TOPOLOGICAL PROPERTIES ASSOCIATED WITH $\mathfrak{m}$-HYPERCONVEXITY

BY

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1. Introduction

Let $\mathfrak{m} \geq 3$ be a cardinal number and let $S(x, r)$ denote the cell

$$\{y : \|x - y\| \leq r\}$$

in a real normed linear space N. The space N is said to be $\mathfrak{m}$-*hyperconvex* [1] if every pairwise-intersecting family $\mathfrak{F}$ of cells in N with card $\mathfrak{F} < \mathfrak{m}$ has non-empty intersection. The $\mathfrak{m}$-hyperconvex normed spaces are exactly those spaces N for which the continuous linear operator T in the diagram

$$\begin{array}{l} L \xrightarrow{T} N \\ \cap| \\ M \end{array}$$

has a norm-preserving extension to M whenever dim $M < \mathfrak{m}$.

In the case $\mathfrak{m} >$ card N the $\mathfrak{m}$-hyperconvex normed spaces were characterised in [6] as the spaces $C(S)$ consisting of all continuous real-valued functions on an extremally disconnected compact Hausdorff space S. It was shown further in [1], for a general $\mathfrak{m}$, that the $\mathfrak{m}$-hyperconvex spaces which are of the form $C(X)$ for some compact Hausdorff space X are those for which X has the topological property $Q(\mathfrak{m}, \mathfrak{m})$. This is the case $\mathfrak{m} = \mathfrak{n}$ of the following:

DEFINITIONS. Let X be a topological space, and let $\mathfrak{m}$ and $\mathfrak{n}$ be cardinal numbers with $\mathfrak{m} \geq 3$ and $\mathfrak{n} \geq 3$.

(a) A pair $(\mathcal{U}, \mathcal{V})$ of disjoint non-empty open subsets of X is a $(\mathfrak{m}, \mathfrak{n})$-*pair* if

$$\mathcal{U} = \bigcup\{\mathcal{U}_i : i \in I\} \quad \text{and} \quad \mathcal{V} = \bigcup\{\mathcal{V}_j : j \in J\},$$

where $\mathcal{U}_i$, $\mathcal{V}_j$ are open for all i and j, cl $\mathcal{U}_i \subseteq \mathcal{U}$ for all i, cl $\mathcal{V}_j \subseteq \mathcal{V}$ for all j, card $I < \mathfrak{m}$ and card $J < \mathfrak{n}$.

(b) The space X has *property* $Q(\mathfrak{m}, \mathfrak{n})$ if each $(\mathfrak{m}, \mathfrak{n})$-pair $(\mathcal{U}, \mathcal{V})$ satisfies cl $\mathcal{U} \cap$ cl $\mathcal{V} \neq \emptyset$.

The present paper considers $\mathfrak{m}$-hyperconvex Banach spaces with $\mathfrak{m} \geq 5$, and the spaces are required to have at least one extreme point on their unit cells. The main result is that every such space is isometrically isomorphic to a normed space of the form $A(K)$, consisting of all real continuous affine functions on a Choquet simplex K with the property that the set EK of ex-

Received October 29, 1968.

treme points of K satisfies $Q(\mathfrak{m}, \mathfrak{m})$ in its structure topology. This topology, introduced by Effros in [4], has for its non-trivial closed sets the intersections with EK of the closed faces of K.

We recall from [1] that every $\mathfrak{m}$-hyperconvex normed space with $\mathfrak{m} > \aleph_0$ is complete.

2. Interpolation properties in partially ordered spaces

(2.1) Definitions. Let V be a partially ordered vector space.

(a) V has the $(\mathfrak{m}, \mathfrak{n})$*-interpolation property*, $(\mathfrak{m}, \mathfrak{n})$-Int, if for every two non-empty subsets A and B of V with card $A < \mathfrak{m}$, card $B < \mathfrak{n}$ and $a \leq b$ for all a in A and b in B, there exists v in V with $a \leq v \leq b$ for all a in A and b in B.

