times (F_{k-n} \wedge S^n) \xrightarrow{1 \times (1 \wedge \tau^n)} E(\mathcal{O}_n) \times (F_{k-n} \wedge S^n)$$

where $\tau^n : S^n \to S^n$ is the compactification of the involution $R^n \to R^n$ by

$y \to -y$. The collection of maps T_k^n is a map of spectra

$$(11.5) \qquad T^n : \mathbf{F}^n \to \mathbf{F}^n.$$

We calculate directly that the transformations

$$(11.6) \qquad (-1)^n T_{\#}^n : h(\ \ ;\mathbf{F}^n) \to h(\ \ ;\mathbf{F}^n)$$

commute with the direct limit maps. Let T in (11.4) be the direct limit of the collection $\{(-1)^n T_{\#}^n\}$. Since $(T^n)^2 = 1$ we have $T^2 = 1$.

(11.7) PROPOSITION. *Let $u \in H_m(X, A, f; \mathbf{F})$. Then $T(u) = u$ modulo 2-torsion.*

Proof. The proof is broken up into several steps.

(a) *The proposition is true when $f = \bar{b}_0$.* We have a commutative diagram

$$\begin{array}{ccc} h^m(X, A, \bar{b}_0; \mathbf{F}^0) & \xrightarrow{T_{\#}^0} & h^m(X, A, \bar{b}_0; \mathbf{F}^0) \\ \downarrow & & \downarrow \\ H^m(X, A, \bar{b}_0; \mathbf{F}) & \xrightarrow{T} & H^m(X, A, \bar{b}_0; \mathbf{F}) \end{array}$$

where the vertical maps are inclusion into the direct limit and by (10.4) are isomorphisms. Since T^0 is the identity map, so is T.

(b) *The proposition is true when f is homotopic to $\bar{b}_0$.* Let $M : X \times I \to B(\mathfrak{O})$ be such that $M_0 = \bar{b}_0$ and $M_1 = f$. Since T is natural we have a commutative diagram

$$\begin{array}{ccc} H^m(X, A, f; \mathbf{F}) & \xrightarrow{T} & H^m(X, A, f; \mathbf{F}) \\ \downarrow M_{\#} & & \downarrow M_{\#} \\ H^m(X, A, \bar{b}_0; \mathbf{F}) & \xrightarrow{T} & H^m(X, A, \bar{b}_0; \mathbf{F}) \end{array}$$

From part (a) we have $M_{\#} T(u) = TM_{\#}(u)$. Since $M_{\#}$ is an isomorphism we have $T(u) = u$.

(c) *If the proposition is true for any two of (X, A, f), (X, f) and (A, f) then it is true for the third.* We have the exact sequence

$$\cdots \xrightarrow{i^*} H^{m-1}(A, f; \mathbf{F}) \xrightarrow{d} H^m(X, A, f; \mathbf{F}) \xrightarrow{j^*} H_m(X, f; \mathbf{F}) \xrightarrow{i^*} \cdots$$

To be specific, suppose it is true for (X, f) and (A, f). Let $u \in H_m(X, A, f; \mathbf{F})$. We have

$$j^*(T(u) - u) = T(j^*(u)) - j^*(u) = 0 \quad \text{modulo 2-torsion.}$$

Hence, there is an integer q such that

$$j^*(2^q(T(u) - u)) = 0.$$

Let $v \in H^{m-1}(A, f; \mathbf{F})$ be such that $d(v) = 2^q(T(u) - u)$. Then

$$d(T(v) - v) = Td(v) - d(v) = 2^{q+1}(u - T(u)).$$

There is an integer r such that $2^r(T(v) - v) = 0$. Then

$$0 = d(2^r(T(v) - v)) = 2^{r+q+1}(u - T(u)).$$

Therefore $T(u) = u$ modulo 2-torsion.

(d) *The proposition is true when A is empty.* Proceed by induction. Let X have dimension s and let $g : I^s \to X$ be a characteristic map for an s-cell of X and let X_1 be the closure of $X - g(I^s)$. We may assume by induction that the proposition is true for (X_1, f). By excision and part (b), the proposition is true for (X, X_1, f). Now apply part (c).

(e) *The proposition is true for arbitrary $(X, A, f) \in \mathcal{P}^*(B(\mathcal{O}))$.* Apply part (c).

(11.8) LEMMA. *For*

$$(X, A, f) \in B(\mathcal{O}_n), \quad u \in h^s(X, A, f; \mathbf{F}^n) \quad \textit{and} \quad v \in h^t(X, A, f; \mathbf{F}^n)$$

we have

$$u \cup v = (-1)^{st+n} v \cup T^n_{\#}(u) = (-1)^{st+n} T^n_{\#}(v) \cup u$$

in $h^{s+t}(X, A, \mu(f \times f); \Omega(\mathbf{F}^{2n}))$.

Proof. From (11.1) there is a homotopy

$$(11.9) \qquad M_r : F_{j-n} \wedge F_{k-n} \to \Omega(F_{j+k+1-2n}), \qquad 0 \leq r \leq 1,$$

such that

$$M_0 = \varepsilon_{j+k-2n}\, \eta_{j-n,k-n} \quad \text{and} \quad M_1 = (-1)^{j-n,k-n} \varepsilon_{j+k-2n}\, \eta_{k-n,j-n}\, \alpha.$$

Now define

$$(11.10) \qquad C_r : S^n \wedge S^n \to S^{2n}, \qquad 0 \leq r \leq 1,$$

by $C_r[z_n, z_n] = z_{2n}$ and

$$C_r[y, y'] = ((1 - r)y + ry', -ry + (1 - r)y'), \qquad y, y' \in R^n.$$

It is easily checked that

$$(11.11) \qquad \lambda_{n,n}(W, W)C_r = C_r \cdot (W, W), \qquad W \in \mathcal{O}_n.$$

