EMBEDDING CONNECTED SUMS OF TORI IN CODIMENSION ONE

BY

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The problem of classifying embeddings of a torus $S^p \times S^q$ in S^{p+q+1} has been settled in the differentiable and piecewise linear (PL) categories by Kosinski [6], Wall [7], and Goldstein [1]. It is the object of this paper to extend these results in the PL category to locally unknotted embeddings of connected sums of tori in a sphere of codimension one. Before stating the main result we make the

DEFINITION. Let M and V be PL manifolds and let $f, g : M \to V$ be locally unknotted PL embeddings. Then f is *pseudo-isotopic* to g, written $f \sim g$, if there is a PL homeomorphism $HiV \times I \to V \times I$ with $H(x, 0) = (x, 0)$ and $H(x, 1) = (h(x), 1)$ such that $hf(M) = g(M)$. Clearly $\sim$ is an equivalence relation. The equivalence class of the locally unknotted embedding $f : M \to V$ is called the *pseudo-isotopy* class of f and the set of all pseudo-isotopy classes is denoted by Pseudo-Iso (M, V).

The main result of this paper is

MAIN THEOREM. *Let $n \geq 5$ and let*

$$M^n = (\#_{i=1}^{r_1} S_i^{p_1} \times S_i^{q_1}) \# (\#_{i=1}^{r_2} S_i^{p_2} \times S_i^{q_2}) \# \cdots \# (\#_{i=1}^{r_s} S_i^{p_s} \times S_i^{q_s})$$

where $2 \leq p_1 < p_2 < \cdots < p_s \leq q_s < \cdots < q_2 < q_1 ; p_j + q_j = n, j = 1, \cdots, s$ *and $\#$ denotes the connected sum. Then* $|\text{Pseudo-Iso } (M^n, S^{n+1})|$

$$= \{\tfrac{1}{2}(r_1 + 1)(r_2 + 1) \cdots (r_s + 1)\} \qquad \textit{if } n \textit{ is odd or if } n \textit{ is even and } p_s \neq n/2$$

$$= \{\tfrac{1}{2}(r_1 + 1)(r_2 + 1) \cdots (r_{s-1} + 1)\} \quad \textit{if } n \textit{ is even and } p_s = n/2$$

where absolute value denotes cardinality and $\{x\}$ is the least integer $\geq x$.

As special cases we have the following corollaries:

COROLLARY 1. *Let $n = p + q \geq 5$ with $p, q \geq 2$. If n is odd or n is even and $p \neq n/2$, then there are $(r + 1)/2$ pseudo-isotopy classes of embeddings of $\#_{i=1}^{r} S_i^p \times S_i^q$ in S^{p+q+1}.*

COROLLARY 2. *Let $n = 2p$ with $p \geq 3$. Then any two embeddings of $\#_{i=1}^{r} S_i^p \times S_i^p$ in S^{2p+1} are pseudo-isotopic.*

It is interesting to note that although there are embeddings

$$f : S^p \to S^{p+q+1}, \quad g : S^q \to S^{p+q+1} - f(S^p)$$

whose images are linked, when it comes to embeddings of $\#_{i=1}^{r} S_i^p \times S_i^q$ in

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S^{p+q+1} such linking phenomena do not occur. Indeed it is this observation that leads to Lemma 3.1 which is essential to the proof of the Main Theorem.

This paper proceeds as follows: In §1, we sketch the proof of the Main Theorem stating the theorems used in its proof; the proofs of these theorems are then given in §2 and §3. Throughout this paper we work in the *PL* category.

1. A sketch of the proof of the main theorem

Let M^n be a simply connected manifold and $f : M^n \to S^{n+1}$ be a locally unknotted embedding. By Alexander duality $S^{n+1} - f(M)$ has two components whose closures we denote throughout the remainder of this paper by A and B. Since f is locally unknotted, A and B are *PL* manifolds and $\partial A = f(M) = \partial B$.

Let $H_i(X)$ denote the homology of X with integer coefficients.

LEMMA 1.1. *Let $f : M^n \to S^{n+1}$ be a locally unknotted embedding. Then*

(*i*) *The inclusions $f(M) \subset A$ and $f(M) \subset B$ induce isomorphisms*

$$H_i(f(M)) \to H_i(A) \oplus H_i(B) \qquad \textit{for } 0 < i < n.$$

(ii) $H_i(A) = H_i(B) = 0$ *for* $i = n, n+1$.

(iii) *If M is $(p-1)$ connected, $p \geq 2$, then so are A and B.*

Proof. (i) and (ii) follow from the Mayer-Victoris sequence of the proper triad $(S^{n+1}; A, B)$ by observing that M must be orientable and that the boundary homomorphism $H_{n+1}(S^{n+1}) \to H_n(M^n)$ is an isomorphism. To see (iii), notice first that M is simply connected; hence, by the Van Kampen Theorem so are A and B. The result then follows from (i) and the Hurewicz isomorphism.

For the remainder of this section we specialize and let

$$M^n = (\#_{i=1}^{r_1} S_i^{p_1} \times S_1^{q_1}) \# \cdots \# (\#_{i=1}^{r_s} S_i^{p_s} \# A_i^{q_s})$$

where p_j, q_j $j = 1, \cdots, s$ are arbitrary, but fixed, integers satisfying $2 \leq p_1 < p_2 < \cdots < p_s \leq q_s < \cdots < q_2 < q_1$, and $p_j + q_j = n \geq 5$ $j = 1, \cdots, s$. Then M is simply connected and

$$\begin{aligned} H_i(M) &= Z && \text{if } i = 0, n \\ &= Z + \cdots + Z \ (r_j \text{ times}) && \text{if } i = p_j, q_j, j = 1, \cdots, s \\ &= 0 && \text{otherwise} \end{aligned}$$