(b) When V has an order unit, V will satisfy the *bounded* $(\mathfrak{m}, \mathfrak{n})$*-interpolation property* $B(\mathfrak{m}, \mathfrak{n})$-Int if the property in (a) holds when the sets A and B are bounded in the order-unit norm of V.

In the above, ∞ will denote "a cardinal number strictly greater than card V". It is clear that $(\mathfrak{n}, \mathfrak{m})$-Int is equivalent to $(\mathfrak{m}, \mathfrak{n})$-Int. If V has an order unit and $\mathfrak{m}$ and $\mathfrak{n}$ are finite, then $B(\mathfrak{m}, \mathfrak{n})$-Int and $(\mathfrak{m}, \mathfrak{n})$-Int are equivalent. Also V is a lattice if and only if it has $(3, \infty)$-Int and V has the Riesz decomposition property if and only if it has $(3, 3)$-Int.

(2.2) Lemma. *Let V be a partially ordered space with order-unit e and the order-unit norm. Let $\mathfrak{m} \geq 3$ and $\mathfrak{n} \geq 3$ be cardinal numbers.*

(a) *If V is $(\mathfrak{m} + \mathfrak{n} - 1)$-hyperconvex, then it has the bounded $(\mathfrak{m}, \mathfrak{n})$-interpolation property.*

(b) *If V has the $(\mathfrak{m}, \mathfrak{n})$-interpolation property, then it is $(\mathfrak{m} \wedge \mathfrak{n})$-hyperconvex.*

Proof. (a) Let V be $(\mathfrak{m} + \mathfrak{n} - 1)$-hyperconvex. Let A and B be bounded subsets of V with card $A < \mathfrak{m}$, card $B < \mathfrak{n}$ and $a \leq b$ for all a in A and b in B, and put

$$t = \sup \{\| x - y \| : x, y \in A \cup B\}.$$

Consider the family

$$\mathfrak{F} = \{S(a + te, t) : a \in A\} \cup \{S(b - te, t) : b \in B\}.$$

We have in all cases that card $\mathfrak{F} < \mathfrak{m} + \mathfrak{n} - 1$

From $0 \leq b - a \leq te$ we obtain

$$-2te \leq (b - te) - (a + te) \leq -te$$

which shows that

$$\| (b - te) - (a + te) \| \leqq 2t \quad \text{and} \quad S(a + te, t) \cap S(b - te, t) \neq \emptyset.$$

Also if a and c are in A, then

$$\| (a + te) - (c + te) \| = \| a - c \| \leq t,$$

showing that

$$S(a + te, t) \cap S(c + te, t) \neq \emptyset.$$

Similarly

$$S(b - te, t) \cap S(d - te, t) \neq \emptyset$$

when b and d are points of B.

Since V is $(\mathfrak{m} + \mathfrak{n} - 1)$-hyperconvex there exists

$$v \in \bigcap \{S(a + te, t) : a \in A\} \cap \bigcap \{S(b - te, t) : b \in B\}.$$

For all a in A and b in B, we have

$$-te \leq v - a - te \quad \text{and} \quad v - b + te \leq te,$$

showing that $a \leq v \leq b$. Hence V has the bounded $(\mathfrak{m}, \mathfrak{n})$-interpolation property.

(b) Suppose V has the $(\mathfrak{m}, \mathfrak{n})$-interpolation property and let

$$\{S(x_i, r_i) : i \in I, \text{card } I < \mathfrak{m} \wedge \mathfrak{n}\}$$

be a pairwise-intersecting family of cells in V. Consider the sets

$$A = \{x_i + r_i e : i \in I\} \quad \text{and} \quad B = \{x_j - r_j e : j \in I\}.$$

For each i and j, $x_j - r_j e \leq x_i + r_i e$. Since card $A < \mathfrak{m}$ and card $B < \mathfrak{n}$ there exists v in V with

$$x_j - r_j e \leq v \leq x_i + r_i e \quad \text{for all } i \text{ and } j.$$

This shows that

$$\bigcap \{S(x_i, r_i) : i \in I, \text{card } I < \mathfrak{m} \wedge n\} \neq \emptyset,$$

and that V is $(\mathfrak{m} \wedge \mathfrak{n})$-hyperconvex.

