RELATIONS IN CATEGORIES

BY

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Introduction

This paper is concerned with relations in general categories. MacLane [1], Puppe [2], Hilton [3] considered the abelian case; namely, the categories of relations extending abelian categories. In order to develop a general theory which still includes the classical case, we need some structure to insure the existence of a good factorization of morphisms; it seems that the bicategory structure in the sense of Isbell [4], Semadeni [5] is adequate. By imposing this structure, we include the classical counterpart in the form of categories with images. Moreover, since a category can generally be made into a bicategory in several ways, the choice may be of importance. To obtain a reasonable system it is likely that only few conditions on the chosen type of category (finitely complete [6] bicategories) may be relaxed.

For completeness some facts about bicategories are established. Relations are introduced and composition defined using the set-theoretical relations as a natural model. Associativity is shown to be false in the general case. The associative case is characterized by a categorical form of the Ore conditions in semigroups and rings. However, in the general exposition associativity is not assumed and it seems that even the nonassociative case can be handled for some purposes. Functors and extensions are considered. In the last section congruences are introduced and a regularity property is proved for congruences with respect to group-like structures.

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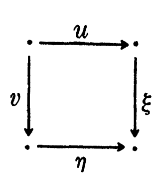
We denote the class of morphisms $A \to B$ in $\mathcal{C}$ by $\mathcal{C}(A, B)$. $|\mathcal{C}|$ denotes the class of objects of $\mathcal{C}$. (Co-) Retractions are (left-) right-invertible morphisms. A product of A, B in $\mathcal{C}$ is usually denoted $(A \times B, \pi_A, \pi_B)$ and the unique morphism ζ into $A \times B$ such that $\pi_A \zeta = \xi$, $\pi_B \zeta = \eta$ is denoted by $\{\xi, \eta\}$. We denote a pullback



by $\downarrow v, u, \eta, \xi \downarrow$.

(0.1) A *bicategory* is a category $\mathcal{C}$ with a structure consisting of two subcategories $\mathcal{I}$ and $\mathcal{S}$ such that the elements of $\mathcal{I}$ are monics, those of $\mathcal{S}$ are epics;

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$\mathcal{I} \cap \mathcal{S}$ is the subcategory of isomorphisms in $\mathcal{C}$; every morphism α in $\mathcal{C}$ is factorizable into $\alpha^i\alpha^s$, $\alpha^i \in \mathcal{I}$, $\alpha^s \in \mathcal{S}$ in an essentially unique way, namely if $\iota\sigma = \iota'\sigma'$ with $\iota, \iota' \in \mathcal{I}, \sigma, \sigma' \in \mathcal{S}$, then there is an invertible ε that satisfies $\iota\varepsilon = \iota'$, $\varepsilon^{-1}\sigma = \sigma'$.

It follows that $\alpha \in \mathcal{I}$ if and only if the epic α^s, in any i-s-factorization of α, is an isomorphism.

(0.2) *If $\alpha\beta \in \mathcal{I}$ then $\beta \in \mathcal{I}$.*

For, $\alpha\beta = (\alpha^i\gamma^i)(\gamma^s\beta^s)$ with $\gamma = \alpha^s\beta^i$ is an i-s-factorization of $\alpha\beta$, so $\gamma^s\beta^s$ is invertible, therefore β^s is a coretraction; but β^s is epic, hence invertible. In particular the class of coretractions is included in $\mathcal{I}$.

In the general case the monics of $\mathcal{I}$ are not necessarily images. The i-s-factorization coincides with image-factorization if and only if $\mathcal{I}$ is the class of monics. The following lemma will show that our theory includes the "classical" case.

(0.3) *If $\mathcal{C}$ is a category with images and with pullbacks and finite products then there is a bicategorical structure on $\mathcal{C}$ with the class of monics as $\mathcal{I}$.*

Proof. If such a structure exists then i-s- and image-factorizations are identical. If $\alpha = \delta\delta'$ and δ is an image of α, then an image of δ' must be invertible. Thus we have to define $\mathcal{S}$ as the class of morphisms with invertible images. Then $\sigma \in \mathcal{S}$ if and only if $\sigma = \mu\nu$ with μ monic implies that μ is invertible. In particular $\mathcal{I} \cap \mathcal{S}$ is the class of isomorphisms. The uniqueness of i-s-factorization follows from the property of images.

The elements of $\mathcal{S}$ are epics: if $\sigma \in \mathcal{S}$ and $\alpha\sigma = \alpha'\sigma$ and if

$$\downarrow v, u, \{1, \alpha'\}, \{1, \alpha\} \downarrow$$

is a pullback, then $v = u$ and it is monic; but $\{1, \alpha\}\sigma = \{1, \alpha'\}\sigma$, so there is a λ that satisfies $u\lambda = \sigma$, hence u is invertible and by $\alpha u = \alpha' u$ we obtain $\alpha = \alpha'$.

To show that $\mathcal{S}$ is a subcategory let $\sigma'\sigma = \mu\nu$ with $\sigma', \sigma \in \mathcal{S}$ and μ monic and we have to show that μ is invertible. Let $\downarrow \psi, \varphi, \mu, \sigma' \downarrow$ and by $\sigma'\sigma = \mu\nu$ there is a λ satisfying $\sigma = \varphi\lambda$. But φ is monic, hence invertible and so $\sigma' = \mu(\psi\varphi^{-1})$ and μ is invertible.

(0.4) *Let $\downarrow \psi, \varphi, \eta, \xi \downarrow$ be a pullback in a bicategory; if $\xi \in \mathcal{I}$ then $\psi \in \mathcal{I}$.*

Proof. By $\xi\varphi = \eta\psi$ there is an ε for which $(\xi\varphi^i)\varepsilon = (\eta\psi^i)^i$ and denoting $\beta = \varepsilon(\eta\psi^i)^s$ we have $\xi(\varphi^i\beta) = \eta\psi^i$. Hence there is a λ that satisfies $\psi^i = \psi\lambda$ and by cancelling ψ^i we obtain $1 = \psi^s\lambda$. But ψ is monic (since ξ is monic), so ψ^s is a monic retraction, hence invertible. Thus $\psi \in \mathcal{I}$.

Consider $\xi : X \to X'$ in $\mathcal{I}$. If $X \times A$, $X' \times A$ exist, then

$$\downarrow \xi \times 1_A, \pi_X, \pi_{X'}, \xi \downarrow .$$

Hence $\xi \times 1_A \in \mathcal{I}$ by (0.4). Applying twice this result we conclude

(0.5) *If $\xi, \eta \in \mathcal{I}$ then $\xi \times \eta \in \mathcal{I}$.* (It holds even without pullbacks [4; 2.3].)

For any ξ, η with common codomain Z in $\mathcal{C}$ and with $\downarrow \psi, \varphi, \eta, \xi \downarrow$ we have

$$\downarrow \{\varphi, \psi\}, \zeta, \xi \times \eta, \{1_Z, 1_Z\} \downarrow$$

where $\zeta = \xi\varphi = \eta\psi$. Since $\{1_Z, 1_Z\}$ is a coretraction, it follows by (0.4) that

(0.6) *If* $\downarrow \psi, \varphi, \eta, \xi \downarrow$, *then* $\{\varphi, \psi\} \in \mathcal{I}$.

1. Relations

(1.1) Let $\mathcal{C}$ be a bicategory with finite products. We say that $[R, \rho_A, \rho_B]$ with $\rho_A \in \mathcal{C}(R, A)$, $\rho_B \in \mathcal{C}(R, B)$ is a *relation* from A to B in $\mathcal{C}$, provided $\{\rho_A, \rho_B\} \in \mathcal{I}$.

Given relations $[R, \rho_A, \rho_B]$, $[S, \sigma_A, \sigma_B]$ from A to B, we declare $[R, \rho_A, \rho_B] = [S, \sigma_A, \sigma_B]$ if an isomorphism $\iota \in \mathcal{C}(R, S)$ exists such that $\sigma_A \iota = \rho_A$ and $\sigma_B \iota = \rho_B$.