Consider a point $[q, q']$ in the internal reduced join $F^n_j \mathbin{\bar{\blacktriangle}} F^n_k$. Since $p^n_j(q) = p^n_k(q')$ we can write

$$q = [e, x, y], q' = [e, x', y']$$

where $e \in E(\mathcal{O}_n)$, $x \in F_{j-n}$, $x' \in F_{k-n}$, $y, y' \in S^n$. Now define

$$(11.12) \qquad K_r : F^n_j \mathbin{\bar{\blacktriangle}} F^n_k \to \Omega(F^{2n}_{j+k+1}), \qquad 0 \leq r \leq 1$$

by $K_r[q, q'](t) = [\nu_n(e, e), M_r[x, x'](t), C_r[y, y']]$. Using (11.11) we see that the definition of K_r is independent of the choice of e in the representation of q and q'. Thus K_r is well defined. We have

$$\begin{aligned} K_0[q, q'] &= [\nu_n(e, e), \varkappa^{1,0}[\varepsilon_{j+k-2n}\, \eta_{j-n,k-n}[x, x'], \zeta_{n,n}[y, y']]] \\ &= \varepsilon^{2n}_{j+k}{}' \eta^n_{j,k}[q, q'], \end{aligned}$$

and

$$K_1[q, q'] = (-1)^{j-n,k-n}[\nu_n(e, e)\varkappa^{1,0}[\varepsilon_{j+k-2n}\,\eta_{k-n,j-n}[x', x]\zeta_{n,n}[y', -y]]]$$
$$= (-1)^{j-n,k-n}\varepsilon^{2n}_{j+k}\,'\eta^{n}_{k,j}[q', T^n_j(q)].$$

Therefore the diagram

$$\begin{tikzcd}
F^n_j \,\bar{\wedge}\, F^n_k \arrow[r, "\tilde{\eta}^n_{j,k}"] \arrow[d, "\alpha"] & \Omega(F^{2n}_{j+k+1}) \arrow[dd, "(-1)^{jk+n}"] \\
F^n_k \,\bar{\wedge}\, F^n_j \arrow[d, "1\,\bar{\wedge}\,T^n_j"] & \\
F^n_k \,\bar{\wedge}\, F^n_j \arrow[r, "\tilde{\eta}^n_{k,j}"] & \Omega(F^{2n}_{j+k+1})
\end{tikzcd} \tag{11.13}$$

is homotopy commutative by a fibre and cross-section preserving homotopy (see (9 6)).

Now let

$$(X, A, f) \in \mathcal{P}^*(B(\mathcal{O}_n)), \qquad u_1 \in \pi_{-s+j}(\mathcal{L}(X, A, f; \mathfrak{F}^n_j)),$$

and

$$v_1 \in \pi_{-t+k}(\mathcal{L}(X, A, f; \mathfrak{F}^n_k)).$$

From the homotopy commutativity of (11.13) we deduce that

$$\begin{aligned}(-1)^{tj}\mathcal{L}^\wedge(\tilde{\eta}^n_{j,k})_*(u_1 \otimes v_1) \\ = (-1)^{st+n}((-1)^{sk}\mathcal{L}(\tilde{\eta}^n_{k,j})_*(v_1 \otimes T^n_\#(u_1))).\end{aligned} \tag{11.14}$$

If u_1 represents $u \in h^s(X, A, f; \mathbf{F}_n)$ and v_1 represents $v \in h^t(X, A, f; \mathbf{F}^n)$, the above equation yields

$$u \cup v = (-1)^{st+n} v \cup T^n_\#(u). \tag{11.15}$$

To show that $u \cup v = (-1)^{st+n}T^n_\#(v) \cup u$, replace C_r in (11.10) by C'_r where $C'_r[z_n, z_n] = z_{2n}$ and

$$C'_r[y, y'] = ((1 - r)y - ry', ry + (1 - r)y'), \qquad y, y' \in R^n,$$

and proceed in the same manner.

Passing to the direct limit we have, by the preceding lemma,

(11.16) THEOREM. *For*

$$(X, A, f) \in B(\mathcal{O}), \quad u \in H^s(X, A, f; \mathbf{F}), \quad v \in H^t(X, A, f; \mathbf{F}),$$

we have

$$u \cup v = (-1)^{st} v \cup T(u) = (-1)^{st}T(v) \cup u.$$

Combining this with (11.7) yields

(11.17) COROLLARY. *For*

$$(X, A, f) \in B(\mathcal{O}), \quad u \in H^s(X, A, f; \mathbf{F}), \quad v \in H^t(X, A, f; \mathbf{F}),$$

we have

$$u \cup v = (-1)^{st} v \cup u \quad \textit{modulo 2-torsion.}$$

12. Thom classes

Let $\mathcal{Y} = (Y, X, r)$ be an $(n - 1)$-sphere bundle with structural group $\mathcal{O}_n$. The *suspension* of $\mathcal{Y}$ is the n-sphere bundle $\Sigma(\mathcal{Y}) = (\Sigma(Y), X, \Sigma(r))$ where $\Sigma(Y) = Y \times I/(\sim)$, the relation $\sim$ defined by

$$(y, t) \sim (y', t) \quad \text{if } r(y) = r(y') \quad \text{and} \quad t = 0 \text{ or } 1.$$

We have $\Sigma(r)[y, t] = r(y)$. There are two cross-sections

$$\zeta_i : X \to \Sigma(Y) \quad \text{by} \quad \zeta_i(x) = [r^{-1}(x), i], \qquad i = 0, 1.$$

We will denote $\zeta_1(X)$ by X^+ and $\zeta_0(X)$ by X^-. We will also identify Y with $Y \times \{\frac{1}{2}\} \subset \Sigma(Y)$. The *Thom space* of $\mathcal{Y}$ is the pair $(\Sigma(Y), X_+)$.