Thus by 1.1,

$$(*) \qquad \begin{aligned} H_i(A) &= Z && \text{if } i = 0 \\ &= Z + \cdots + Z \ (u_j \text{ times}) && \text{if } i = p_j, j = 1, \cdots, s \\ &= Z + \cdots + Z \ (v_j \text{ times}) && \text{if } i = q_j, j = 1, \cdots, s \\ &= 0 && \text{otherwise} \end{aligned}$$

where $0 \leq u_j \leq r_j$ and $0 \leq v_j \leq r_j$, $j = 1, \cdots, s$. It then follows from Alexander duality that $u_j + v_j = r_j, j = 1, \cdots, s$; and from 1.1 that

$$\begin{aligned} H_i(B) &= Z & &\text{if } i = 0 \\ &= Z + \cdots + Z \ (v_j \text{ times}) & &\text{if } i = p_j, j = 1, \cdots, s \\ &= Z + \cdots + Z \ (u_j \text{ times}) & &\text{if } 1 = q_j, j = 1, \cdots, s \\ &= 0 & &\text{otherwise} \end{aligned}$$

Let $0 \leq u_j \leq r_j$ and $0 \leq v_j \leq r_j, j = 1, \cdots, s$ and construct a polyhderon

$$X(u_1, \cdots, u_s, v_1 \cdots, v_s)$$

as follows: Let $S_1^{p_1}, \cdots, S_{u_1}^{p_1}, S_1^{p_2}, \cdots, S_{u_s}^{p_s}, S_1^{q_s}, \cdots, S_{v_s}^{q_s}, S_1^{q_{s-1}}, \cdots, S_{v_1}^{q_1}$ be spheres of the indicated dimensions with base points $b_1, \cdots, b_{u_1}, b_{u_1+1}, \cdots,$ $b_{u_1+\cdots+u_s}, b_{u_1+\cdots+u_s+1}, b_{u_1+\cdots+u_s+v_s}, b_{u_1+\cdots+u_s+v_s+1}, \cdots, b_{u_1+\cdots+u_s+v_s+\cdots+v_1}$ respectively and let I be the closed interval

$$[1, u_1 + \cdots + u_s + v_s + \cdots + v_1].$$

Then $X(u_1, \cdots, u_s, v_1, \cdots, v_s)$ is obtained from the disjoint union of the spheres and the interval by identifying the base point b_i with the point $i \in I, i = 1, \cdots, u_1 + \cdots + u_s + v_s + \cdots + v_1$. Thus

$$X(u_1, \cdots, u_s, v_1, \cdots, v_s)$$

has the homotopy type of the one point union of u_j spheres of dimension p_j and v_j spheres of dimension $q_j, j = 1, \cdots, s$. In the sequel we shall often identify the spheres $S_1^{p_1}$, or $S_1^{p_2}$ and the interval [1, 2] with their images in $X(u_1, \cdots, u_s, v_1, \cdots, v_s)$. We shall also suppress the arguments $u_1, \cdots,$ $u_s, v_1, \cdots, v_s$ and simply write X when no confusion can arise.

The first step in the proof of the Main Theorem is

THEOREM 1.2. (i) *Let $h : M^n \to S^{n+1}$ be a locally unknotted embedding and suppose the homology of A is as in* $(*)$. *Then there is an embedding*

$$f : X(u_1, \cdots, u_s, v_1, \cdots, v_s) \to S^{n+1}$$

such that A is a regular neighborhood of $f(X)$.

(ii) *Conversely if $f : X(u_1, \cdots, u_s, v_1, \cdots, v_s) \to S^{n+1}$ is an embedding and N is a regular neighborhood of $f(X)$, then*

$$\begin{aligned} N = (\natural_{i=1}^{u_i} S_i^{p_1} \times B_i^{q_1+1}) \natural \cdots \natural (\natural_{i=1}^{u_s} S_i^{p_s} \times B_i^{q_s+1}) \natural (\natural_{i=1}^{v_s} S_i^{q_s} \times B_i^{p_s+1}) \\ \natural \cdots \natural (\natural_{i=1}^{v_1} S_i^{q_1} \times B_i^{p_1+1}) \end{aligned}$$

where $\natural$ denotes boundary connected sum and the B_i are balls of the indicated dimensions. Hence $\partial N = M$.

The proof of Theorem 1.2 is given in §2.

The next step in the proof is

THEOREM 1.3. *Let $f, g : X(u_1, \cdots, u_s, v_1, \cdots, v_s) \to S^{n+1}$ be embeddings* *and let P and Q be regular neighborhoods of $f(X)$ and $g(X)$ respectively. Then there is a homeomorphism $h : S^{n+1} \to S^{n+1}$ of degree 1 such that $h(P) = Q$.*

The proof of Theorem 1.3 is given in §3.

The final important step in the proof of the Main Theorem is

THEOREM 1.4. *Let*

$$f : X(u_1, \cdots, u_s, v_1, \cdots, v_s) \to S^{n+1}$$

and

$$g : X'(u_1', \cdots, u_s', v_1', \cdots, v_s') \to S^{n+1}$$

be embeddings and let P and Q be regular neighborhoods of $f(X)$ and $g(X')$ respectively. In order that there be a homeomorphism $h : S^{n+1} \to S^{n+1}$ of degree 1 such that $h(\partial P) = \partial Q$ it is necessary and sufficient that $u_j + v_j = u_j' + v_j'$, $j = 1, \cdots, s$ and that one of the following be true:

(i) *If n is odd, $u_j = u_j'$ and $v_j = v_j'$, $j = 1, \cdots, s$; or $u_j = v_j'$ and $v_j = u_j'$, $j = 1, \cdots, s$.*

(ii) *If n is even and $p_s \neq n/2$, $u_j = u_j'$ and $v_j = v_j'$, $j = 1, \cdots, s$; or $u_j = v_j'$ and $v_j = u_j'$, $j = 1, \cdots, s$.*

(iii) *If n is even and $p_s = n/2$, $u_j = u_j'$ and $v_j = v_j'$, $j = 1, \cdots, s - 1$; or $u_j = v_j'$ and $v_j = u_j'$ $j = 1, \cdots, s - 1$.*

Proof. We prove only case (i) since the proofs of (ii) and (iii) are similar.