(2.3) Corollary. *The following are equivalent:*

(a) *V is 5-hyperconvex,*

(b) *V has $(3, 3)$-Int,*

(c) *V has $(\mathfrak{m}, \mathfrak{n})$-Int for all $\mathfrak{m}$ and $\mathfrak{n}$ with $3 \leqq \mathfrak{m} \leqq \aleph_0$ and $3 \leq \mathfrak{n} \leq \aleph_0$,*

(d) *V is $\mathfrak{m}$-hyperconvex for all $\mathfrak{m}$ with $5 \leq \mathfrak{m} \leq \aleph_0$.*

Proof. By Lemma 2.2(a), (a) $\Rightarrow$ (b). We may show by induction that for all finite $\mathfrak{m}, \mathfrak{n} \geq 3$

$$(\mathfrak{m}, \mathfrak{n})\text{-Int} \Rightarrow (\mathfrak{m}, \mathfrak{n} + 1)\text{-Int}.$$

This shows (b) $\Rightarrow$ (c). That (c) $\Rightarrow$ (d) now follows from Lemma 2.2(b) and the implication (d) $\Rightarrow$ (a) is trivial.

(2.4) Corollary. *Let V be a partially ordered normed space with order-unit and the order-unit norm. Then for any cardinal $\mathfrak{m} \geq 5$,*

V has the $(\mathfrak{m}, \mathfrak{m})$-interpolation property

$\Rightarrow$ V is $\mathfrak{m}$-hyperconvex

$\Rightarrow$ V has the bounded $(\mathfrak{m}, \mathfrak{m})$-interpolation property.

Proof. The first implication is a consequence of Lemma 2.2(b). The second implication follows in the case of finite $\mathfrak{m}$ from Corollary 2.3. In the case $\mathfrak{m} \geq \aleph_0$, we observe that $2\mathfrak{m} - 1 = \mathfrak{m}$ and use Lemma 2.2(a).

The following result, part of [7, Theorem 4.7], relates the above to our as yet un-ordered 5-hyperconvex normed spaces.

(2.5) PROPOSITION. *Let N be a 4-hyperconvex normed space whose unit cell U has an extreme point e. When N is partially ordered by the cone $\mathbf{R}^+(e + U)$, the order-unit norm derived from e coincides with the original norm.*

3. The property $Q(\mathfrak{m}, \mathfrak{n})$

Let $\mathfrak{m}$ and $\mathfrak{n}$ be cardinal numbers with $\mathfrak{m} \geq 3$ and $\mathfrak{n} \geq 3$. We shall prove that if a Choquet simplex is such that $A(K)$ has the bounded $(\mathfrak{m}, \mathfrak{n})$-interpolation property, then EK has the property $Q(\mathfrak{m}, \mathfrak{n})$ in the structure topology. This then gives a representation theorem for $\mathfrak{m}$-hyperconvex Banach spaces whose unit cells possess an extreme point.

The following known results (3.1)–(3.5) concerning Choquet simplexes will be required. For further details see [2], [4], [8].