For $A, B \in |\mathcal{C}|$ we denote by $\mathcal{R}_\mathcal{C}(A, B)$ the class of relations from A to B in $\mathcal{C}$. It is not necessarily a set and it is not empty since $[A \times B, \pi_A, \pi_B] \in \mathcal{R}_\mathcal{C}(A, B)$. For different pairs of objects in $\mathcal{C}$ the respective classes of relations are disjoint (since relations are introduced as some "triples" and not as subobjects of products).

(1.2) For $\alpha \in \mathcal{C}(A, B)$ the morphism $\{1_A, \alpha\} : A \to A \times B$ is in $\mathcal{I}$ (it is a coretraction since $\pi_A\{1_A, \alpha\} = 1_A$), hence $[A, 1_A, \alpha] \in \mathcal{R}_\mathcal{C}(A, B)$. We shall denote this relation by Γ_α and call it *the graph of* α.

A relation $[R, \rho_A, \rho_B]$ is a graph if and only if ρ_A is invertible in $\mathcal{C}$ and in this case $[R, \rho_A, \rho_B] = \Gamma_{\rho_B}\Gamma_{\rho_A^{-1}}$.

The correspondence between morphisms in $\mathcal{C}(A, B)$ and their graphs in $\mathcal{R}_\mathcal{C}(A, B)$ is one-to-one. It is surjective if and only if every morphism having A as codomain is invertible. Note that changes in the bicategorical structure of $\mathcal{C}$ do not affect the class of graphs, since $\mathcal{I}$ always includes the coretractions.

The graph Γ_{1_A} will be called *the diagonal of* A and denoted by Δ_A.

(1.3) We introduce the following natural partial order into each $\mathcal{R}_\mathcal{C}(A, B)$. $[R, \rho_A, \rho_B) \leq [S, \sigma_A, \sigma_B]$ if there is a $\delta : R \to S$ for which $\sigma_A \delta = \rho_A$, $\sigma_B \delta = \rho_B$ (hence $\delta \in \mathcal{I}$).

$[A \times B]$ is the greatest element of $\mathcal{R}_\mathcal{C}(A, B)$. Different graphs cannot be compared by this partial order.

(1.4) With $[R, \rho_A, \rho_B] \in \mathcal{R}_\mathcal{C}(A, B)$ we associate the relation

$$[R, \rho_B, \rho_A] \in \mathcal{R}_\mathcal{C}(B, A)$$

denoting it $[R]^-$ and naming it the *converse* of $[R]$. The converse mapping is an order-isomorphism between the classes $\mathcal{R}_\mathcal{C}(A, B)$ and $\mathcal{R}_\mathcal{C}(B, A)$.

(1.5) Now we add the assumption that $\mathcal{C}$ has pullbacks also, thus $\mathcal{C}$ is a finitely complete [6] bicategory. We define *composition* of relations.

Let $[R, \rho_X, \rho_Y] \in \mathcal{R}_\mathcal{C}(X, Y)$, $[S, \sigma_Y, \sigma_Z] \in \mathcal{R}_\mathcal{C}(Y, Z)$. We construct a pullback $\downarrow \psi, \varphi, \sigma_Y, \rho_Y \downarrow$, and we *i-s*-factorize the morphism $\{\rho_X \varphi, \sigma_Z \psi\}$, so

$$\{\rho_X \varphi, \sigma_Z \psi\} = \{\mu_X, \mu_Z\}\zeta : \cdot \to M \to A \times C$$

with $\{\mu_X, \mu_Z\} \in \mathcal{I}$, $\zeta \in \mathcal{S}$. Hence $\{M, \mu_X, \mu_Z\} \in \mathcal{R}_\mathcal{C}(X, Z)$, and we call it the *composite* of the two relations, writing simply $[M] = [S][R]$.

By the definition of equality (1.1) composition is obviously well-defined.

Let us use the term *near-category* for a system subject to the axioms of category except the associativity of composition.

(1.6) *With the defined composition the class* $\mathcal{R}_\mathcal{C}$ *of relations over* $\mathcal{C}$ *is a near-category with* $|\mathcal{R}_\mathcal{C}| = |\mathcal{C}|$ *and with the relations as morphisms. The diagonals are the identities of this near-category.* $\Gamma : \mathcal{C} \to \mathcal{R}$ *which takes each* α *to its graph* Γ_α *is a covariant embedding functor which is identity on* $|\mathcal{C}|$.

Strictly speaking another assumption is not generally fulfilled, namely the classes $\mathcal{R}_\mathcal{C}(-, -)$ are not always sets, but this assumption about categories is not important in the present paper. However, we note that the assumption holds for $\mathcal{R}_\mathcal{C}$ if and only if $\mathcal{I}$ is locally small in the sense of [6].

The converse operator $[R] \to [R]^-$ is an anti-isomorphism of $\mathcal{R}_\mathcal{C}$ with itself, showing that the near-category $\mathcal{R}_\mathcal{C}$ is dual to itself. $\Gamma^- : \mathcal{C} \to \mathcal{R}_\mathcal{C}$ which takes each α to Γ_α^- is a contravariant embedding.

We remark that since we have pullbacks, we have finite intersections in $\mathcal{C}$. An intersection of monics belonging to $\mathcal{I}$ is in $\mathcal{I}$. It follows that each $\mathcal{R}_\mathcal{C}(-, -)$ has finite intersections.

(1.7) For $\xi : A \to X$, $\eta : A \to Y$ the composition $\Gamma_\eta \Gamma_\xi^-$ is defined and it is a relation from X to Y represented by the components of $\{\xi, \eta\}^i$.

(1.8) For $\xi : X \to A$, $\eta : Y \to A$ the composition $\Gamma_\eta^- \Gamma_\xi$ is a relation from X to Y and by (0.6) it follows that $\Gamma_\eta^- \Gamma_\xi = [\cdot, \varphi, \psi]$ where $\downarrow \psi, \varphi, \eta, \xi \downarrow$ is a pullback.

A consequence of (1.7) is that $\mathcal{R}_\mathcal{C}$ is generated by the graphs and their converses:

(1.9) THEOREM. $[R, \rho_A, \rho_B] = \Gamma_{\rho_B} \Gamma_{\rho_A}^-$ *for every* $[R, \rho_A, \rho_B] \in \mathcal{R}_\mathcal{C}(A, B)$.

Proof. $\{\rho_A, \rho_B\} \in \mathcal{I}$.

This factorization is not necessarily unique; if $[R, \rho_A, \rho_B] = \Gamma_\beta \Gamma_\alpha^-$, then a $\zeta \in \mathcal{S}$ exists such that $\alpha = \rho_A \zeta$, $\beta = \rho_B \zeta$, and ζ is not necessarily invertible.

(1.10) THEOREM. *For* $[R, \rho_A, \rho_B] \in \mathcal{R}_\mathcal{C}(A, B)$ *we have*: $[R][R]^- \leq \Delta_B$ *if and only if* ρ_A *is monic*: $[R]^-[R] \geq \Delta_A$ *if and only if* $\rho_A \in \mathcal{S}$.

Proof. If ρ_A is monic, then $\downarrow 1_R, 1_R, \rho_A, \rho_A \downarrow$, hence $[R][R]^-$ is obtained by factorizing $\{\rho_B, \rho_B\}$. Since $\{\overset{i}{\rho}_B, \overset{i}{\rho}_B\} \in \mathcal{I}$ (by (0.2), due to $\pi\{\overset{i}{\rho}_B, \overset{i}{\rho}_B\} = \overset{i}{\rho}_B$), it follows that $\{\overset{i}{\rho}_B, \overset{i}{\rho}_B\}\overset{s}{\rho}_B$ is an i-s-factorization of $\{\rho_B, \rho_B\}$. Thus

$$[R][R]^- = [\cdot, \overset{i}{\rho}_B, \overset{i}{\rho}_B] \leq [B, 1_B, 1_B].$$

Conversely, let

$$[R][R]^- = [M, \mu_1, \mu_2] \leq \Delta_B.$$

Then $\mu_1 = \mu_2$. If $\downarrow \psi, \varphi, \rho_A, \rho_A \downarrow$ then $\{\rho_B\varphi, \rho_B\psi\} = \{\mu_1, \mu_2\}\zeta$, hence $\rho_B\varphi = \rho_B\psi$, so $\{\rho_A, \rho_B\}\varphi = \{\rho_A, \rho_B\}\psi$. It follows that $\varphi = \psi$ since $\{\rho_A, \rho_B\}$ is monic. Finally ρ_A is monic since $\rho_A \gamma = \rho_A \delta$ implies the existence of a λ that satisfies $\varphi\lambda = \gamma$, $\psi\lambda = \delta$, thus $\gamma = \delta$.