Suppose there is a homeomorphism $h : S^{n+1} \to S^{n+1}$ of degree 1 such that $h(\partial P) = \partial Q$. Then ∂P and ∂Q have isomorphic homology and since, by 1.2, $H_{p_j}(\partial P)$ is free abelian of rank $u_j + v_j$ while $H_{p_j}(\partial Q)$ is free abelian of rank $u_j' + v_j'$, $u_j + v_j = u_j' + v_j'$, $j = 1, \cdots, s$. Also since h is a homeomorphism and ∂P and ∂Q separate S^{n+1} into two components, either $h(P) = Q$ or $h(P) = S^{n+1} - Q^\circ$. If $h(P) = Q$, a simple argument using the homology of P and Q shows that $u_j = u_j'$ and $v_j = v_j'$, $j = 1, \cdots, s$. If $h(P) = S^{n+1} - Q^\circ$, an equally simple argument using the homology of P and $S^{n+1} - Q^\circ$, and 1.1 shows that $u_i = v_i'$ and $v_i = u_i'$, $i = 1, \cdots, s$.

Suppose now that (i) holds. Then if $u_j = u_j'$ and $v_j = v_j'$, $j = 1, \cdots, s$, X and X' are homeomorphic and there is a homeomorphism $h : S^{n+1} \to S^{n+1}$ of degree 1 with $h(\partial P) = \partial Q$ by 1.3. Suppose $u_j = v_j'$ and $v_j = u_j'$, $j = 1, \cdots, s$. Then by 1.1, $H_{p_j}(S^{n+1} - Q^\circ)$ and $H_{q_j}(S^{n+1} - Q^\circ)$ are free abelian of ranks u_j and v_j, respectively, $j = 1, \cdots, s$; for $H_{p_j}(\partial Q)$ and $H_{q_j}(\partial Q)$ are abelian of rank $u_j' + v_j'$ by 1.2 and $H_{p_j}(Q)$ and $H_{q_j}(Q)$ are free abelian of ranks u_j' and v_j' respectively. Hence by 1.2 there is an embedding

$$g' = X(u_1, \cdots, u_s, v_1, \cdots, v_s) \to S^{n+1}$$

such that $S^{n+1} - Q^\circ$ is a regular neighborhood of $g'(X)$. Another application of 1.3 gives the homeomorphism h.

We are now ready to give the

Proof of the Main Theorem. Since any homeomorphism $h : S^{n+1} \to S^{n+1}$of degree 1 is ambient isotopic to the identity, Theorems 1.2–1.4 establish a one to one correspondence between elements of Pseudo-Iso (M^n, S^{n+1}) and sequences of integers $u_1, \cdots, u_s, v_1, \cdots, v_s$, $0 \leq u_j \leq r_j$, $0 \leq v_j \leq r_j$, $j = 1, \cdots, s$ satisfying only the relations $u_j + v_j = r_j$, $j = 1, \cdots, s$, and (i), (ii), or (iii) of 1.4. A simple calculation completes the proof.

2. Proof of Theorem 1.2

Throughout this section let $X = X(u_1, \cdots, u_s, v_1, \cdots, v_s)$, $X_1 = S^{p_1} \subset X$ if $u_1 \geq 1$ and $X_1 = S_1^{p_2} \subset X$ if $u_1 = 0$, $X_2 = X - (X_1 \cup [1, 2])$. The proof of Theorem 1.2 requires some lemmas to which we now turn.

LEMMA 2.1. *Let $f : X \rightarrow S^{n+1}$ be an embedding and let N be a regular neighborhood of $f(X)$ in S^{n+1}. Then there exists regular neighborhoods N_i of $f(X_i)$ in N, $i = 1, 2$, and a ball $B^{n+1} \subset N$ such that*

(i) $N_1 \cap N_2 = \emptyset$,

(ii) *$N_i \cap B$ is a face of B in ∂N_i, $i = 1, 2$,*

(iii) $N = N_1 \cup B \cup N_2$.

Proof. By the uniqueness of regular neighborhoods it suffices to prove the lemma in one case. Therefore let $(J; K, K_1, K_2, L)$ be a triangulation of $(S^{n+1}; f(X), f(X_1), f(X_2), f[1, 2])$ with K, K_1, K_2 and L full in J and let

$$N = \bigcup_{v \in K'} \text{st } (v, J''),$$

$$N_i = \bigcup_{v \in K_i'} \text{st } (v, J''), \quad i = 1, 2,$$

$$B = \bigcup_{v \in L' - (f(1) \cup f(2))} \text{st } (v, J'')$$

where the single (double) prime denotes a first (second) derived complex and st (v, J'') is the closed star of the vertex v in J''. Then N, N_1, and N_2 are regular neighborhoods of $f(X)$, $f(X_1)$, and $f(X_2)$ respectively, and $N_1 \cap N_2 = \emptyset$. But also B is a regular neighborhood of the complex L_1 obtained from L' by deleting $f(1)$, $f(2)$, and the open 1-simplices σ_1, σ_2 containing these points. Thus, since L_1 is either a point or homeomorphic to a closed interval, L_1 is collapsible and B is a ball.

Now $N_1 \cap B = \bigcup(\text{st } (v, J'') \cap \text{st } (w, J''))$ where the union runs over all vertices $v \in K_1'$ and $w \in L' - (f(1) \cup f(2))$. Since $v, w \in J'$, st $(v, J'') \cap$ st $(w, J'') \neq \emptyset$ if and only if v and w span a 1-simplex of J'. But if $v = \hat{\sigma}$ and $w = \hat{\tau}$ where $\sigma, \tau \in J$ and $\hat{\sigma}$ $(\hat{\tau})$ is the point of σ° (τ°) at which σ (τ) is starred in forming J', then v and w span a 1-simplex of J' if and only if $\tau < \sigma$ or $\sigma < \tau$ where $<$ means "is a face of".

Suppose $\tau < \sigma$. Then since a point of τ°, namely $w = \hat{\tau}$, is in $L' - (f(1) \cup f(2)$, $\tau \in L$. Similarly $\sigma \in K_1$. Thus $\tau \in K_1 \cap L$ and since $K_1 \cap L = f(1)$, $\tau = f(1)$. Hence $w = \hat{\tau} = f(1) \notin L' - (f(1) \cup f(2))$ which is a contradiction. Thus $\sigma < \tau$.