(3.1) THEOREM (Edwards [3]). *Let K be a compact convex set in a locally convex Hausdorff space, and let C be the set of lower semicontinuous concave real functions on K.*

The following are equivalent:

(i) *K is a Choquet simplex;*

(ii) *For all f and g with $-f$, g in C and $f \leq g$, there exists a in $A(K)$ with $f \leq a \leq g$;*

(iii) *$A(K)$ has $(3, 3)$-Int;*

(iv) *$A(K)$ has the Riesz decomposition property.*

(3.2) COROLLARIES. *Let F and G be closed faces of a Choquet simplex K.*

(a) (Urysohn's Lemma for simplexes) *If $F \cap G = \emptyset$, there exists a in $A(K)$ with*

$$0 \leq a \leq e,\ a \mid F = 0 \quad \text{and} \quad a \mid G = 1.$$

(b) *The set $H = \text{co}\,(F \cup G)$ is a closed face of K and*

$$H \cap EK = (F \cup G) \cap EK.$$

Proof. (a) Apply Edwards' Theorem with $f = \chi_G$ and $g = e - \chi_F$, where χ_G and χ_F are the characteristic functions of F and G.

(b) The last assertion and the fact that H is closed follow by elementary arguments.

It remains to show that H is a face of K. Suppose k is a point in $EK \setminus (F \cup G)$. By part (a), there exist f_k and g_k in $A(K)$ with $0 \leq f_k \leq e$, $0 \leq g_k \leq e$, $f_k(k) = g_k(k) = 1$ and $f_k(F) = g_k(G) = \{0\}$.

Now let δ_k be the function with

$$\delta_k(x) = 0 \ (x \neq k), \qquad \delta_k(k) = 1.$$

The functions $f = \delta_k$ and $g = f_k \wedge g_k$ satisfy the conditions of Theorem 3.1, and so there exists h_k in $A(K)$ with

$$h_k(k) = 1, \quad h_k \mid (F \cup G) = 0 \quad \text{and} \quad 0 \leq h_k \leq e.$$

The sets

$$H_k = \{x \in K : h_k(x) = 0\} \quad \text{and} \quad H' = \bigcap \{H_k : k \in EK \setminus (F \cup G)\}$$

are closed faces of K containing H. But $H' \cap EK = H \cap EK$, and so $H' = H$ and H is a face of K.

Corollary 3.2(b) gives directly the non-trivial part of the proof that the structure topology is a topology. We recall that with the structure topology EK is compact, but may not be Hausdorff. With the relative topology as a subset of K, EK is a Hausdorff space.

(3.3) PROPOSITION. *Let K be a Choquet simplex. The following are equivalent*:

(i) *EK is closed in K;*
(ii) *EK is a Hausdorff space in the structure topology;*
(iii) *the relative topology and the structure topology of K coincide;*
(iv) *$A(K)$ is a lattice;*
(v) *$A(K) \cong C(EK)$.*

The following is a consequence of Lemma 4.3 of [5].

(3.4) PROPOSITION. *Let V be a partially ordered vector space with order-unit e and the order-unit norm. Let K be the positive face of the unit cell in the dual space V^*. If V is complete, then it is isometrically isomorphic to $A(K)$, where K is taken with the relative weak*-topology.*

(3.5) THEOREM. *Let N be a 5-hyperconvex Banach space whose unit cell has an extreme point e. Then N is isometrically isomorphic to a space $A(K)$ where K is a Choquet simplex.*

Proof. Since N is 4-hyperconvex, Proposition 2.5 shows that it may be regarded as a partially ordered normed space with order-unit e and with the order unit norm coinciding with the original norm. By Proposition 3.4, using the completeness of N, N is isometrically isomorphic to $A(K)$, where K is the positive face of the unit cell in N^*, with the relative weak*-topology.

By Corollary 2.3, N has the (3, 3)-Int property, so by Theorem 3.1 K is a simplex.

(3.6) THEOREM. *Let K be a Choquet simplex. If $A(K)$ has the bounded $(\mathfrak{m}, \mathfrak{n})$-interpolation property $\mathfrak{m} \geq 3$ and $\mathfrak{n} \geq 3$, then the set EK has property $Q(\mathfrak{m}, \mathfrak{n})$ in the structure topology.*

Proof. Let

$$\mathfrak{U} = \bigcup \{\mathfrak{U}_i : i \in I\} \quad \text{and} \quad \mathfrak{V} = \bigcup \{\mathfrak{V}_j : j \in J\}$$

be a $(\mathfrak{m}, \mathfrak{n})$-pair in the structure topology of EK.