Now let $\rho_A \in \mathcal{S}$. Forming $\downarrow \psi', \varphi', \rho_B, \rho_B \downarrow$ and i-s-factorizing

$$\{\rho_A \varphi', \rho_A \psi'\} = \{\nu_1, \nu_2\}\zeta',$$

we have $[R]^{-}[R] = [\cdot, \nu_1, \nu_2]$. By $\rho_B 1_R = \rho_B 1_R$ there is a λ' such that $\varphi'\lambda' = 1_R = \psi'\lambda'$. This implies $\{\rho_A, \rho_A\} = \{\rho_A \varphi', \rho_A \psi'\}\lambda'$, hence $\{1_A, 1_A\}\rho_A = \{\nu_1, \nu_2\}\zeta'\lambda'$ and here the left-hand side is i-s-factorized since $\rho_A \in \mathcal{S}$. It follows that $\{1_A, 1_A\} \leq \{\nu_1, \nu_2\}$.

Finally suppose $[R]^{-}[R] \geq \Delta_A$. Then ν_1, ν_2 are retractions. But $\rho_A \varphi' = \nu_1 \zeta'$ and $\zeta' \in \mathcal{S}$, so $\rho_A \varphi' \in \mathcal{S}$, hence $\rho_A \in \mathcal{S}$ (0.2).

(1.11) COROLLARY. *$[R]^{-}[R] = \Delta_A$ if and only if $\rho_A \in \mathcal{S}$ and ρ_B is monic.*

It follows that for a bicategorical structure with all monics in $\mathcal{I}$ both equalities $[R]^{-}[R] = \Delta$ and $[R][R]^{-} = \Delta$ hold, if and only if $[R]$ is the graph of an isomorphism.

(1.12) COROLLARY. *If $\alpha \in \mathcal{C}(A, B)$ then $\Gamma_\alpha \Gamma_\alpha^{-} \leq \Delta_B$, $\Gamma_\alpha^{-} \Gamma_\alpha \geq \Delta_A$.*

(1.13) COROLLARY. *For $\alpha \in \mathcal{C}(A, B)$ we have $\Gamma_\alpha^{-} \Gamma_\alpha = \Delta_A$ if and only if α is monic; $\Gamma_\alpha \Gamma_\alpha^{-} = \Delta_B$ if and only if $\alpha \in \mathcal{S}$.*

Proof. By (1.10) with $[R] = \Gamma_\alpha^{-}$ and by (1.12).

Thus $\Gamma : \mathcal{C} \to \mathcal{R}_\mathcal{C}$ *assigns to each monic of $\mathcal{C}$ a coretraction in $\mathcal{R}_\mathcal{C}$ and to each element of $\mathcal{S}$ a retraction in $\mathcal{R}_\mathcal{C}$.*

The last statement can be improved: *Γ_α is a retraction if and only if $\alpha \in \mathcal{S}$.* Suppose $[A, 1_A, \alpha][R, \rho_B, \rho_A] = \Delta_B$; then since $\Gamma_\alpha[R]$ is constructed via $\downarrow \rho_A, 1_R, 1_A, \rho_A \downarrow$ by i-s-factorizing $\{\rho_B, \alpha\rho_A\}$, it follows that $\alpha\rho_A \in \mathcal{S}$, hence $\alpha \in \mathcal{S}$ (0.2).

In the last proof we actually obtained $\Gamma_\alpha(\Gamma_{\rho_A} \Gamma_{\rho_B}^{-}) = \Gamma_{\alpha\rho_A} \Gamma_{\rho_B}^{-}$. Hence graphs and converses appearing in the order $\Gamma, \Gamma, \Gamma^{-}$ (or $\Gamma^{-}, \Gamma, \Gamma$) are "associative". The trouble arises with $\Gamma, \Gamma^{-}, \Gamma$ (or $\Gamma^{-}, \Gamma, \Gamma^{-}$) (2.1).

Thus far we have a near-category of relations over a finitely complete bicategory. Some words about the chosen frame-work. Composition of relations in set-theory is actually an image of a certain morphism constructed via an intersection or a pullback. Hence the immediate generalization seems to be toward categories with images and to define relations with the aid of the class of monics. However, (0.3) shows that we have included this case even by further generalization to bicategories and by defining relations with the aid of the classes $\mathcal{I}$ of special monics in the bicategorical structure.

2. Associativity

(2.1) A counterexample. Consider the category of topological Hausdorff groups $\mathfrak{I}gh$ with $\mathcal{I}$ the class of homeomorphisms onto closed subspaces and $\mathcal{S}$ the class of mappings with dense ranges. $\mathcal{R}_{\mathfrak{I}gh}$ is well defined. The rational line Q and the real line R are elements of $|\mathfrak{I}gh|$ and the immersion $\xi : Q \rightarrow R$ is an element of $\mathcal{S}$, hence $\Gamma_\xi \Gamma_\xi^- = \Delta_R$ (1.13). Choose an irrational x and define $\eta : Q \rightarrow R$ by $\eta(q) = xq$ $(q \in Q)$. So $(\Gamma_\xi \Gamma_\xi^-)\Gamma_\eta = \Gamma_\eta$. Since obviously the constructed sides of a pullback of ξ, η are $0 \rightarrow Q$, it follows (1.8) that $\Gamma_\xi^- \Gamma_\eta = [0]$, hence $\Gamma_\xi(\Gamma_\xi^- \Gamma_\eta) = [0]$. Thus $\Gamma_\xi(\Gamma_\xi^- \Gamma_\eta) \neq (\Gamma_\xi \Gamma_\xi^-)\Gamma_\eta$.

Assuming that $\mathcal{R}_\mathcal{C}$ is a category and identifying $\mathcal{C}$ with the image of Γ, thus writing α instead of Γ_α, we have that every $\rho \in \mathcal{R}_\mathcal{C}$ can be written in the form $\rho = \beta\alpha^-$ with $\alpha, \beta \in \mathcal{C}$ (1.9), and the elements of $\mathcal{S}$ are exactly those α of $\mathcal{C}$ for which $\alpha\alpha^- =$ identity (1.13). So the elements of $\mathcal{R}_\mathcal{C}$ look like "right-quotients" over $\mathcal{C}$ and the elements of $\mathcal{S}$ are exactly the "right-regular" elements. In the well-known theories in semigroups and rings, the "Ore conditions" are sufficient for embedding a semigroup (or ring) into a semigroup (or ring) of right-quotients such that a prescribed class of elements will become right-regular. Hence we formulate an Ore-like categorical condition

(A) *For every ξ, η in $\mathcal{C}$ with common codomain and with $\xi \in \mathcal{S}$, there is a common right multiple $\xi u = \eta v$ with $v \in \mathcal{S}$.*

Common multiples were already assumed in the form of pullbacks. The condition (A) does not require that $\xi u = \eta v$ should be a pullback. However, if we have pullbacks in $\mathcal{C}$ and if (A) holds, then (by (0.2)) it holds in particular in a pullback.

Turning back to the example in $\mathfrak{I}gh$ we observe that (A) does not hold in that system (with the particular ξ, η of the example, v is $0 \rightarrow Q$, so $v \notin \mathcal{S}$). The necessity of (A) for associativity could be predicted with the aid of the counterexample, as follows.

(2.2) Lemma. *Let $\downarrow v, u, \eta, \xi \downarrow$ be a pullback in $\mathcal{C}$ and suppose $\xi \in \mathcal{S}$ and $\Gamma_\xi(\Gamma_\xi^- \Gamma_\eta) = (\Gamma_\xi \Gamma_\xi^-)\Gamma_\eta$. Then $v \in \mathcal{S}$.*

Proof. $\xi \in \mathcal{S}$ implies $(\Gamma_\xi \Gamma_\xi^-)\Gamma_\eta = \Gamma_\eta$ (1.13). $\Gamma_\xi^- \Gamma_\eta = [\cdot, v, u]$ (1.8), hence the assumptions imply $\Gamma_\xi[\cdot, v, u] = \Gamma_\eta$. The composite $\Gamma_\xi[\cdot, v, u]$ is obtained by factorizing $\{v, \xi u\}$, and, since the result has to be Γ_η, a $\zeta \in \mathcal{S}$ exists satisfying $\{v, \xi u\} = \{1, \eta\}\zeta$. Thus $v = \zeta \in \mathcal{S}$.