By an argument similar to the one above $\sigma = f(1)$. Therefore since $\sigma < \tau \in L$ and since L is a triangulation of $f([1, 2])$, τ is a 1-simplex of J.

Finally since $f : [1, 2] \to S^{n+1}$ is an embedding there is a unique 1-simplex τ of L with $f(1) < \tau$. Thus

$$B \cap N_1 = \text{st}\,(f(1), J'') \cap \text{st}\,(\hat{\tau}, J'')$$

which is just the cell B^n dual to the 1-simplex of J' spanned by $f(1)$ and $\hat{\tau}$. Since it is clear that $B^n \subset \partial B$ and $B^n \subset \partial N_1$, (ii) holds for $N_1 \cap B$. Similarly (ii) holds for $N_2 \cap B$. Since (iii) follows from the construction of N, N_1, N_2, and B, the proof of the lemma is complete.

COROLLARY 2.2. *Let $f : X \to S^{n+1}$ be an embedding and let N be a regular neighborhood of $f(X)$ in S^{n+1}. Then*

$$N = (\natural_{i=1}^{u_1} S_i^{p_1} \times B^{q_1+1}) \natural \cdots \natural (\natural_{i=1}^{u_s} S_i^{p_s} \times B^{q_s+1})$$
$$\natural (\natural_{i=1}^{v_s} S_i^{q_s} \times B^{p_s+1}) \natural \cdots \natural (\natural_{i=1}^{v_1} S_i^{q_1} \times B^{p_1+1}).$$

Proof. The proof is by induction on $r = u_1 + \cdots + u_s + v_s + \cdots + v_1$. If $r = 1$, since $p, q \geq 2$ the corollary follows from Zeeman's unknotting theorem [8].

Suppose the corollary is true (for) $(r - 1)$. In the lemma then $N_1 = S_1^{p_1} \times B^{q_1+1}$ (or $S_1^{p_2} \times B^{q_2+1}$ if $u_1 = 0$) by [8] and

$$N_2 = (\natural_{i=2}^{u_1} S_i^{p_1} \times B^{q_1+1}) \natural \cdots \natural (\natural_{i=1}^{u_s} S_i^{p_s} \times B^{q_s+1})$$
$$\natural (\natural_{i=1}^{v_s} S_i^{q_s} \times B^{p_s+1}) \natural \cdots \natural (\natural_{i=1}^{v_1} S_i^{q_1} \times B^{p_1+1}).$$

The corollary then follows from 2.1 and the definition of the boundary connected sum.

LEMMA 2.3. *Let M^n be an orientable manifold and $B_i \subset M^\circ$, $i = 1, 2$, be n-balls. Let $P \subset M - (B_1 \cup B_2)$ be a polyhderon which does not disconnect M. If $h : B_1 \to B_2$ is a homeomorphism of degree 1, then there is a homeomorphism $k : M \to M$ isotopic to 1_M extending h such that $k \mid P = 1_P$ where 1_M, 1_P denote the identity maps of M and P.*

Proof. This is Theorem 3 of [2].

LEMMA 2.4. *Let M^n be an orientable manifold and let $B_i \subset \partial M$, $i = 1, 2$, be $(n - 1)$ balls with $B_1 \cap B_2 = \emptyset$. If $h : M \to M$ is a homeomorphism of degree 1, then there is a homeomorphism k of M isotopic to h such that $k \mid B_i = 1_{B_i}$, $i = 1, 2$.*

Proof. By 2.3 since $h^{-1} \mid h(B_1) : h(B_1) \to B_1$ is of degree 1, there is a homeomorphism $k_1 : \partial M \to \partial M$ isotopic to $1_{\partial M}$ with $k_1 \mid h(B_1) = h^{-1} \mid h(B_1)$. Let

$$K : \partial M \times [0, 1] \to \partial M \times [0, 1]$$

be an isotopy between k_1 and $1_{\partial M}$ and extend k_1 to a homeomorphism $k_1' : M \to M$ by setting $k_1' = K$ on a collar around ∂M and $k_1' =$ identity outside the collar. Clearly k_1' is isotopic to 1_M.

By a second application of 2.3, there is a homeomorphism $k_2 : \partial M \to \partial M$ isotopic to $1_{\partial M}$ such that $k_2 \mid k_1 h(B_2) = (k_1 h)^{-1} \mid B_2$ and $k_2 \mid B_1 = 1_{B_1}$. Extend k_2 to a homeomorphism $k_2' : M \to M$ isotopic to 1_M as above. Letting $k = k_2' k_1' h$ completes the proof.

LEMMA 2.5. *Let $V = \natural_{i=1}^{r} S^q \times B^p$ where $p \geq 3$ and $q \geq 2$. Then any automorphism $\sigma : H_q(V) \to H_q(V)$ can be realized by a homeomorphism $h : V \to V$.*

Proof. Let $e_1, \cdots, e_r \in H_q(V)$ be the system of generators corresponding to the zero sections $S_i^q \times 0 \subset S_i^q \times B_i^p \subset V$, $i = 1, \cdots, r$, and let R, S, and T_i, $i = 1, \cdots, r-1$, be the automorphisms of $H_q(V)$ satisfying $R(e_1) = -e_1$, $R(e_j) = e_j, j \neq 1$; $S(e_1) = e_1 + e_2$, $S(e_j) = e_j, j \neq 1$; and $T_i(e_i) = e_{i+1}$, $T_i(e_{i+1}) = e_i$, $T_i(e_j) = e_j$, $j \neq 1, i+1$. We prove the lemma first for σ equal R, S, or T_i. To simplify the proof in these cases, we assume without loss of generality that in V, $(S_i^q \times B_i^p) \cap (S_j^q \times B_j^p) = C_{i,j}$ is a $(p + q - 1)$ ball in $\partial(S_i^q \times B_i^p) \cap \partial(S_j^q \times B_j^p)$ if $j = i - 1$ or $i + 1$, and is empty if $j \neq i - 1, i, i + 1$.