For $x \in X$, let $i_x : (S^n, z_n) \to (\Sigma(r)^{-1}(x), \zeta_1(x))$ be an identification. Let $\mathbf{F}$ be a ring spectrum with map $\omega : \mathbf{S} \to \mathbf{F}$ and pairing $\eta : (\mathbf{F}, \mathbf{F}) \to \mathbf{F}$. For $f : X \to B(\mathcal{O})$ consider the sequence

$$(12.1) \quad H^n(\Sigma(Y), X^+, f\Sigma(r); \mathbf{F} \xrightarrow{i_x^*} H^n(S^n, z_n; \mathbf{F}) \xleftarrow{\omega_\#} H^n(S^n, z_n; \mathbf{S}).$$

A *Thom class* for the pair $(\mathcal{Y}, f)$ relative to $\mathbf{F}$ is an element

$$(12.2) \qquad u \in H^n(\Sigma(Y), X^+, f\Sigma(r); \mathbf{F})$$

with the property that for each $x \in X$ there is a generator

$$v_0 \in H^n(S^n, z_n; \mathbf{S})$$

such that $i_x^*(x) = \omega_\#(v_0)$. It is clear that the existence of a Thom class for f depends only on the homotopy class of f.

(12.3) THEOREM. *If $f : X \to B(\mathcal{O})$ is homotopic to a classifying map for $\mathcal{Y}$ then there exists a Thom class for $(\mathcal{Y}, \mathrm{f})$.*

Proof. We may assume that $f(X) \subset B(\mathcal{O}_n)$. Further, it is sufficient to prove the theorem when $\mathbf{F} = \mathbf{S}$ since, if u is a Thom class for $\mathbf{S}$ then $\omega_\#(u)$ will be a Thom class for $\mathbf{F}$.

Let $\mathcal{O}_n$ act on $\Sigma(S^{n-1})$ by $W[x, t] = [Wx, t]$, $W \in \mathcal{O}_n$. Recall the action of $\mathcal{O}_n$ on S^n described in Section 8. Evidently, we can construct an equivariant homeomorphism

$$q : (\Sigma(S^{n-1}), +) \to (S^n, z_n).$$

Here $+$ denotes the point $S^{n-1} \times \{1\}$. Now let

$$(12.4) \qquad \tilde{f} : Y \to E(\mathcal{O}_n) \times S^{n-1}/\mathcal{O}_n$$

be a bundle map covering f and let $\hat{f}$ be the composition

$$\Sigma(Y) \xrightarrow{\Sigma(\tilde{f})} \Sigma(E(\mathcal{O}_n) \times S^{n-1}/\mathcal{O}_n)$$
$$= E(\mathcal{O}_n) \times \Sigma(S^{n-1})/\mathcal{O}_n \xrightarrow{[1 \times q]} E(\mathcal{O}_n) \times S^n/\mathcal{O}_n.$$

For $\mathbf{S}$ we have the sectioned fibration $\mathcal{S}_n^n$ as in (8.1). Here

$$\mathcal{S}_n^n = (\mathrm{E}(\mathcal{O}_n) \times (S^0 \wedge S^n)/\mathcal{O}_n, \mathrm{B}(\mathcal{O}_n), p_n^n, \Delta_n^n)$$
$$= (\mathrm{E}(\mathcal{O}_n) \times S^n/\mathcal{O}_n, \mathrm{B}(\mathcal{O}_n), p_n^n, \Delta_n^n).$$

Thus,

$$\hat{f} \in \mathcal{L}(\Sigma(Y), X^+, f\Sigma(r), \mathcal{S}_n^n). \tag{12.5}$$

Now let u be the image of $[\hat{f}]$ under the composition

$$\pi_0(\mathcal{L}(\Sigma(Y), X^+, f\Sigma(r); \mathcal{S}_n^n)) \to h^n(\Sigma(Y), X^+, f\Sigma(r); \mathbf{S}^n)$$
$$\to H^n(\Sigma(Y), X^+, f\Sigma(r); \mathbf{S}),$$

each map being inclusion into the direct limit. A representative of $i_x^*(u)$ is

$$S^n \xrightarrow{i_x} \Sigma(r)^{-1}(x) \xrightarrow{\hat{f}} (p_n^n)^{-1}(f(x)) = S^n,$$

which, being a homeomorphism, represents a generator of

$$\pi_0(\mathfrak{M}(S^n, z_n; S^n)).$$

It follows that u is a Thom class for $(\mathcal{Y}, f)$ relative to $\mathbf{S}$.

Let $\mathbf{D}$ be an $\mathbf{F}$-module. That is, there is a pairing $\theta : (\mathbf{F}, \mathbf{D}) \to \mathbf{D}$ such that the diagram

$$\begin{tikzcd}
S^s \wedge D_t \arrow[d, "\omega_s \wedge 1"'] \arrow[dr, "\lambda^2_{s,t}"] & \\
F_s \wedge D_t \arrow[r, "\eta_{s,t}"'] & D_{s,t}
\end{tikzcd} \tag{12.6}$$

is homotopy commutative, $-\infty < s, t < \infty$. For example, every spectrum $\mathbf{D}$ with the natural pairing $\lambda^2 : (\mathbf{S}, \mathbf{D}) \to \mathbf{D}$ is an $\mathbf{S}$-module.