(2.3) The definition of composition is easily generalized to a "ternary" (or even n-ary) composition. For $[R, \rho_X, \rho_Y]$, $[S, \sigma_Y, \sigma_Z]$, $[T, \tau_Z, \tau_U]$ we construct

$$\downarrow \psi, \varphi, \sigma_Y, \rho_Y \downarrow \quad \text{and} \quad \downarrow \psi', \varphi', \tau_Z, \sigma_Z \downarrow \quad \text{and then} \quad \downarrow \theta', \theta, \varphi', \psi \downarrow .$$

Let

$$\{\rho_X \varphi\theta, \tau_U \psi'\theta'\}^i = \{\nu_X, \nu_U\} : N \to X \times U.$$

We define $[T]\circ[S]\circ[R] = [N, \nu_X, \nu_U]$.

(2.4) LEMMA. *Binary composition of relations is associative if and only if* $[T] \circ [S] \circ [R] = [T]([S][R])$ *for every three composable relations.*

Proof. If binary composition is associative, then with the notation of (2.3) and by (1.8), (1.9) we have

$$\begin{aligned}[T]([S][R]) &= \Gamma_{\tau_U}(\Gamma_{\tau_Z}^- \Gamma_{\sigma_Z})(\Gamma_{\sigma_Y}^- \Gamma_{\rho_Y})\Gamma_{\rho_X}^- \\ &= \Gamma_{\tau_U} \Gamma_{\psi'}(\Gamma_{\varphi'}^- \Gamma_{\psi})\Gamma_{\varphi}^- \Gamma_{\rho_X}^- = \Gamma_{\tau_U} \Gamma_{\psi'} \Gamma_{\theta'} \Gamma_{\theta}^- \Gamma_{\varphi}^- \Gamma_{\rho_X}^- \\ &= \Gamma_{\tau_U\psi'\theta'} \Gamma_{\rho_X\varphi\theta}^- = [N, \nu_X, \nu_U] = [T] \circ [S] \circ [R].\end{aligned}$$

If the stated condition holds generally, then

$$\begin{aligned}([T][S])[R] &= ([R]^-([S]^-[T]^-))^- = ([R]^- \circ [S]^- \circ [T]^-)^- \\ &= [T] \circ [S] \circ [R] = [T]([S][R]).\end{aligned}$$

(2.5) THEOREM. $\mathfrak{R}_\mathfrak{C}$ *is a category if and only if* (A) *holds in* $\mathfrak{C}$.

Proof. One part of the assertion is proved by Lemma (2.2). To establish the other part we assume (A) and use (2.4), thus trying to prove $[T]([S][R]) = [T] \circ [S] \circ [R]$, (using again the notation of (2.3)). If $\{\rho_X \varphi, \sigma_Z \psi\} = \{\mu_X, \mu_Z\}\zeta$ is an i-s-factorization then $[S][R] = [\cdot, \mu_X, \mu_Z]$. Now we take pullbacks

$$\downarrow \psi'', \varphi'', \tau_Z, \mu_Z \downarrow \quad \text{and} \quad \downarrow v, u, \varphi'', \zeta \downarrow .$$

We obtain two pullbacks

$$\downarrow \psi''v, u, \tau_Z, \mu_Z \zeta \downarrow \quad \text{and} \quad \downarrow \psi'\theta', \theta, \tau_Z, \sigma_Z \psi \downarrow ,$$

the first by juxtaposition of the previous two pullbacks and the second by juxtaposition of the last two in (2.3). The two pullbacks just obtained, are both constructed on the same pair of morphisms τ_Z and $\sigma_Z \psi = \mu_Z \zeta$. Hence there is an invertible λ that satisfies

$$\psi''v\lambda = \psi'\theta' \quad \text{and} \quad u\lambda = \theta.$$

So we have

$$\rho_X \varphi\theta = \rho_X \varphi u\lambda = \mu_X \zeta u\lambda = \mu_X \varphi'' v\lambda, \qquad \tau_U \psi'\theta' = \tau_U \psi'' v\lambda$$

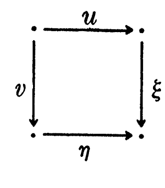
implying

$$\{\rho_X \varphi\theta, \tau_U \psi'\theta'\} = \{\mu_X \varphi'', \tau_U \psi''\}v\lambda.$$

Now we use (A): $\zeta \in \mathcal{S}$ implies $v \in \mathcal{S}$. Thus $v\lambda \in \mathcal{S}$ and so the last equality implies $\{\rho_X \varphi\theta, \tau_U \psi'\theta'\}^i = \{\mu_X \varphi'', \tau_U \psi''\}^i\varepsilon$ with ε invertible. Here the left-hand side represents $[T] \circ [S] \circ [R]$ and the right-hand side represents $[T]([S][R])$ proving the assertion.

Remark. In the general case $[T] \circ [S] \circ [R] \leq [T]([S][R])$ (and thus $\leq ([T][S])[R]$ also); this follows from $\{\rho_X \varphi\theta, \tau_U \psi'\theta'\} = \{\mu_X \varphi'', \tau_U \psi''\}(\nu\lambda)$ in the proof above.

(2.6) We have proved that $\mathcal{R}_\mathcal{C}$ is a category if and only if the bicategorical structure on $\mathcal{C}$ is connected with the pullbacks in such a manner that in a pullback



if $\xi \in S$ then $v \in \mathcal{S}$.

We saw that this associativity condition (A) does not hold in $\mathcal{T}gh$ with a certain bicategorical structure. But it holds in $\mathcal{T}gh$ if we take for $\mathcal{S}$ the class of quotient maps. (If ξ is an open map and

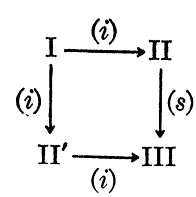
$$K \xrightarrow{u} X \xrightarrow{\xi} A = K \xrightarrow{v} Y \xrightarrow{\eta} A$$

is a pullback, then v is open. For, we can take $K = \{(x, y) \mid \xi(x) = \eta(y)\}$ with the topology induced by $X \times Y$ and, to show that v is open, let U be open in K and $y_0 \in vU$. There is an x_0 in X for which $(x_0, y_0) \in U$, hence there are open sets $C \subset X$, $D \subset Y$ such that

$$(x_0, y_0) \in (C \times D) \cap K \subset U.$$

Then $\eta^{-1}(\xi C) \cap D$ is open in Y and $y_0 \in \eta^{-1}(\xi C) \cap D \subset vU$.)

A simple example shows that (A) may hold for two bicategorical structures on the same category, yet fail to hold for an intermediate structure. With the usual way of regarding ordered classes as categories, a partially-ordered class $\mathcal{O}$ with finite intersections is a finitely complete category. Such an $\mathcal{O}$ carries two obvious bicategorical structures ($\mathcal{I} = \mathcal{O}$, $\mathcal{S} =$ Identities; $\mathcal{S} = \mathcal{O}$, $\mathcal{I} =$ Identities) satisfying (A). Then look at the following $\mathcal{O}$ of four objects and with the bicategorical structure as indicated



Of course (A) does not hold in the only non-trivial square, even though the bicategorical structure lies between the two extremal structures.

(A) enables us to prove the following: *If* $[R]$ *is invertible in* $\mathcal{R}_\mathcal{C}$ *then* $[R]^{-1} = [R]^{\smile}$. Let $[R, \rho_A, \rho_B]^{-1} = [S, \sigma_B, \sigma_A]$ and consider $\downarrow \psi, \varphi, \sigma_B, \rho_B \downarrow$. By $[S][R] = \Delta_A$ we have $\{\rho_A \varphi, \sigma_A \psi\} = \{1_A, 1_A\}\zeta$ with $\zeta \in \mathcal{S}$. Therefore $\rho_A, \sigma_A \in \mathcal{S}$.