Suppose that $\sigma = R$. Let $f : S_1^q \to S_1^q$ and $g : B_1^p \to B_1^p$ be homeomorphisms of degree -1. Then $f \times g : S_1^q \times B_1^p \to S_1^q \times B_1^p$ has degree 1 and by the proof of 2.4, there is a homeomorphism $k : S_1^q \times B_1^p \to S_1^q \times B_1^p$ such that $k \mid C_{1,2} = 1_{C_{1,2}}$. Then k extends to a homeomorphism $h : V \to V$ by setting h equal to the identity outside $S_1^q \times B_1^p$. Clearly h realizes R.

Suppose that $\sigma = S$ and that $V = S_1^q \times B_1^p \natural S_2^q \times B_2^p$. Let $a : [1, 2] \to V$ be an embedding such that $a([1, 2]) \cap (S_i^q \times 0) = a(i)$, $i = 1, 2$; and let

$$K = S_1^q \times 0 \cup S_2^q \times 0 \cup a([1, 2]).$$

Let $N \subset V^\circ$ be a regular neighborhood of K. Then by the proof of 2.1, $N = N_1 \cup D \cup N_2$ where $N_i = S_i^q \times B^p$ is a regular neighborhood of $S_i^q \times 0$ in V, $i = 1, 2$; $N_1 \cap N_2 = \emptyset$; D is a $(p + q)$ ball; and $D \cap N_1$ is a face of D in ∂N_i, $i = 1, 2$.

Let $B^p = E_1^p \cup E_2^p$ where E_i^p $i = 1, 2$ are balls such that $E_1^p \cap E_2^p$ is a common face. Then there is a homoemorphism $f : S^q \times B^p \to S_2^q \times B_2^p$ such that $C_{1,2} \subset f(S^q \times E_1^p)$ and such that $f(S^q \times 0)$ represents e_2. Then

$$(S_1^q \times B_1^p) \cup f(S^q \times E_1^p) = V'$$

is homeomorphic to V. Thus by Irwin's Theorem [5], there is an embedding $g_1 : S^q \to V$ with $g_1(S^q) \subset V^{\circ\prime}$ which represents $e_1 + e_2$. Let M_1 be a regular neighborhood of $g_1(S^q)$ in $V^{\circ\prime}$. Then since V' embeds in S^{p+q} (i.e., in codimension 0), M_1 is also a regular neighborhood of $g_1(S^q)$ in S^{p+q}. Since $p \geq 3$, [8, Theorem 2] implies that M_1 is homeomorphic to $S^q \times B^p$. Let $h_1 : N_1 \to M_1$ be a homeomorphism of degree 1 such that $h_1 \mid S_1^q \times 0 = g_1$.

Now let $g_2 : S^q \quad V$ be the composite

$$S^q = S^q \times 0 \subset S^q \times E_2 \xrightarrow{h} V.$$

By an argument similar to the one above there is a homeomorphism $h_2 : N_2 \to M_2$ of degree 1 such that $h_2 \mid S_2^q \times 0 = g_2$ where $M_2 \subset V^\circ$ is a regular neighborhood of $g_2(S^q)$ not meeting M_1.

Let $x_i \in (N_i \cap D)^\circ$ $i = 1, 2$ and let

$$b : [1, 2] \to \overline{V - (M_1 \cup M_2)} = W$$

be an embedding such that $b(i) = h_i(x_i)$ $i = 1, 2$. Let D' be a regular neighborhood of $b([1, 2])$ in W which meets ∂W regularly. Then D' is a $(p + q)$ ball and it follows from regular neighborhood theory (see, for example [3, Lemma 2.19]) that we can assume that $D' \cap \partial M_i = h_i(N_i \cap D)$ $i = 1, 2$. Now since $h_i \mid N_i \cap D$ is of degree 1, it is easy to see that these maps can be extended to a homeomorphism $h_3 : D \to D'$.

Let $h_4 : N_1 \cup D \cup N_2 \to M_1 \cup D' \cup M_2$ be the homeomorphism obtained by patching h_1, h_2, and h_3 together. Then h_4 has degree 1. Finally since

$$\overline{V - (N_1 \cup D \cup N_2)} \quad \text{and} \quad \overline{V - (M_1 \cup D' \cup M_2)}$$

are both homeomorphic to $\partial V \times [0, 1]$ by the H-cobordism theorem, h_4 can be extended to a homeomorphism $h : V \to V$ of degree 1.

If $V = \natural_{i=1}^r S_i^q \times B_i^p$ with $r \geq 2$, we can use the proof of 2.4 to assume that the homeomorphism h constructed above leaves $C_{2,3}$ pointwise fixed. Then h can be extended to a homeomorphism of V by setting $h(x) = x$ if

$$x \notin (S_1^q \times B_1^p) \cup (S_2^q \times B_2^p).$$

It follows from the construction that h realizes S.

Suppose $\sigma = T_i$ and let $V' = S_i^q \times B_i^p \natural S_{i+1}^q \times B_{i+1}^p \subset V$. Then there is obviously a homeomorphism $k : V' \to V'$ of degree 1 that interchanges $S_i^q \times B_i^p$ and $S_{i+1}^q \times B_{i+1}^p$. By 2.4, there is such a homeomorphism with

$$k \mid C_{i-1,1} \cup C_{i,i+1}$$

the identity. Extend k to a homeomorphism $h : V \to V$ by setting $h(x) = x$ if $x \in V - V'$ and $h \mid V' = k$. Then h realizes T_i.

The lemma now follows by noting that R, S, and T_i $i = 1, \cdots, r - 1$ generate Aut $(H_q(V))$.

COROLLARY 2.6. *Any system of generators $g_1, \cdots, g_r \in H_q(V)$ can be represented by embeddings $f_i : S^q \to V$ with mutually disjoint images, $i = 1, \cdots, r$.*

Remark. It is easy to see that each g_i can be represented by an embedded q-sphere. It is the mutual disjointness which requires some effort to prove.