Now suppose that $(\mathcal{Y}, f)$ has a Thom class u relative to $\mathbf{F}$. For $X_1 \subset X$ let $Y_1 = r^{-1}(X_1)$ and for $g : X \to B(\mathcal{O})$ define

$$\Phi_u : H^k(X, X_1, g; \mathbf{D}) \to H^{n+k}(\Sigma(Y), X^+ \cup \Sigma(Y_1), \mu(f \times g)\Sigma(r); \mathbf{D}) \tag{12.7}$$

by

$$\Phi_u(v) = u \cup \Sigma(r)^*(v).$$

(12.8) THEOREM (Thom isomorphism theorem). *The mapping Φ_u is an isomorphism.*

Proof. The method of proof is the same as that of (11.7).

(a) *The theorem is true when $f = g = \bar{b}_0$.* Since the transformation ψ in

(10.7) preserves pairings we have a commutative diagram

$$\begin{array}{ccc} j^k(X, X_1; \mathbf{D}) & \xrightarrow{\Phi'} & j^{n+k}(\Sigma(Y), X^+ \cup \Sigma(Y_1); \mathbf{D}) \\ \downarrow \psi & & \downarrow \psi \\ H^k(X, X_1, \bar{b}_0; \mathbf{D}) & \xrightarrow{\Phi_u} & H^{n+k}(\Sigma(Y), Y^+ \cup \Sigma(Y_1)\, \bar{b}_0; \mathbf{D}) \end{array}$$

where $\Phi'(v) = \psi^{-1}(u) \cup \Sigma(r)^*(v)$. Now Dold [3] has established a Thom isomorphism theorem for cohomology theories on $\mathcal{P}^*$. Clearly $\psi^{-1}(u)$ is a Thom class in Dold's sense so that Φ' is an isomorphism. Therefore Φ_u is an isomorphism.

(b) *The theorem is true when* f and g are homotopic to $\bar{b}_0$. Let

$$M, N : X \times I \to B(\mathcal{O})$$

be such that $M_0 = \bar{b}_0, M_1 = f$ and $N_0 = \bar{b}_0, N_1 = g$. We have

$$\Sigma(Y) \times I \xrightarrow{\Sigma(Y) \times 1} X \times I \xrightarrow{M} B(\mathcal{O}).$$

Let $u' = (M(\Sigma(r)) \times 1))_{\#}(u)$. By (6.17) we have a commutative diagram

$$\begin{array}{ccc} H^k(X, X_1, f; \mathbf{D}) & \xrightarrow{\Phi_u} & H^{n+k}(\Sigma(Y), X^+ \cup \Sigma(Y_1), \mu(f \times g)\Sigma(r); \mathbf{D}) \\ \downarrow F_{\#} & & \downarrow \mu(M \times N)(\Sigma(r) \times 1)_{\#} \\ H^k(X, X_1\, \bar{b}_0; \mathbf{D}) & \xrightarrow{\Phi_{u'}} & H^{n+k}(\Sigma(Y), X^+ \cup \Sigma(Y_1), \bar{b}_0; \mathbf{D}). \end{array}$$

Clearly, u' is a Thom class for $(\mathcal{Y}, \bar{b}_0)$. Now apply part (a).

(c) *If the theorem is true for any two of* (X, X_1, g), (X, g) *and* (X_1, g) *then it is true for the third.*

We have a commutative diagram

$$\begin{array}{ccccccc} \cdots \xrightarrow{d} & H^{n+k}(\Sigma(Y), X^+ \cup \Sigma(Y_1), \tau) & \xrightarrow{j^*} & H^{n+k}(\Sigma(Y), X^+, \tau; \mathbf{D}) & \xrightarrow{i^*} & H^{n+k}(\Sigma(Y_1), X_1^+, \tau; \mathbf{D}) & \xrightarrow{d} \cdots \\ & \uparrow \Phi_u & & \uparrow \Phi_u & & \uparrow \Phi_{u'} & \\ \cdots \xrightarrow{d} & H^k(X, X_1, f;) & \xrightarrow{j^*} & H^k(X, f; \mathbf{D}) & \xrightarrow{i^*} & H^k(X_1, f; \mathbf{D}) & \xrightarrow{d} \cdots \end{array}$$

where

$$u' = i^*(u) \in H^n(\Sigma(Y_1), X_1^+, f\Sigma(t); \mathbf{D})$$

and τ denotes the map $\mu(f \times g)\Sigma(r)$. Now use the "Five Lemma."

Steps (d) and (e) are precisely the same as in (11.7). This completes the proof.

(12.9) *Remark.* MacAlpine [9] has defined, for an $(n-1)$-sphere bundle $\mathcal{Y} = (Y, X, r)$ and point spectrum $\mathbf{D}$, the *k-th* cohomology group $j^k(X; \mathbf{D}(\mathcal{Y}))$ of X with coefficients in $\mathbf{D}$ twisted by $\mathcal{Y}$ to be

$$j^k(X; \mathbf{D}(\mathcal{Y})) = j^{n+k}(\Sigma(Y), X^+; \mathbf{D}).$$

Now let $\mathcal{Y}^*$ be an inverse for $\mathcal{Y}$. That is, $\mathcal{Y} \circ \mathcal{Y}^*$ is trivial. Let

$$f, f^* : X \to B(\mathcal{O})$$

be classifying maps for $\mathcal{Y}$, $\mathcal{Y}^*$ respectively. Then $\mu(f \times f^*)$ is homotopic to $\bar{b}_0$. Let $M : X \times I \to B(\mathcal{O})$ be such that $M_0 = \bar{b}_0$ and $M_1 = \mu(f \times f^*)$. Let u be a Thom class for $(\mathcal{Y}, f)$ relative to $\mathbf{S}$. We have identifications

$$H^k(X, f^*; \mathbf{D}) \xrightarrow{\Phi_u} H^{n+k}(\Sigma(Y), X^+, \mu(f \times f^*)\Sigma(r); \mathbf{D})$$
$$\cdot \xrightarrow{M(\Sigma(r) \times 1)_\#} H^{n+k}(\Sigma(Y), X^+, \bar{b}_0; \mathbf{D}) \xrightarrow{\Psi^{-1}} j^{n+k}(\Sigma(Y), X^+; \mathbf{D}).$$

Thus, MacAlpine's group $j^k(X; \mathbf{D}(\mathcal{Y}))$ is our group $H^k(X, f^*; \mathbf{D})$, where f^* is a classifying map for an inverse $\mathcal{Y}^*$ of $\mathcal{Y}$.