And so $[R][S] = \Delta_B$ implies $\rho_B, \sigma_B \in \mathcal{S}$. Hence $\varphi, \psi \in \mathcal{S}$ by (A). Now $\rho_A \varphi = \sigma_A \psi$ and $\rho_B \varphi = \sigma_B \psi$, hence $\{\rho_A, \rho_B\}\varphi = \{\sigma_A, \sigma_B\}\psi$ and both sides are i-s-factorized. This implies $[R, \rho_A, \rho_B] = [S, \sigma_A, \sigma_B]$, that is $[R]^{\smile} = [S]$.

We remark that by (1.11) it follows that $[R]$ is invertible if and only if $\rho_A, \rho_B \in \mathcal{S}$ and are monics. If $\mathcal{I}$ includes all the monics, then this occurs if and only if $[R]$ is the graph of an isomorphism in $\mathcal{C}$, hence in this case the class of isomorphisms is not enlarged in passing from $\mathcal{C}$ to $\mathcal{R}_{\mathcal{C}}$. This happens in particular if $\mathcal{C}$ is balanced.

(2.7) Dualization of the definitions yields near-categories $\mathcal{R}^0_{\mathcal{C}}$ associated with finitely co-complete bicategories $\mathcal{C}$. A morphism in $\mathcal{R}^0_{\mathcal{C}}(A, B)$ is a co-relation $[\rho_B, \rho_A, R]^0$ with $\rho_A \in \mathcal{C}(A, R)$, $\rho_B \in \mathcal{C}(B, R)$ and $\langle \rho_A, \rho_B \rangle \in \mathcal{S}$. A covariant embedding of $\mathcal{C}$ into $\mathcal{R}^0_{\mathcal{C}}$ is obtained by assigning to every $\alpha \in \mathcal{C}(A, B)$ the co-graph $[1_B, \alpha, B]^0$.

For the category $\mathcal{E}ns$ (sets and functions, $\emptyset$ as initial object), both extensions $\mathcal{R}_{\mathcal{E}ns}$ and $\mathcal{R}^0_{\mathcal{E}ns}$ are categories ($(\mathrm{A})^0$ is easy). $\mathcal{G}r$ (groups) is an example with $\mathcal{R}_{\mathcal{G}r}$ being a category, whereas $\mathcal{R}^0_{\mathcal{G}r}$ is only a near-category. To see that $(\mathrm{A})^0$ does not hold in $\mathcal{G}r$, observe that a pushout in $\mathcal{G}r$ is represented in the form

$$\begin{array}{ccc} G & \xrightarrow{\ \xi\ } & X \\ {\scriptstyle \eta}\downarrow & & \downarrow{\scriptstyle u} \\ Y & \xrightarrow[\ v\ ]{} & X * Y/M \end{array}$$

where M is the normal subgroup generated by the elements $\xi g * \eta g^{-1}$ $(g \in G)$ in the free product $X * Y$, and u, v take elements into respective cosets. We need a pushout with ξ monic and v not. So, let G, X, Y be free groups, G with two generators g_1, g_2, X with two generators x_1, x_2 and Y with one generator y. We define ξ, η by

$$\xi g_1 = x_1, \quad \xi g_2 = x_2^{-1} x_1 x_2; \qquad \eta a_1 = y, \quad \eta a_2 = 1.$$

Hence ξ is monic. $X * Y$ is free with generators x_1, x_2, y and M is a normal subgroup containing $\xi a_1 * \eta a_1^{-1} = x_1 y^{-1}$, $\xi a_2 * \eta a_2^{-1} = x_2^{-1} x_1 x_2$. Hence $x_1 \in M$. so $y \in M$ and $v(y) = 1$, showing that v is not monic.

3. Functors

If $\mathcal{C}$, $\mathcal{C}'$ are bicategories we say that a covariant functor $F : \mathcal{C} \to \mathcal{C}'$ is an i-functor (s-functor) if $F\mathcal{I}_{\mathcal{C}} \subset \mathcal{I}_{\mathcal{C}'}$ $(FS_{\mathcal{C}} \subset S_{\mathcal{C}'})$.

(3.1) Let $\mathcal{C}$, $\mathcal{C}'$ be finitely complete bicategories and $F : \mathcal{C} \to \mathcal{C}'$ an i-functor preserving products. Then F obviously induces a map $\mathcal{R}_F$ from $\mathcal{R}_{\mathcal{C}}$ to $\mathcal{R}_{\mathcal{C}'}$ defined by

$$\mathcal{R}_F[R, \rho_A, \rho_B] = [FR, F\rho_A, F\rho_B].$$

$\mathcal{R}_F$ takes graphs into graphs $\mathcal{R}_F \Gamma_\alpha = \Gamma_{F\alpha}$ $(\alpha \in \mathcal{C})$, it commutes with the conversion functor and it is isotonic, but it is not necessarily a functor.

$\mathcal{R}_F$ *"decreases" composition, (namely* $\mathcal{R}_F([S][R]) \geq (\mathcal{R}_F[S])(\mathcal{R}_F[R])$ *whenever* $[S][R]$ *is defined), if and only if F preserves pullbacks.*

Proof. Suppose F preserves pullbacks. For $[R, \rho_X, \rho_Y]$ and $[S, \sigma_Y, \sigma_Z]$ in $\mathcal{C}$ let $\downarrow \psi, \varphi, \sigma_Y, \rho_Y \downarrow$ and $\{\rho_X \varphi, \sigma_Z \psi\} = \{\mu_X, \mu_Z\}\zeta$ an i-s-factorization. Then

$$[\cdot\, , F\mu_X, F\mu_Z] = \mathcal{R}_F([S][R]).$$

Since $\downarrow F\psi, F\varphi, F\sigma_Y, F\rho_Y \downarrow$ in $\mathcal{C}'$, it follows that $\{F(\rho_X \varphi), F(\sigma_Z \psi)\}^i$ represents $(\mathcal{R}_F[S])(\mathcal{R}_F[R])$. The asserted inequality then follows from

$$\{F(\rho_X \varphi), F(\sigma_Z \psi)\} = \{F\mu_X, F\mu_Z\}F\zeta$$

since F is an i-functor. On the other hand, if $\mathcal{R}_F$ decreases composition and if $\downarrow v, u, \eta, \xi \downarrow$ is a pullback in $\mathcal{C}$, it follows by $\Gamma_\eta^- \Gamma_\xi = [\cdot\, , u, v]$ (1.8) that

$$\Gamma_{F\eta} \Gamma_{F\xi}^- \leq \mathcal{R}_F[\cdot\, , u, v] = [\cdot\, , Fu, Fv].$$

But $\Gamma_{F\eta}^- \Gamma_{F\xi} = [\cdot\, , \bar{u}, \bar{v}]$ where $\downarrow \bar{v}, \bar{u}, F\eta, F\xi \downarrow$ is a pullback in $\mathcal{C}'$, so $\{Fu, Fv\} \geq \{\bar{u}, \bar{v}\}$. Since $(F\xi)(Fu) = (F\eta)(Fv)$ implies $\{Fu, Fv\} \leq \{\bar{u}, \bar{v}\}$, we conclude that $\downarrow Fv, Fu, F\eta, F\xi \downarrow$ is a pullback in $\mathcal{C}'$.

$\mathcal{R}_F$ *"increases" composition, if and only if F is an s-functor.*

Proof. Suppose F is an s-functor. With $[S]$, $[R]$ and pullback of the previous proof, let $\downarrow \psi', \varphi', F\sigma_Y, F\rho_Y \downarrow$. Hence by $(F\sigma_Y)(F\psi) = (F\rho_Y)(F\varphi)$ there is a λ' in $\mathcal{C}'$ that satisfies $\varphi'\lambda' = F\varphi$, $\psi'\lambda' = F\psi$, so

$$\{(F\rho_X)\varphi', (F\sigma_Z)\psi'\}\lambda' = \{F\mu_X, \mathrm{F}\mu_Z\}(F\zeta).$$

Here the right-hand side is i-s-factorized, hence

$$\{(F\rho_X)\varphi', (F\sigma_Z)\psi'\}^i \geq \{F\mu_X, F\mu_Z\},$$

proving the assertion. On the other hand, if $\mathcal{R}_F$ increases composition, then in particular for $\sigma \in \mathcal{S}_\mathcal{C}$, by $\Gamma_\sigma \Gamma_\sigma^- = \Delta$ (1.13), we obtain $\Gamma_{F\sigma} \Gamma_{F\sigma}^- \geq \mathcal{R}_F(\Gamma_\sigma \Gamma_\sigma^-) = \Delta$ in $\mathcal{R}_{\mathcal{C}'}$. But $\Gamma_{F\sigma} \Gamma_{F\sigma}^- \leq \Delta$ in any case (1.12), so $F\sigma \in \mathcal{S}_{\mathcal{C}'}$ (1.13).