Proof of 2.6. Let $e_1, \cdots, e_r \in H_q(V)$ be the set of generators described above and let σ be the automorphism of $H_q(V)$ that sends e_i to g_i, $i = 1, \cdots, r$. Then σ can be realized by a homeomorphism h of V. Therefore since the e_i $i = 1, \cdots, r$ can be represented by mutually disjoint embedded q-spheres, so can the g_i, $i = 1, \cdots, r$.

We are now ready for the

Proof of Theorem 1.2. The key step in the proof is to show that if $e_1, \cdots, e_r \in \tilde{H}_*(A)$ generate $\tilde{H}_*(A)$, where $r = u_1 + \cdots + u_s + v_1 + \cdots + v_s$ and $\tilde{H}_*(A)$ denotes the reduced homology of A, then these generators can be represented by mutually disjoint spheres embedded in A°. (Note that since $\tilde{H}_*(A)$ is of rank r this is a minimal system of generators.) In proving this key step we distinguish between the generators of dimension $\leq p_s$ and the generators of dimension $> p_s$.

Let $e_1, \cdots, e_t \in \tilde{H}_*(A)$ be the generators of $\tilde{H}_*(A)$ of dimension $\leq p_s$. Thus $t = u_1 + \cdots + u_s$ if $p_s \neq q_s$, and $t = u_1 + \cdots + u_s + v_s$ if $p_s = q_s$. Since $\tilde{H}_*(M)$ maps onto $\tilde{H}_*(A)$ and since every class in $\tilde{H}_*(M)$ is spherical, every class in $\tilde{H}_*(A)$ is spherical. Thus there are maps $f_i : S_i^{p_j} \to A$, $i = 1, \cdots, u_j; j = 1, \cdots, s$ (or if $p_s = q_s$, $i = 1, \cdots, u_s + v_s$ for $j = s$) representing the generators $e_1 \cdots, e_t$. By Irwin's results [5], we may assume that the f_i are embeddings. Since $\dim A \geq 2p_s + 1$, these embeddings can be made mutually disjoint by general position arguments.

Suppose now that $f_i : S_i \to A^\circ$ $i = 1, \cdots, t$ are embeddings with mutually disjoint images representing the generators $e_1, \cdots, e_t \in \tilde{H}_*(A)$ of dimension less than m where $m \geq p_s$ and that $N_1, \cdots, N_t$ are mutually disjoint regular neighborhoods of the images. Let $e_{t+1}, \cdots, e_{t+w} \in H_m(A)$ be the generators of dimension m. (Note that $w = v_j$ if $m = q_j$).

If some care is exercised in forming the connected sum, it is clear that the embeddings $S_i^{q_j} = x_i \times S_i^{q_j} \subset S_i^{p_j} \times S_i^{q_j}$ where $x_i \in S_i^{p_j}$ yield embeddings $S_i^{q_j} \to M^n$, $i = 1, \cdots, r_j$, representing a system of generators of $H_{q_j}(M)$ which are mutually disjoint. Use a collar of M in

$$A - (N_1^\circ \cup \cdots \cup N_t^\circ) = A'$$

to obtain embeddings $g_i : S^{q_j} \to A^{\circ\prime}$ with mutually disjoint images, $i = 1, \cdots, r_j$. Now let $y_i \in g_i(S^{q_j})$, $i = 1, \cdots, r_j$, be any point and let $a_i : [i, i + 1] \to A^{\circ\prime}$, $i = 1, \cdots, r_j - 1$, be an embedding such that

$$a_i([i, i + 1]) \cap g_k(S^{q_j}) = y_i$$

if $k = i$, y_{i+1} if $k = i + 1$, and is empty otherwise. Since $\dim A \geq 6$, it is possible to select the embeddings a_i such that $a_i([i, i + 1]) \cap a_k([k, k + 1])$ is y_i if $k = i - 1$, is y_{i+1} if $k = i + 1$, and is empty if $k \neq i - 1, i, i + 1$. Thus the embeddings g_i, $i = 1, \cdots, r_j$, and a_i, $i = 1, \cdots, r_j - 1$, fit together to give an embedding $g : X(r_j) \to A'$.

Let N be a regular neighborhood of $g(X(r_j))$ in $A^{\circ\prime}$. Then N is also a regular neighborhood of $g(X(r_j))$ in S^{n+1}. Hence by 2.2,

$$N = \natural_{i=1}^{r} S_i^{q_j} \times B^{p_j+1}.$$

Since the inclusion $M \subset A$ induces an epimorphism $H_{q_j}(M) \to H_{q_j}(A)$, it follows from the construction of N, that $H_{q_j}(N) \to H_{q_j}(A)$ is also an epimorphism. Now let $e_1', \cdots, e_{r_j}' \in H_{q_j}(N)$ be a set of generators such that e_i' projects to $e_{t+i} \in H_{q_j}(A)$ for $i = 1, \cdots, v_j$ and to 0 otherwise. But 2.6 shows that $e_i', \cdots, e_{r_j}'$ can be represented by mutually disjoint embedded q_j spheres in N. Hence, there are embeddings $f_i : S_i^{q_j} \to A^\circ$, $i = t + 1, \cdots, t + v_j$ representing $e_{t+1}, \cdots, e_{t+v_j}$ whose images are mutually disjoint and do not meet any of the other embedded spheres $f_i(S_i)$, $i = 1, \cdots, t$.

By induction, therefore, there are embeddings $f_i : S_i \to A^\circ$ representing e_i, $i = 1, \cdots, r$, with mutually disjoint images. The embedding

$$f : X(u_1, \cdots, u_s, v_1 \cdots, v_s) \to A^\circ$$

is obtained from the f_i by an argument similar to the one used above to construct the embedding $g : X(r_j) \to A^{\circ\prime}$. Note that it follows from the construction of f that $f_* : H_*(X) \to H_*(A)$ is an isomorphism.