Since $\text{cl}\, \mathfrak{U}_i \subseteq \mathfrak{U}$ for all i in I, the sets $\text{cl}\, \mathfrak{U}_i$ and $EK \setminus \mathfrak{U}$ are disjoint closed sets. By Corollary 3.2(a) there exist functions f_i in $A(K)$ with

$$0 \leq f_i \leq e, \quad f_i \,|\, \text{cl}\, \mathfrak{U}_i = 1 \quad \text{and} \quad f_i \,|\, (EK \setminus \mathfrak{U}) = 0.$$

Similarly, for each j in J, there exists g_j in $A(K)$ with

$$0 \leq g_j \leq e, \quad g_j \,|\, \text{cl}\, \mathfrak{V}_j = 0 \quad \text{and} \quad g_j \,|\, (EK \setminus \mathfrak{V}) = 1.$$

The sets $A = \{f_i : i \in I\}$ and $B = \{g_j : j \in J\}$ satisfy the requirements of property $B(\mathfrak{m}, \mathfrak{n})$-Int, since $f_i \leq g_j$ for all i in I and j in J, card $A < \mathfrak{m}$, card $B < \mathfrak{n}$, and $A \cup B \subseteq S(0, 1)$. Thus there exists h in $A(K)$ with $f_i \leq h \leq g_j$ for all i in I and j in J.

Now $h(u) = 1$ for u in $\mathfrak{U}$ and $h(v) = 0$ for v in $\mathfrak{V}$, so that the sets $h^{-1}(\{1\})$ and $h^{-1}(\{0\})$ are disjoint closed faces of K containing $\mathfrak{U}$ and $\mathfrak{V}$ respectively. This shows that in the structure topology the closures $\text{cl}\, \mathfrak{U}$ and $\text{cl}\, \mathfrak{V}$ are disjoint and EK has property $Q(\mathfrak{m}, \mathfrak{n})$.

(3.7) Theorem. *Let $\mathfrak{m} \geq 5$. If N is a $\mathfrak{m}$-hyperconvex Banach space whose unit cell has an extreme point, then N is isometrically isomorphic to a space $A(K)$, where K is a Choquet simplex such that EK satisfies $Q(\mathfrak{m}, \mathfrak{m})$ in the structure topology.*

Proof. By Theorem 3.5, N is of the form $A(K)$ for a suitable Choquet simplex K. By Corollary 2.4 it has property $B(\mathfrak{m}, \mathfrak{m})$-Int and the result now follows from Theorem 3.6 .

(3.8) Proposition. *Let $\mathfrak{m} \geq 5$ and suppose that N is a $\mathfrak{m}$-hyperconvex Banach space whose unit cell has an extreme point e.*

(a) *If N is isometrically isomorphic to $C(X)$ where X is a compact Hausdorff space, then X satisfies $Q(\mathfrak{m}, \mathfrak{m})$.*

(b) *If N is a lattice under the natural ordering given by e, then the set EK is closed in K and satisfies $Q(\mathfrak{m}, \mathfrak{m})$.*

Proof. Let K be the simplex given by Theorem 3.7. In case (b), $A(K)$ is a lattice and by Proposition 3.3, $A(K) \cong C(EK)$, where EK is closed in K. In case (a), X is homeomorphic to EK with the relative topology. Using Proposition 3.3 again, the two topologies on EK coincide. So since EK satisifies Q $(\mathfrak{m}, \mathfrak{m})$ ints structure topology, EK and hence X satisfy $Q(\mathfrak{m}, \mathfrak{m})$ in their induced topologies.

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