RANGES OF QUASI-NILPOTENT OPERATORS

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1. Introduction

Suppose T is a quasi-nilpotent, but not nilpotent, operator on a Banach Space E (that is, $\lim \| T^n \|^{1/n} = 0$ but no $T^n = 0$). The main result of this paper, Theorem 2, is that $T^k(E)$ properly contains $T^{k+1}(E)$ for all nonegative integers k. We prove this by considering the collection of formal power series which converge, in the strong operator topology, when the indeterminate is replaced by T. We include a few additional remarks on the relation between the properties of T and this set of power series. More sophisticated results along these lines, which seem to require additional hypotheses on T, will appear elsewhere.

2. Ranges

Let $\| T^n \| = c_n$, $\phi \in E$, and $f = \sum_{n=1}^{\infty} \lambda_n z^n$, where z is an indeterminate. We denote the two series $\sum_{n=1}^{\infty} \lambda_n T^n$ and $\sum_{n=1}^{\infty} \lambda_n T^n \phi$ by $\bar{f}(T)$ and $\bar{f}(T)\phi$, respectively. To preserve the point of view of [2], we consider only series with zero constant term. The following definition describes all the collections of series needed in the proof of Theorem 2.

DEFINITION 1. *Let f and T be as above and let k be a nonnegative integer. The collections of formal power series K, K^∞, B, and S_{-k} are defined as follows.*

(A) $f \in K \Leftrightarrow \bar{f}(T)$ *converges absolutely in the uniform operator topology* $\Leftrightarrow \sum_{n=1}^{\infty} | \lambda_n c_n | < \infty$.

(B) $f \in K^\infty \Leftrightarrow \{ | \lambda_n c_n | \}_{n=1}^{\infty}$ *is bounded.*

(C) $f \in B \Leftrightarrow \bar{f}(T)$ *converges in the strong operator topology* $\Leftrightarrow \bar{f}(T)\phi$ *converges strongly for all* $\phi \in E$.

(D) $f \in S_{-k} \Leftrightarrow fz^k \in B \Leftrightarrow \bar{f}(T)\phi$ *converges strongly for all* $\phi \in T^k(E)$.

THEOREM 2. *If T is a quasi-nilpotent; but not nilpotent, operator on a Banach space E, and if k is a nonnegative integer, then $T^k(E)$ properly contains $T^{k+1}(E)$.*

Proof. In view of Definition 1(D), it will be enough to show that $S_{-(k+1)}$ properly contains S_{-k}. In fact, we need only find an f not in B for which $fz \in B$, because we can discard low order terms and divide by z^k to obtain a series in $S_{-(k+1)}$ but not S_{-k}. Notice also that $K \subseteq B \subseteq K^\infty$, where the last inequality follows from the Banach-Steinhaus Uniform Boundedness Theorem.

We will complete the proof by finding an f not in K^∞ for which $fz \in K$. Let $\| T^n \| = c_n$; then $\lim_n (c_n)^{1/n} = 0$. Hence $\lim_n \inf c_n/c_{n-1} = 0$ [1, prob. 12-4,

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p. 383]. Choose an increasing sequence of positive integers $\{n_k\}_{k=1}^{\infty}$ such that $n_1 \geq 2$ and $c_{n_k}/c_{n_k-1} < 1/k^3$. Then

$$f = \sum_{k=1}^{\infty} z^{n_k-1}/k^2 c_{n_k}$$

has the desired property.

Theorem 2 yields the following two simple corollaries, whose proofs we omit:

COROLLARY 3. *If F is a nonzero closed subspace of E, then $T(F) \neq F$. If $T(F) \subseteq F$ and k is a nonnegative integer, then either $T^k(F) = \{0\}$ or $T^k(F)$ properly contains $T^{k+1}(F)$.*

COROLLARY 4. *Suppose x is a quasi-nilpotent, but not nilpotent, element of a Banach algebra R, and suppose L is R or a closed proper left ideal of R. Then:*

(A) *$\{Lx^k\}_{k=1}^{\infty}$ is an infinite chain of left ideals whenever $x \in L$. In particular, if L is a minimal ideal, $x \notin L$.*

(B) *For each nonnegative integer k, either $x^kL = \{0\}$ or x^kL properly contains $x^{k+1}L$.*

Since T is a right topological divisor of zero in the ring of bounded operators on E (this is the statement $\lim_n \inf c_n/c_{n-1} = 0$), the case $k = 0$ in Theorem 2 is a special case of a known result [3, p. 279], [4, p. 494, Th. 3.6].

No analogue of Theorem 2 relates the closures of the sets $T^k(E)$. This can be shown by various examples. For instance, let E be a Hilbert Space with orthonormal basis $\{b_n\}_{n=1}^{\infty}$ and define T by $Tb_n = b_{n+1}/n$. Then the intersection of the closures of the $T^k(E)$ is $\{0\}$, while the intersection of the $T^{*k}(E)$ is dense.

We should also point out that the property in Definition 1(D) *cannot* be used to characterize $T^k(E)$. If we let E_k be the set of all $\phi \in E$ for which $\bar{f}(T)\phi$ converges whenever $f \in S_{-k}$, various examples show that no other simple relation between E_k and $T^k(E)$ seems to hold in general. For instance, for the T of the previous paragraph $E_k = \text{cl}\ (T^k(E)) \neq T^k(E)$; while for T^*, $E_k = T^{*k}(E)$. Of course Definition 1(D) does show that, for all T, $E_k \supseteq T^k(E)$; and the argument in Theorem 2 does prove that each E_k properly contains E_{k+1}.

The next theorem gives another example of the use of power series to obtain information about T. The result is well known and can be easily proved without the use of power series, by either the Banach-Steinhaus Theorem or the Baire Category theorem. In the hypothesis, we do not assume T to be quasi-nilpotent.

THEOREM 5. *Suppose T is a bounded linear operator on a Banach space E, and suppose that for all $\phi \in E$ there is a positive integer n with $T^n\phi = 0$; then T is algebraically nilpotent.*

Proof. If f is any formal power series, then $\bar{f}(T)$ converges in the strong operator topology. Therefore K^{∞} contains every power series without constant term, which is impossible unless some $T^n = 0$.

Throughout this paper we have considered only strong convergence of the series $\bar{f}(T)\phi$. However, no essential differences arise if we substitute some other type of convergence, like weak convergence or strong-absolute convergence.

References

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