If $\mathcal{C}$ is a finitely complete bicategory then for every $A \in |\mathcal{C}|$ the functor $\mathcal{C}(A, -) : \mathcal{C} \to \mathcal{E}ns$ (sets) is an i-functor preserving limits. So, the extended mapping $\mathcal{R}_{\mathcal{C}(A,-)}$ from $\mathcal{R}_\mathcal{C}$ to $\mathcal{R}_{\mathcal{E}ns}$ takes relations in $\mathcal{C}$ into relations of sets and it decreases composition. Hence $\mathcal{R}_{\mathcal{C}(A,-)}$ is a functor if and only if $\mathcal{C}(A, -)$ is an s-functor, that is what we should call a functor of bicategories. We conclude

(3.2) $\mathcal{R}_{\mathcal{C}(P,-)} : \mathcal{R}_\mathcal{C} \to \mathcal{R}_{\mathcal{E}ns}$ *is a functor if and only if the object P of* $\mathcal{C}$ *is projective in the bicategorical sense of* [5] ($\mathcal{S}$-projective in the sense of [6]).

$\mathcal{C} \times \mathcal{C}$ is a finitely complete category and carries a natural bicategorical structure given by $\mathcal{I}_{\mathcal{C}\times\mathcal{C}} = \mathcal{I}_\mathcal{C} \times \mathcal{I}_\mathcal{C}$, $\mathcal{S}_{\mathcal{C}\times\mathcal{C}} = \mathcal{S}_\mathcal{C} \times \mathcal{S}_\mathcal{C}$. Fixing the products in $\mathcal{C}$, we have a product functor $F : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, $\mathrm{F}(\alpha, \beta) = \alpha \times \beta$. If the condition (A) holds in $\mathcal{C}$ then F is a functor of bicategories. To prove this we remark that F is an i-functor, even without (A) (0.5). Then for $\xi : X \to X'$,

$\eta : Y \to Y'$ both in $S_\mathcal{C}$, we show $\xi \times \eta \in S$. Since $\downarrow \xi \times 1_Y, \pi_X, \pi_{X'}, \xi \downarrow$, it follows that $\xi \times 1_Y \in \mathcal{S}_\mathcal{C}$ by (A); similarly $1_{X'} \times \eta \in S_\mathcal{C}$, hence $\xi \times \eta \in \mathcal{S}_\mathcal{C}$.

Since $F : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves products and pullbacks we conclude

(3.3) *If* (A) *holds in* $\mathcal{C}$ *then* $\mathcal{R}_F : \mathcal{R}_{\mathcal{C}\times\mathcal{C}} \to \mathcal{R}_\mathcal{C}$ *is a functor of categories.*

Let $\mathcal{K}$ be any (small) category and consider $\mathcal{C}^\mathcal{K}$, the category of functors $\mathcal{K} \to \mathcal{C}$ and natural morphisms. Denote by $E : \mathcal{C} \to \mathcal{C}^\mathcal{K}$ the obvious embedding taking elements of $\mathcal{C}$ to "constant" elements of $\mathcal{C}^\mathcal{K}$. For each $A \in |\mathcal{C}|$ there is a "projection" functor $\Pi_A : \mathcal{C}^\mathcal{K} \to \mathcal{C}$, $\Pi_A\, t = tA$.

There is a unique bicategorical structure on $\mathcal{C}^\mathcal{K}$ such that E and all the Π_A's are functors of bicategories. It is constructed by taking $\mathcal{I}_{\mathcal{C}^\mathcal{K}}$ = the class of all $t \in \mathcal{C}^\mathcal{K}$ with $tA \in \mathcal{I}_\mathcal{C}$ for each $A \in |\mathcal{C}|$, and $S_{\mathcal{C}^\mathcal{K}}$ is defined similarly.

Since $\mathcal{C}^\mathcal{K}$ is a finitely complete category, we have

(3.4) $\mathcal{R}_{\mathcal{C}^\mathcal{K}}$ *exists and* $\mathcal{R}_E : \mathcal{R}_\mathcal{C} \to \mathcal{R}_{\mathcal{C}^\mathcal{K}}$, $\mathcal{R}_{\Pi_A} : \mathcal{R}_{\mathcal{C}^\mathcal{K}} \to \mathcal{R}_\mathcal{C}$ *are functors.*

If (A) holds in $\mathcal{C}$ then it holds in $\mathcal{C}^\mathcal{K}$ since pullbacks in $\mathcal{C}^\mathcal{K}$ can be constructed "pointwise". Thus, if $\mathcal{R}_\mathcal{C}$ is a category then the $\mathcal{R}_{\mathcal{C}^\mathcal{K}}$'s are categories.

Obviously, if $F : \mathcal{C} \to \mathcal{C}'$ is a functor of bicategories and if it preserves pullbacks, then $\mathcal{R}_F : \mathcal{R}_\mathcal{C} \to \mathcal{R}_{\mathcal{C}'}$ is a functor. It is the only functor $\bar{F} : \mathcal{R}_\mathcal{C} \to \mathcal{R}_{\mathcal{C}'}$ for which $\bar{F}\Gamma = \Gamma F$ and $\bar{F}\Gamma^- = \Gamma^- F$. The only functor $\bar{F}$ satisfying both last equalities is necessarily defined by $\bar{F}[R, \rho_A, \rho_B] = \Gamma_{F\rho_A}\Gamma^-_{F\rho_B}$. This suggests to try to extend more F's to functors from $\mathcal{R}_\mathcal{C}$ to $\mathcal{R}_{\mathcal{C}'}$, by using this definition. We do this for instance when associativity holds at least in $\mathcal{R}_\mathcal{C}$.

For a (near-) category $\mathcal{A}$ we call *involution* to a contravariant functor $(\text{-})^*$ of $\mathcal{A}$ onto itself satisfying $(\alpha^*)^* = \alpha$ for every $\alpha \in \mathcal{A}$ and $1^* = 1$ for every identity 1 in $\mathcal{A}$. The conversion functor of $\mathcal{R}_\mathcal{C}$ is an involution.

For a functor G from $\mathcal{C}$ to a category with involution $\mathcal{A}$, we ask about functors $\tilde{G}$ from the (near-) category $\mathcal{R}_\mathcal{C}$ to the category $\mathcal{A}$ which (1) extend G, namely $\tilde{G}\Gamma = G$ and, (2) commute with the involutions, namely $\tilde{G}([R]^-) = (\tilde{G}[R])^*$ for every relation $[R]$.

Suppose that such a $\tilde{G}$ exists. First, since for $\sigma \in \mathcal{S}$ we have $\Gamma_\sigma \Gamma_\sigma^-$ = Identity in $\mathcal{R}_\mathcal{C}$ (1.13), this would imply

$$(G\sigma)(G\sigma)^* = (\tilde{G}\Gamma\sigma)(\tilde{G}\Gamma\sigma)^* = (\tilde{G}\Gamma_\sigma)(\tilde{G}\Gamma_\sigma^-) = \tilde{G}(\Gamma_\sigma\Gamma_\sigma^-) = 1.$$

Then, for $\downarrow \psi, \varphi, \eta, \xi \downarrow$ in $\mathcal{C}$ we have $\Gamma_\eta^-\Gamma_\xi = \Gamma_\psi\Gamma_\varphi^-$ (1.8, 1.9), and this would imply

$$(G\eta)^*(G\xi) = (\tilde{G}\Gamma\eta)^*(\tilde{G}\Gamma\xi) = (\tilde{G}\Gamma_\eta^-)(\tilde{G}\Gamma_\xi) = \tilde{G}(\Gamma_\eta^-\Gamma_\xi)$$
$$= \tilde{G}(\Gamma_\psi\Gamma_\varphi^-) = (G\psi)(G\varphi)^*.$$