13. Euler classes

Suppose that the pair $(\mathcal{Y}, f)$ has a Thom class

$$u \in H^n(\Sigma(Y), X^+, f\Sigma(r); \mathbf{F}).$$

The cross-section $\zeta_0 : X \to \Sigma(Y)$ yields

(13.1) $$\zeta_0^* : H^n(\Sigma(Y), X^+, f\Sigma(r); \mathbf{F}) \to H^n(X, f; \mathbf{F}).$$

The element $\mathfrak{X} = \zeta_0^*(u)$ will be called an *Euler class* for $(\mathcal{Y}, f)$ relative to $\mathbf{F}$. For example, if $\mathcal{Y}$ is orientable in the ordinary sense (i.e., $w_1 = 0$), $f = \bar{b}_0$ and $\mathbf{F} = \mathbf{K}(Z)$, then $\mathfrak{X}$ is the usual integral Euler class. A number of properties of the integral Euler class hold true in general and the proofs are essentially the same. We will now list them.

(13.2) *For* $r^* : H^n(X, f; \mathbf{F}) \to H^n(Y, rf; \mathbf{F})$ *we have* $r^*(\mathfrak{X}) = 0$.

(13.3) $\Phi_u(\mathfrak{X}) = u \cup u \in H^n(\Sigma(Y), X, \mu(f \times f); \mathbf{F})$.

(13.4) *If n is odd then* $\mathfrak{X} = 0$ *modulo 2-torsion.*

(13.5) *Suppose* $(\tilde{g}, g) : \mathcal{Y}_1 \to \mathcal{Y}_2$ *is a bundle map of* $(n-1)$*-sphere bundles and* $\mathfrak{X}$ *is an Euler class for* $(\mathcal{Y}_2, f)$. *Then* $g^*(\mathfrak{X})$ *is an Euler class for* $[\mathcal{Y}_1, fg)$.

(13.6) *Suppose* $\mathcal{Y}_1$, $\mathcal{Y}_2$ *are* $(n_1 - 1)$ *and* $(n_2 - 1)$*-sphere bundles over* X *respectively. Let* $\mathfrak{X}_1$, $\mathfrak{X}_2$ *be Euler classes for* $(\mathcal{Y}_1, f_1)$ *and* $(\mathcal{Y}_2, f_2)$ *respectively. Then*

$$\mathfrak{X}_1 \cup \mathfrak{X}_2 \in H^{n_1+n_2}(X, \mu(f_1 \times f_2); \mathbf{F})$$

is an Euler class for $(\mathcal{Y}_1 \circ \mathcal{Y}_2, \mu(f_1 \times f_2))$.

Here $\mathcal{Y}_1 \circ \mathcal{Y}_2$ denotes the Whitney join of $\mathcal{Y}_1$ and $\mathcal{Y}_2$.

The proof of (13.2) and of (13.5) is left to the reader. For the proof of (13.3) we must show that

$$(13.7) \qquad u \cup u = u \cup \Sigma(r)^* \zeta_0^*(u).$$

Consider the inclusions

$$(13.8) \quad (\Sigma(Y), X^+) \xrightarrow{i_1} (\Sigma(Y), \Sigma^+(Y)) \xleftarrow{i_2} (\Sigma_-(Y), Y) \xleftarrow{j} \Sigma_-(Y),$$

where $\Sigma^+(Y)$ and $\Sigma_-(Y)$ are the images of $Y \times [\frac{1}{2}, 1]$ and $Y \times [0, \frac{1}{2}]$ in $\Sigma(Y)$ respectively. Evidently

$$\Sigma(r)^* \zeta_0^* : H^n(\Sigma_-(Y), f\Sigma(r); \mathbf{F}) \to H^n(\Sigma_-(Y), f\Sigma(r); \mathbf{F})$$

is the identity. Let $\hat{u} = i_2^* i_1^{*-1}(u)$. We have

$$\begin{aligned} \hat{u} \cup \Sigma(r)^* \zeta_0^*(\hat{u}) &= \hat{u} \cup j^* \Sigma(\mathrm{r})^* \zeta_0^*(\hat{u}) \\ &= \hat{u} \cup \Sigma(\mathrm{r})^* \zeta_0^* j^*(\hat{u}) \\ &= \hat{u} \cup j^*(\hat{u}) = \hat{u} \cup \hat{u}. \end{aligned}$$

Applying $i_1^* i_2^{*-1}$ to this equation yields (13.7).

By (11.17) we have

$$(13.9) \qquad u \cup u = (-1)^{n^2} u \cup u \quad \text{modulo 2-torsion.}$$

Thus, if n is odd, $2(u \cup u)$ and hence $u \cup u$ is zero modulo 2-torsion. This together with (13.3) yields (13.4).