Now let N be a regular neighborhood of $f(X)$ in A°. Then $\pi_1(\partial N) = 0$ by 2.2. Since $\pi_1(A)$ and $\pi_1(N)$ also vanish, it follows from Van Kampen's Theorem that $\pi_1(A - N^\circ) = 0$. But also $H_i(A - N^\circ, \partial N) = H_i(A, N) = 0$ for all i since f_* is an isomorphism. Hence the inclusion $\partial N \subset A - N^\circ$ is a homotopy equivalence. Since $\pi_1(M) = 0$ and $H_i(A - N^\circ, M) = 0$ by Lefschetz duality, the inclusion $M \subset A - N^\circ$ is also a homotopy equivalence. Hence by the H-Cobordism Theorem, $A - N^\circ \approx \partial N \times I$ and A is a regular neighborhood of $f(X)$. Thus (i) holds. Since (ii) follows from 2.2, the proof is complete.

3. Proof of Theorem 1.2

Before turning to the proof of Theorem 1.3, we fix our notation and prove two lemmas. In this section $X = X(u_1 \cdots, u_s, v_1, \cdots, v_s)$ and if $u_1 \geq 1$, $X_1 = S_1^{p_1} \subset X$ and $X_2 = X - (X_1 \cup [1, 2))$.

LEMMA 3.1. *Let $u_1 \geq 1$. Let $f : X \to S^{n+1}$ be an embedding and let $N(N_2)$ be a regular neighborhood of $f(X)$ in $S^{n+1}(f(X_2)$ in $N^\circ)$. Then there is an embedding $f' : X \to N$ and a ball $B^{n+1} \subset S^{n+1}$ such that*

(i) *N is a regular neighborhood of $f'(X)$ in S^{n+1}; and*

(ii) *$f(X_1) \subset B$ and $B \cap N_2 = \emptyset$.*

Proof. Since N (respectively N_2) collapses to X (respectively X_2) and $u_1 \geq 1$, the exact sequence of the pair (N, N_2) shows that $H_i(N, N_2) = Z$ if $i = p_1$ and 0 if $i \neq p_1$. By Theorem 1.1, $H_{p_1-1}(\partial N_2) = 0$. Thus in the commutative diagram

$$\begin{array}{ccccccc}
0 \to H_{p_1}(N_2) & \xrightarrow{i_{1*}} & H_{p_1}(N) & \xrightarrow{j_{1*}} & H_{p_1}(N, N_2) \to 0 \\
\uparrow k_{1*} & & \uparrow k_{2*} & & \uparrow k_{3*} \\
0 \to H_{p_1}(\partial N_2) & \xrightarrow{i_{2*}} & H_{p_1}(N - N_2^\circ) & \xrightarrow{j_{2*}} & H_{p_1}(N - N_2^\circ, \partial N_2) \to 0 \\
\| & & \downarrow k_{4*} & & \\
H_{p_1}(\partial N_2) & \xrightarrow{i_{3*}} & H_{p_1}(S^{n+1} - N_2^\circ) & &
\end{array}$$

the top two rows are exact. Note that k_{3*} is an isomorphism by excision and that k_{1*} and i_{3*} are epimorphisms by Lemma 2.1. A diagram chase then shows that there is a class $x \in H_{p_1}(N - N_2^\circ)$ such that $k_{4*}x = 0$ and $j_{1*}k_{2*}x$ is a generator of $H_{p_1}(N, N_2)$. Hence by the exactness of the top row, $H_{p_1}(N)$ is generated by the image of i_{1*} and $k_{2*}x$.

The embedding $f' : X \to N$ is now obtained by noticing that since $N - N_2^\circ$ is $(p_1 - 1)$-connected, x is spherical and can be represented by an embedding $g : S^{p_1} \to (N - N_2^\circ)^\circ$; by setting $f' \mid S_1^{p_1} = g, f' \mid X_2 = f \mid X_2$; and by extending f' over the interval $[1, 2] \subset X$ as in the proof of 1.2. Then clearly $f'_* : H_i(X) \to H_i(N)$ is an isomorphism if $i \neq p_1$ and has image generated by the image of i_{1*} and $k_{2*}x$. Thus $f'_* : H_{p_1}(X) \to H_{p_1}(N)$ is an epimorphism. Since both of these groups are free abelian on u_1 generators, this implies that f'_* is an isomorphism in dimension p_1 also. Then (i) follows from the H-Cobordism Theorem.

To prove (ii), we notice first that $S^{n+1} - N_2^\circ$ is $(p_1 - 1)$-connected. Thus since $k_{4*}x = 0$, the composite embedding

$$g : S^{p_1} \to N - N_2^\circ \subset S^{n+1} - N_2^\circ$$

is null homotopic. Since $q_1 \geq 2$, the codimension of this embedding is ≥ 3 $(n = p_1 + q_1)$. The proof of the engulfing theorem [3, Theorem 7.4, p. 163] then shows that there is a ball $B^{n+1} \subset S^{n+1} - N_2^\circ$ such that

$$f'(X_1) = g(S^{p_1}) \subset B^{n+1}$$

proving (ii).

LEMMA 3.2. *Let $u_1 \geq 1$, $n \geq 5$ and let $f, g : X \to S^{n+1}$ be embeddings such that $f \mid X_i = g \mid X_i$, $i = 1, 2$; and let P, Q be regular neighborhoods of $f(X)$ and $g(X)$ respectively. Then there is a homeomorphism $h : S^{n+1} \to S^{n+1}$ of degree* 1 *such that $h(P) = Q$.*