(3.5) THEOREM. $\tilde{G}$ *satisfying* (1) *and* (2) *exists if and only if* $(G\sigma)(G\sigma)^* = 1$ *for every* $\sigma \in \mathcal{S}$ *and* $(G\eta)^*(G\xi) = (G\psi)(G\varphi)^*$ *for every pullback* $\downarrow \psi, \varphi, \eta, \xi \downarrow$ *in* $\mathcal{C}$. *If such a* $\tilde{G}$ *exists then it is unique.*

Proof. We have already established necessity of the conditions. We prove sufficiency. The unique way to define $\tilde{G}$ in order to fulfill (1) and (2) is by

$$\tilde{G}[R, \rho_X, \rho_Y] = (G\rho_Y)(G\rho_X)^* \quad \text{for } [R, \rho_X, \rho_Y] \in \mathcal{R}_{\mathcal{C}}(X, Y),$$

since $\tilde{G}\Gamma$ should be $= G$ and $[R] = (\Gamma\rho_Y)(\Gamma\rho_X)^{-}$. This $\tilde{G}$ is a well-defined mapping: if $[R] = [R', \rho'_X, \rho'_Y]$ then $\rho'_X = \rho_X \varepsilon$, $\rho'_Y = \rho_Y \varepsilon$ with ε invertible, hence $\varepsilon \in \mathcal{S}$, and by the associativity of $\mathcal{A}$ we have

$$\tilde{G}[R'] = (G\rho_Y)(G\varepsilon)(G\varepsilon)^*(G\rho_X)^* = (G\rho_Y)(G\rho_X)^* = \tilde{G}[R].$$

We show that $\tilde{G}$ is a functor. For $X \in |\mathcal{C}|$,

$$\tilde{G}\Delta_X = (G1_X)(G1_X)^* = 1_{GX}\, 1_{GX} = 1_{GX}.$$

Now, for a composite $[S, \sigma_Y, \sigma_Z][R, \rho_X, \rho_Y]$ we go through $\downarrow \psi, \varphi, \sigma_Y, \rho_Y \downarrow$ and i-s-factorize $\{\rho_X\varphi, \sigma_Z\psi\} = \{\mu_X, \mu_Z\}\zeta$; hence, we have $(G\psi)(G\varphi)^* = (G\sigma_Y)^*(G\rho_Y)$ for the pullback and $(G\zeta)(G\zeta)^* = 1$ since $\zeta \in \mathcal{S}$, thus

$$\begin{aligned}\tilde{G}([S][R]) &= (G\mu_Z)(G\mu_X)^* = G(\mu_Z\,\zeta)(G(\mu_X\,\zeta))^* = G(\sigma_Z\,\psi)(G(\rho_X\,\varphi))^* \\ &= (G\sigma_Z)(G\psi)(G\varphi)^*(G\rho_X)^* = G(\sigma_Z)G(\sigma_Y)^*(G\rho_Y)(G\rho_X)^* \\ &= (\tilde{G}[S])(\tilde{G}[R]).\end{aligned}$$

(3.6) Corollary. *Assume that $\mathcal{C}$, $\mathcal{C}'$ are finitely complete bicategories and* (A) *holds in $\mathcal{C}'$. If F is an s-functor from $\mathcal{C}$ to $\mathcal{C}'$ which preserves pullbacks, then $\mathcal{R}_F : \mathcal{R}_{\mathcal{C}} \to \mathcal{R}_{\mathcal{C}'}$ defined by $\mathcal{R}_F[\cdot, \rho_A, \rho_B] = \Gamma_{F\rho_B}\Gamma^{-}_{F\rho_A}$ is a functor.*

Proof. If $\sigma \in \mathcal{S}_{\mathcal{C}}$ then $F\sigma \in \mathcal{S}_{\mathcal{C}'}$, hence $\Gamma_{F\sigma}\Gamma^{-}_{F\sigma} = \Delta$. If $\downarrow \psi, \varphi, \eta, \xi \downarrow$ in $\mathcal{C}$ then $\downarrow F\psi, F\varphi, F\eta, F\xi \downarrow$ in C', hence $\Gamma^{-}_{F\eta}\Gamma_{F\xi} = \Gamma_{F\psi}\Gamma^{-}_{F\varphi}$. The assertion then follows from (3.5) with $\mathcal{A} = \mathcal{R}_{\mathcal{C}'}$ and $G = \Gamma F$.

Example. If $\mathcal{C}_1$, $\mathcal{C}_2$ are two finitely complete bicategories with the same underlying category $\mathcal{C}$ and if $\mathcal{I}_1 \subset_{\neq} \mathcal{I}_2$, then the extension of the "identity" functor $\mathcal{C}_1 \to \mathcal{C}_2$ is not a functor since $\mathcal{C}_1 \to \mathcal{C}_2$ is not an s-functor ($\mathcal{S}_1 \supset_{\neq} \mathcal{S}_2$); but if (A) holds in $\mathcal{C}_1$, then, by the corollary, the extension of the "identity" $\mathcal{C}_2 \to \mathcal{C}_1$ is a functor from the near-category $\mathcal{R}_{\mathcal{C}_2}$ onto the category $\mathcal{R}_{\mathcal{C}_1}$.

4. Congruences and rectangles

In the abelian case a basic fact is that the relations are "regular" in the sense of Von Neumann, $\rho\rho^{-}\rho = \rho$ ([1], [2], [3]), but this is obviously false in the general case. However, we show here that this is true for "congruences" with respect to group-like structure (in the sense of Kan [7] and Eckmann-Hilton [8]). So we start with introducing congruences. Through this section $\mathcal{C}$ will be a finitely complete bicategory.

Following [7], [8] we say that (A, m) with $m : A \times A \to A$ is a monoid-structure on A and given monoids (A, m), (A', m') we call $\alpha \in \mathcal{C}(A, A')$ a homomorphism if $m'(\alpha \times \alpha) = \alpha m$.

(4.1) Let (A, m_A), (B, m_B) be monoid-structures. We say that

$$[R, \rho_A, \rho_B] \in \mathcal{R}_{\mathcal{C}}(A, B)$$

is an m_A, m_B-*congruence* if there is a morphism $u : R \times R \to R$ satisfying

$$\rho_A u = m_A(\rho_A \times \rho_A), \qquad \rho_B u = m_B(\rho_B \times \rho_B),$$

that is if and only if ρ_A, ρ_B are homomorphisms from (R, u) to (A, m_A), (B, m_B) respectively. The morphism u is then uniquely determined since

$$\{\rho_A, \rho_B\}u = \{m_A(\rho_A \times \rho_A), m_B(\rho_B \times \rho_B)\}$$

and $\{\rho_A, \rho_B\}$ is monic.

We can make the following general definition: for $[R, \rho_A, \rho_B] \in \mathcal{R}_\mathcal{C}(A, B)$ and $[S, \sigma_{A'}, \sigma_{B'}] \in \mathcal{R}_\mathcal{C}(A', B')$ and $\alpha \in \mathcal{C}(A, A')$, $\beta \in \mathcal{C}(B, B')$ we say that the pair α, β maps $[R]$ into $[S]$ if there is a $u \in \mathcal{C}(R, S)$ satisfying

$$(\alpha \times \beta)\{\rho_A, \rho_B\} = \{\sigma_{A'}, \sigma_{B'}\}u$$

(hence u is unique). By this definition $[R]$ is an m_A, m_B-congruence if and only if the pair m_A, m_B maps

$$[R \times R, \rho_A \times \rho_A, \rho_B \times \rho_B] \in \mathcal{R}_\mathcal{C}(A \times A, B \times B) \quad \text{into } [R] \in \mathcal{R}_\mathcal{C}(A, B).$$

Γ_α is an m_A, m_B-congruence if and only if α is a homomorphism from (A, m_A) to (B, m_B). An intersection of m_A, m_B-congruences is an m_A, m_B-congruence. It can be shown that if $\zeta \in \mathcal{S}$ implies $\zeta \times \zeta \in \mathcal{S}$ then compositions of congruences are congruences. This implies the following.