We will now outline a proof of (13.6). The Whitney join

$$\mathcal{Y}_1 \circ \mathcal{Y}_2 = (Y_1 \circ Y_2, X, r_1 \circ r_2)$$

of $\mathcal{Y}_1 = (Y_1, X, r_1)$ and $\mathcal{Y}_2 = (Y_2, X, r_2)$ is defined as follows. Let $\mathrm{Y}_1 \circ Y_2$ be the quotient of $(Y_1 \oplus Y_2) \times I$ after the identifications

$$(y_1, y_2, 0) \sim (y_1, y_2', 0) \quad \text{and} \quad (y_1, y_2, 1) \sim (y_1', y_2, 1).$$

The map $r_1 \circ r_2$ is given by $r_1 \circ r_2[y_1, y_2, t] = r_1(y_1) = r_2(y_2)$. A bundle equivalence

$$(13.10) \qquad \gamma : \Sigma(Y_1 \circ Y_2) \to \Sigma(Y_1) \,\overline{\blacktriangle}\, \Sigma(Y_2)$$

is defined by

$$\begin{aligned} \gamma[y_1, y_2, t, \lambda] &= [y_1, 2\lambda t, y_2, \lambda], & 0 \le t \le \tfrac{1}{2} \\ &= [y_1, \lambda, y_2, 2\lambda(1-t)], & \tfrac{1}{2} \le t \le 1. \end{aligned}$$

Consider the diagram

$$(13.11)\qquad \begin{tikzcd}
 & (\Sigma(Y_1) \times \Sigma(Y_2), \Sigma(Y_1) \times X^+ \cup X^+ \times \Sigma(Y_2)) \\
 & (\Sigma(Y_1) \oplus \Sigma(Y_2), \Sigma(Y_1) \oplus X^+ \cup X^+ \oplus \Sigma(Y_2)) \arrow[u, "j"] \arrow[d, "\omega"] \\
X \arrow[uur, "\zeta_0 \times \zeta_0"] \arrow[ur, "\zeta_0 \oplus \zeta_0"] \arrow[r, "\zeta_0 \mathbin{\overline{\blacktriangle}} \zeta_0"] \arrow[dr, "\zeta_0"] \arrow[d, "\delta"] & (\Sigma(Y_1) \mathbin{\overline{\blacktriangle}} \Sigma(Y_2), X^+ \mathbin{\overline{\blacktriangle}} X^+) \\
X \times X & (\Sigma(Y_1 \circ Y_2), X^+) \arrow[u, "\gamma"]
\end{tikzcd}$$

where j is inclusion, ω is projection, and δ is the diagonal map. Let

$$(13.12)\qquad u_i \in H^{n_i}(\Sigma(Y_i), X^+, f\Sigma(r_i); \mathbf{F})$$

be a Thom class for $\mathcal{Y}_i$ such that $\mathcal{X}_i = \zeta_0^*(u_i)$, $i = 1, 2$, and let

$$(13.13)\qquad u = \gamma^*\omega^{*-1}j^*(u_1 \times u_2) \\ \in H^{n_1+n_2}(\Sigma(Y_1 \circ Y_2), X^+, \Sigma(r_1 \circ r_2)\mu(f_1 \times f_2); \mathbf{F})$$

It is easy to check that u is a Thom class for

$$(\mathcal{Y}_1 \circ \mathcal{Y}_2, \mu(f_1 \times f_2)).$$

Now, by the commutativity of (13.11)

$$(13.14)\qquad \zeta_0^*(u) = \delta^*(\zeta_0^*(u_1) \times \zeta_0^*(u_2)) = \mathcal{X}_1 \cup \mathcal{X}_2,$$

which shows that $\mathcal{X}_1 \cup \mathcal{X}_2$ is an Euler class for

$$(\mathcal{Y}_1 \circ \mathcal{Y}_2, \mu(f_1 \times f_2)).$$

Suppose $(\mathcal{Y}, f)$ has a Thom class u relative to $\mathbf{F}$ and $\mathbf{D}$ is an $\mathbf{F}$-module. Given $g : X \to B(\mathcal{O})$, let $\tau = \mu(f \times g)\Sigma(r)$ and consider the exact sequence

$$(13.15)\qquad \cdots \xrightarrow{d} H^k(\Sigma_-(y), Y, \tau; \mathbf{D}) \xrightarrow{j^*} H^k(\Sigma_-(y), \tau; \mathbf{D}) \xrightarrow{i^*} H^k(Y, \tau; \mathbf{D}) \xrightarrow{d} \cdots.$$

We have identifications

$$(13.16)\qquad H^k(X, \mu(f \times g); \mathbf{D}) \xrightarrow{\Sigma(r)^*} H^k(\Sigma_-(Y), \tau; \mathbf{D}),$$

and

$$(13.17)\qquad H^{k-n}(X, g; \mathbf{D}) \xrightarrow{\Phi_u} H^k(\Sigma(Y), X^+, \tau; \mathbf{D}) \xrightarrow{(i_1i_2)^{*-1}} H^k(\Sigma_-(\mathrm{y}), Y, \tau; \mathbf{D})$$

(see (13.8)). Substituting in (13.15) yields the exact *Gysin sequence*

$$(13.18) \quad \cdots \xrightarrow{\nu} H^{k-n}(X, g; \mathbf{D}) \xrightarrow{\Psi} H^k(X, \mu(f \times g); \mathbf{D}) \xrightarrow{r^*} H^k(Y, \mu(f \times g)r; \mathbf{D}) \xrightarrow{\nu} \cdots$$

and we check that

$$(13.19) \qquad \Psi(v) = \mathfrak{X} \cup v, \qquad v \in H^{k-n}(X, g; \mathbf{D}),$$

where $\mathfrak{X}$ is the Euler class associated with u.

(13.20) LEMMA. *If one Euler class relative to* $\mathbf{F}$ *is zero then every Euler class relative to* $\mathbf{F}$ *is zero.*

Proof. Since μ is a multiplication map for the canonical group structure on $[X, B(\mathcal{O})]$, every map $f' : X \to B(\mathcal{O})$ is homotopic to one of the form $\mu(f \times g)$. If $\mathfrak{X} = 0$, the map Ψ in the Gysin sequence is zero. Therefore

$$r^* : H^n(X, f'; \mathbf{F}) \to H^n(Y, f'r\rangle \mathbf{F})$$

is injective for every map f'. Then by (13.2) every Euler class $\mathfrak{X}' \in H^n(X, f'; \mathbf{F})$ must be zero. This completes the proof.