(4.2) *Let* $\mathfrak{M} \subset |\mathcal{C}|$ *and for each* $A \in \mathfrak{M}$ *let a monoid-structure* (A, m_A) *be given. If* (A) *holds in* $\mathcal{C}$ *then the class of* m_A, m_B-*congruences with* A, B *running through* $\mathfrak{M}$ *is a subcategory of the full subcategory of* $\mathcal{R}_\mathcal{C}$ *with objects in* $\mathfrak{M}$.

We call $\delta \in \mathcal{C}(X, Y)$ a constant if $\delta\xi = \delta\eta$ for every ξ, η with codomain X and with common domain. (With our notations, δ is a constant if and only if $\Gamma_\delta \Gamma_\delta^- = [X \times X]$.) For a monoid-structure (A, m), we call $e : A \to A$ a unit if e is a constant and if $m\{1_A, e\} = 1_A = m\{e, 1_A\}$. Then $h : A \to A$ is called an inverse if $m\{1_A, h\} = e = m\{h, 1_A\}$.

An associative monoid-structure with unit and with inverse will be called a *group-structure*. (We recall that (A, m) is associative if $m(m \times 1) = m(1 \times m)\alpha$ where $\alpha : (A \times A) \times A \to A \times (A \times A)$ is the natural isomorphism [8].)

Let e_A, e_B be units for (A, m_A), (B, m_B) respectively and let $[R, \rho_A, \rho_B]$ be an m_A, m_B-congruence with induced structure (R, u); if the pair e_A, e_B maps $[R]$ into $[R]$, then there exists a unit f for (R, u), namely the unique $f : R \to R$ that satisfies $(e_A \times e_B)\{\rho_A, \rho_B\} = \{\rho_A, \rho_B\}f$. A similar fact holds for inverses.

(4.3) LEMMA. *For every* $[R, \rho_A, \rho_B] \in \mathcal{R}_\mathcal{C}(A, B)$ *the inequality* $[R] \circ [R]^- \circ [R] \geq [R]$ *holds.* (Ternary composition was defined in (2.3).)

Proof. If $\downarrow \psi, \varphi, \rho_B, \rho_B \downarrow$, $\downarrow \psi', \varphi', \rho_A, \rho_A \downarrow$ and $\downarrow \theta', \theta, \varphi', \psi \downarrow$ then $[R] \circ [R]^- \circ [R]$ is represented by $\{\rho_A \varphi\theta, \rho_B \psi'\theta'\}^{\natural}$. The pullback property implies existence of λ such that $\varphi\lambda = 1_R = \psi\lambda$ and a λ' such that $\varphi'\lambda' = 1_R = \psi'\lambda'$, then a λ'' such that $\theta\lambda'' = \lambda$, $\theta'\lambda'' = \lambda'$. Hence $\{\rho_A\varphi\theta, \rho_B\,\psi'\theta'\}\lambda'' = \{\rho_A, \rho_B\}$ yielding the assertion.

(4.4) The inverse inequality is generally false, and (using a term justified in sets) we say that $[R]$ is *rectangular* if $[R] \circ [R]^{-} \circ [R] \leq [R]$, that is to say if $[R] \circ [R]^{-} \circ [R] = [R]$.

For instance if ρ_A is monic than $[R, \rho_A, \rho_B]$ is rectangular ((1.10) and (2.5) remark); in particular graphs are rectangular. If for α, β in $\mathcal{C}$ the product $\alpha \times \beta$ is in $\mathcal{I}$, then the components of $\alpha \times \beta$ represent a rectangular relation, so "rectangles" are rectangular (with $\rho_A = \alpha\pi_1$, $\rho_B = \beta\pi_2$ in the proof of (4.3) we have

$$\{\rho_A\,\varphi\theta, \rho_B\,\psi'\theta'\} = (\alpha \times \beta)\{\pi_1\,\varphi\theta, \pi_2\,\psi'\theta'\},$$

hence $\{\rho_A\,\varphi\theta, \rho_B\,\psi'\theta'\}^i \leq \alpha \times \beta$ yielding the assertion).

For an i-functor F preserving products and pullbacks, the mapping $\mathcal{R}_F$ "decreases" ternary composition (proved as with binary composition in (3.1)), hence it preserves rectangularity (by (4.3)).

In the category $\mathcal{E}ns$ the group-structures are the groups and the congruences which are subgroups are rectangular. We establish a general

(4.5) Theorem. *Let (A, m_A), (B, m_B) be group-structures with units e_A, e_B and inverses h_A, h_B, and let $[R, \rho_A, \rho_R]$ be an m_A, m_B-congruence mapped into itself by the pair h_A, h_B. Then $[R]$ is rectangular.*

Proof. We remark that the induced structure (R, u) is a group-structure: it is easily shown to be associative and, if k is the morphism which by assumption satisfies

$$(h_A \times h_B)\{\rho_A, \rho_B\} = \{\rho_A, \rho_B\}k,$$

then $f = u\{1_R, k\}$ satisfies $(e_A \times e_B)\{\rho_A, \rho_B\} = \{\rho_A, \rho_B\}f$ and so f is a unit for (R, u) and k is an inverse.

To prove the theorem let

$$\downarrow \psi, \varphi, \rho_B, \rho_B \downarrow, \qquad \downarrow \psi', \varphi', \rho_A, \rho_A \downarrow, \qquad \downarrow \theta', \theta, \varphi', \psi \downarrow$$

and denote $\varphi\theta = \beta$, $\psi\theta = \gamma = \varphi'\theta'$, $\psi'\theta' = \delta$. Then $\rho_A\,\gamma = \rho_A\,\delta$ and $\rho_B\,\beta = \rho_B\,\gamma$. The relation $[R] \circ [R]^{-} \circ [R]$ is represented by $\{\rho_A\,\beta, \rho_B\,\delta\}^i$, hence it suffices to construct a v such that $\{\rho_A, \rho_B\}v = \{\rho_A\,\beta, \rho_B\,\delta\}$.

We have $\rho_A\,u = m_A(\rho_A \times \rho_A)$, $\rho_B\,u = m_B(\rho_B \times \rho_B)$ and $h_A\,\rho_A = \rho_A\,k$, $h_B\,\rho_B = \rho_B\,k$. Set $v = u\{u\{\beta, k\gamma\}, \delta\}$ and for both ρ_A, ρ_B we obtain (indices not marked)

$$\rho v = m(\rho \times \rho)\{u\{\beta, k\gamma\}, \delta\} = m\{\rho u\{\beta, k\gamma\}, \rho\delta\}$$

$$= m\{m(\rho \times \rho)\{\beta, k\gamma\}, 1\rho\delta\} = m(m \times 1)\{\{\rho\beta, h\rho\gamma\}, \rho\delta\}.$$

With the index A, we use the "associator" α_A and obtain

$$\begin{aligned}\rho_A v &= m_A(1_A \times m_A)\alpha_A\{\{\rho_A \beta, h_A \rho_A \gamma\}, \rho_A \delta\} \\ &= m_A(1_A \times m_A)\{\rho_A \beta, \{h_A \rho_A \gamma, \rho_A \delta\}\} \\ &= m_A\{\rho_A \beta, m_A\{h_A \rho_A \gamma, \rho_A \delta\}\}.\end{aligned}$$

But $\rho_A \gamma = \rho_A \delta$, hence

$$\{h_A \rho_A \gamma, \rho_A \delta\} = \{h_A, 1_A\}\rho_A \gamma \quad \text{and} \quad m_A\{h_A \rho_A \gamma, \rho_A \delta\} = e_A \rho_A \gamma$$

since $m_A\{h_A, 1_A\} = e_A$, and since e_A is constant we have

$$\rho_A v = m_A\{\rho_A \beta, e_A \rho_A \gamma\} = m_A\{\rho_A \beta, e_A \rho_A \beta\} = m_A\{1_A, e_A\}\rho_A \beta = \rho_A \beta.$$

With B we obtain

$$\begin{aligned}\rho_B v &= m_B\{m_B\{\rho_B \beta, h_B \rho_B \gamma\}, \rho_B \delta\} = m_B\{m_B\{\rho_B \beta, h_B \rho_B \beta\}, p_B \delta\} \\ &= m_B\{e_B \rho_B \beta, \rho_B \delta\} = m_B\{e_B \rho_B \delta, \rho_B \delta\} = m_B\{e_B, 1_B\}\rho_B \delta = \rho_b \delta.\end{aligned}$$

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