A necessary condition that the bundle $\mathcal{Y}$ admit a cross-section is that every Euler class $\mathfrak{X}$ relative to $\mathbf{F}$ be zero. This is a consequence of (13.2).

In a moment we will need the following well known fact.

(13.21) LEMMA. *Suppose* $\mathcal{Y}_i = (Y_i, X_i, r_i)$ *is a fibre space with fibre* F_i, $i = 1, 2$, *and* $(\bar{f}, f) : \mathcal{Y}_1 \to \mathcal{Y}_2$ *is a map such that* $\bar{f}_{\#} : \pi_k(F_1) \to \pi_k(F_2)$ *is an isomorphism,* $k < m$. *Let* $Z \in \mathcal{P}^*$ *and* $g : Z \to X$ *be given with* Z *q-coconnected. Then*

$$\bar{f}_{\#} : L(Z, g; \mathcal{Y}_1) \to L(Z, fg; \mathcal{Y}_2)$$

is one-one if $q < m + 1$ *and onto if* $q \leq m + 1$.

(13.22) LEMMA. *Let* $\mathcal{Y} = (Y, X, r)$ *be an* $(n - 1)$*-sphere bundle and suppose* X *is* $(2n - 2)$*-coconnected. If* $\zeta_0, \zeta_1 : X \to \Sigma(Y)$ *are homotopic as cross-sections of* $\Sigma(r)$, *then* $\mathcal{Y}$ *admits a cross-section.*

Proof. Let

$$\Lambda(\Sigma(Y)) = \{\sigma : I \to \Sigma(Y) \mid \sigma(I) \subset \Sigma(r)^{-1}(x), \text{ some } x \in X, \text{ and } \sigma(i) = \zeta_i(x), i = 0, 1\}$$

and let $\hat{r} : \Lambda(\Sigma(Y)) \to X$ be given by $\hat{r}(\sigma) = r\sigma(0)$. Then $\Lambda(\Sigma(\mathcal{Y})) = (\Lambda(\Sigma(Y)), X, \hat{r})$ is a fibre space and we have a map

$$(\lambda, 1) : (Y, X, \pi) \to (\Lambda(\Sigma(Y)), X, \hat{r})$$

by $\lambda(y)(t) = [y, t], 0 \leq t \leq 1$. The fibre $\hat{r}^{-1}(X)$ is the space $\Lambda(S^n)$ of paths on S^n from the south to the north pole and it is well known that

$$\lambda_{\#} : \pi_k(S^{n-1}) \to \pi_k(\Lambda(S^n))$$

is an isomorphism, $k < 2n - 3$. Now let $F : X \times I \to \Sigma(Y)$ be a homotopy such that $F_0 = \zeta_0$, $F_1 = \zeta_1$ and $\Sigma(r)F_t = 1$, $0 \leq t \leq 1$. Then

$$\hat{F} : X \to \Lambda(\Sigma(Y)) \quad \text{by} \quad \hat{F}(x)(t) = F(x, t)$$

is a cross-section to $\hat{r}$. By the previous lemma, there exists a cross-section to r.

(13.23) **Theorem.** *Let $\mathcal{Y} = (Y, X, r)$ be an $(n - 1)$-sphere bundle and suppose X is $(2n - 2)$-coconnected. Let $\mathfrak{X}$ be an Euler class for $(\mathcal{Y}, f)$ relative to the sphere spectrum* $\mathbf{S}$. *Then $\mathcal{Y}$ admits a cross-section if and only if $\mathfrak{X} = 0$.*

Proof. Suppose that $\mathfrak{X} = 0$. By Lemma (13.2) we may assume that $f : X \to B(\mathcal{O}_n)$ is a classifying map for $\mathcal{Y}$ and $\mathfrak{X} = \zeta_0^*(u)$, where u is the Thom class constructed in the proof of Theorem (12.3). This class u is represented by a map $\hat{f} : \Sigma(Y) \to E(\mathcal{O}_n) \times S^n/\mathcal{O}_n$ such that the diagrams

$$\begin{array}{ccc} \Sigma(Y) & \xrightarrow{\hat{f}} & E(\mathcal{O}_n) \times S^n/\mathcal{O}_n \\ \downarrow{\scriptstyle \Sigma(r)} & & \downarrow{\scriptstyle p_n^n} \\ X & \xrightarrow{f} & B(\mathcal{O}_n) \end{array} \qquad \begin{array}{ccc} \Sigma(Y) & \xrightarrow{\hat{f}} & E(\mathcal{O}_n) \times S^n/\mathcal{O}_n \\ \uparrow{\scriptstyle \zeta_1} & & \uparrow{\scriptstyle \Delta_n^n} \\ X & \xrightarrow{f} & B(\mathcal{O}_n) \end{array}$$

are commutative. Now $\mathfrak{X} = \zeta_0^*(u)$ is the image of $[\hat{f}\zeta_0]$ under

$$\pi_0(\mathfrak{L}(X, f; \mathbb{S}_n^n)) \to h^n(X, f; \mathbf{S}_n) \to H^n(X, f; \mathbf{S}),$$

each map being inclusion into the direct limit. The first map is one-one and onto by (13.21) and the fact that X is $(2n - 2)$-coconnected. The second is an isomorphism by (10.4). Therefore, since $\mathfrak{X} = 0$, we have

$$[\hat{f}\zeta_0] = [\Delta_n^n f] = [\hat{f}\zeta_1] \in L(X, f; \mathbb{S}_n^n).$$

In other words, for

$$\hat{f}_{\#} : L(X, 1, \Sigma(\mathcal{Y})) \to L(X, f, \mathbb{S}_n^n))$$

we have $\hat{f}_{\#}[\zeta_0] = \hat{f}_{\#}[\zeta_1]$. By (13.21), $\hat{f}_{\#}$ is one-one. Consequently ζ_0 and ζ_1 are homotopic as cross-sections to $\Sigma(r)$. Now apply the preceding lemma.

14. Equivariant maps

Let A be a fixed point free cellular involution on the CW-complex Y and let $f : Y/A \to B(\mathcal{O})$ be a classifying map for the 0-sphere bundle

$$\mathcal{Y} = (Y_1, T/A, r)$$

where $r : Y \to Y/A$ is the quotient map. Let $c \in H^1(Y/A, f; \mathbf{S})$ be an Euler class for $(\mathcal{Y}, f)$. Define $f^n : Y/A \to B(\mathcal{O})$ inductively by $f^1 = f$ and $f^n = \mu(f \times f^{n-1})$, $n > 1$. Next, define

$$c^n \in H^n(Y/A, f^n; \mathbf{S}) \tag{14.1}$$

by $c^1 = c$ and $c^n = c \cup c^{n-1}$. $n > 1$.

Let $\hat{A}$ denote the involution on $Y \times S^{n-1}$ given by

$$(y, z) \to (A(y), -z)$$

and form the $(n-1)$-sphere bundle $\mathcal{Y}^\wedge = (Y \times S^{n-1}/A, Y/A, \hat{r})$ where $\hat{r}[y, z] = [y]$. There exists an equivariant map $Y \to S^{n-1}$ if and only if $\mathcal{Y}^\wedge$ admits a cross-section. Now, $\mathcal{Y}^\wedge$ may be identified with the n-fold Whitney join of $\mathcal{Y}$ with itself. Therefore c^n is an Euler class for $(\mathcal{Y}^\wedge, f^n)$. By Theorem (13.23) we have

(14.2) Theorem. *Suppose Y is $(2n-2)$-coconnected. There is an equivariant map $Y \to S^{n-1}$ if and only if $c^n = 0$.*

Now let G be a finite group of odd order which is free and cellular on the finite CW-complex Z. Let $\pi : Z \to Z/G$ be the quotient map.

(14.3) Lemma. *For $f : Z/G \to B(\Theta)$ and a spectrum $\mathbf{D}$,*

$$\pi^* : H^k(Z/G, f; \mathbf{D}) \to H^k(Z, f\pi; \mathbf{D})$$

is injective modulo the class of odd torsion groups, $-\infty < k < \infty$.

This is well known for ordinary cohomology. The spectral sequence argument in the general case is exactly the same (see [7, section 10.8]).

Now suppose G is a group of odd order which acts on the closed n-dimensional C^∞-manifold M, the action being free and C^∞. Then M/G inherits a C^∞-structure such that the quotient map $\pi : M \to M/G$ is an immersion.

(14.4) Theorem. *Assume $2k > n + 1$. Then M/G immerses in R^{n+k} if and only if M immerses in R^{n+k}.*

Proof. If there is an immersion $\varphi : M/G \to R^{n+k}$, the composition $\varphi\pi : M \to R^{n+k}$ is an immersion of M in R^{n+k}. On the other hand suppose M immerses in R^{n+k}. Let $N = M/G$ and let $T(M)$ and $T(N)$ denote the tangent sphere bundles of M and N respectively. Each has an involution A defined by sending a unit vector v to $-v$. According to a theorem of Hirsch and Haefliger [5], it is sufficient to show that there is an equivariant map $T(N) \to S^{n+k-1}$ We have a commutative diagram

$$\begin{array}{ccc}
\dfrac{T(M) \times S^{n+k-1}}{\hat{A}} & \xrightarrow{[T(\pi) \times 1]} & \dfrac{T(N) \times S^{n+k-1}}{\hat{A}} \\
\Big\downarrow \hat{r} & & \Big\downarrow \hat{r} \\
T(M)/A & \xrightarrow{[T(\pi)]} & T(N)/A.
\end{array}$$

Let $f : T(N)/A \to B(\Theta)$ be a classifying map for the 0-sphere bundle $T(N) \to T(N)/A$ and let

$$c^{n+k} \in H^{n+k}(T(N)/A, f^{n+k}; \mathbf{S})$$

be as described above. By (9.5), $[T(\pi)]^*(c^{n+k})$ is an Euler class for the sphere bundle

$$\left(\frac{T(M) \times S^{n+k-1}}{\hat{A}}, T(M)/A, \hat{r}\right)$$

and map $f^{n+k}[T(\pi)]$. Since M immerses in R^{n+k}, $[T(\pi)]^*(c^{n+k}) = 0$.

The action of G on M lifts to an action of G on $T(M)$ by setting $g \cdot v = T(g)(v)$, $v \in T(M)$, $g \in G$. Since $g \cdot (-v) = -(g \cdot v)$, $T(M)/A$ inherits an action of G by $g \cdot [v] = [g \cdot v]$ and we can exhibit an identification between $T(M)/A/G$ and $T(M/G)/A = T(N)/A$. Then by the previous lemma

$$[T(\pi)]^* : H^{n+k}(T(N)/A, f^{n+k}; \mathrm{S}) \to H^{n+\kappa}(T(M)/A, f^{n+k}[T(\pi)]; \mathrm{S})$$

is injective modulo odd torsion. Thus, $c^{n+k} = 0$ modulo odd torsion. However, by (13.4), c and hence c^{n+k} is zero modulo 2-torsion. We must conclude that $c^{n+k} = 0$. Now apply (14.2).

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