Proof. By altering the embeddings $f \mid [1, 2]$ and $g \mid [1, 2]$ if necessary, we may assume that $f([1, 2]) \cap g([1, 2])$ consists of the two points $f(1) = g(1)$ and $f(2) = g(2)$. (Since S^{n+1}, P, and Q have dimension ≥ 6, it is always possible to alter f and g in this way.) Hence $f \mid [1, 2]$ and $g \mid [1, 2]$ combine to give an embedding $e : S^1 \to S^{n+1}$. Let $H : S^1 \times I \to S^{n+1}$ be an embedding with $H \mid S^1 \times 0 = e$ and

$$H(S^1 \times 1) \subset S^{n+1} - (f(X_1) \cup f(X_2)).$$

(Clearly such an embedding H exists.) Then since $S^{n+1} - (f(X_1) \cup f(x_2))$ is simply connected, $H \mid S^1 \times 1$ extends to a map

$$H^1 : B^2 \to S^{n+1} - (f(X_1) \cup F(X_2)).$$

Again since $n + 1 \geq 6$, we may assume that H^1 is an embedding and that

$$H(S^1 \times I) \cap H^1(B^2) = H(S^1 \times 1).$$

Thus H and H^1 fit together to give an embedding $G : B^2 \to S^{n+1}$ with

$$G(\partial B^2) = f([1, 2]) \cup g([1, 2]).$$

Let B^{n+1} be a regular neighborhood of $G(B^2)$ relative to $f(X_1) \cup f(X_2)$. Then B^{n+1} is a ball [4, Lemma 1]; and

$$f \mid [1, 2] : [1, 2] \to B^{n+1} \quad \text{and} \quad g \mid [1, 2] : [1, 2] \to B^{n+1}$$

are proper embeddings. By the unknotting of balls in balls [8], then, there is a homeomorphism $h_1 : B^{n+1} \to B^{n+1}$ of degree 1 such that

$$h_1 f([1, 2]) = g([1, 2]) \quad \text{and} \quad h_1 \mid \partial B^{n+1} = \text{identity}.$$

Extend h_1 to a homeomorphism of S^{n+1} by setting $h_1 = $ identity outside B^{n+1}. Clearly h_1 has degree 1. Then since $f(X_i) \cap B^{n+1} = f(i)$ and $g \mid X_i = f \mid X_i$. $i = 1, 2$, $h_1 f(X) = g(X)$. Thus $h_1(P)$ is a regular neighborhood of $g(X)$. By the uniqueness of regular neighborhoods, then, there is a homeomorphism $h_2 : S^{n+1} \to S^{n+1}$ of degree 1 with $h_2(h_1(P)) = Q$. Letting $h = h_2 h_1$ completes the proof.

We turn now to the

Proof of Theorem 1.3. The proof is by induction on the number

$$r (= u_1 + \cdots + u_s + v_1 + \cdots + v_s)$$

of spheres in $X(u_1, \cdots, u_s, v_1, \cdots, v_s)$. If $r = 1$, the theorem follows from [8, Theorem 2] and the uniqueness of regular neighborhoods. Suppose the theorem is true for $(r - 1)$ spheres and let X contain r spheres. There are now two cases.

Case I. $X = X(u_1, \cdots, u_s, v_1, \cdots, v_s)$ with $u_1 \geq 1$. In this case we assume without loss of generality that f and g satisfy (i) and (ii) of 3.1 and that as in 2.1, $P = P_1 \cup C \cup P_2$, $Q = Q_1 \cup D \cup Q_2$ where $P_1(Q_1)$ is a regular neighborhood of $f(X_1)(g(X_1))$; $P_2(Q_2)$ is a regular neighborhood of $f(X_2)(g(X_2))$; C and D are balls; and $P_1 \cap P_2 = \emptyset$ $(Q_1 \cap Q_2 = \emptyset)$. Since P_1 collapses to $f(X_1)$ and there is an $(n + 1)$ ball B_1 with $f(X_1) \subset B_1$ we may engulf P_1 in B_1 and assume by [3, Lemma 7.1] that $P_1 \subset B_1$. Furthermore we may assume that $B_1 \cap P_2 = \emptyset$. Similarly we assume that there is an $(n + 1)$ ball B_2 with $Q_1 \subset B_2$ and $B_2 \cap Q_2 = \emptyset$.

Now by 2.3, there is a homeomorphism $h_1 : S^{n+1} \to S^{n+1}$ of degree 1 with $h_1(B_1) = B_2$. By the induction hypothesis there is a homeomorphism $h_2 : S^{n+1} \to S^{n+1}$ of degree 1 with $h_2 h_1(P_2) = Q_2$. We assert that h_2 can be chosen such that $h_2 \mid B_2 = $ identity; for if $h_2 \mid B_2 \neq$ identity, then since Q_2 does not disconnect S^{n+1} and $Q_2 \subset S^{n+1} - (B_2 \cup h_2(B_2))$ by 2.3 there is a homeomorphism $k : S^{n+1} \to S^{n+1}$ of degree 1 such that $k \mid Q_2 = $ identity and $k \mid h_2(B_2) = h_2^{-1}$. Thus

$$k h_2 h_1(P_2) = k(Q_2) = Q_2 \quad \text{and} \quad k h_2 \mid B_2 = h_2^{-1} h_2 = \text{identity}.$$

Replacing h_2 by $k h_2$ verifies the assertion. Similarly since $S^{n+1} - \mathring{B}_2$ is an $(n + 1)$ ball, there is a homeomorphism $h_3 : S^{n+1} \to S^{n+1}$ of degree 1 such that $h_3 h_1(P_1) = Q_1$ and $h_3 \mid S^{n+1} - \mathring{B}_2$ is the identity. Thus if $h' = h_3 h_2 h_1$, $h'(P_1) = Q_1$ and $h'(P_2) = Q_2$.

Define an embedding $f' : X \to h'(P)$ by setting $f' \mid X_i = g \mid X_i$, $i = 1, 2$; and letting $f' \mid [1, 2]$ describe a path in $h'(P)$ between $g(1)$ and $g(2)$ not meeting $g(X_1) \cup g(X_2)$. Since $h'(P_i) = Q_i$ collapses to $g(X_i)$, $i = 1, 2$, it is easy to check that $f' : X \to h'(P)$ is a homotopy equivalence and that $h'(P)$ is therefore a regular neighborhood of $f'(X)$. But then by 3.2 there is a homeomorphism $h_4 : S^{n+1} \to S^{n+1}$ of degree 1 such that $h_4 h'(P) = Q$. Letting $h = h_4 h'$ completes the proof of Case I.

Case II. $X = X(u_1, \cdots, u_s, v_1 \cdots, v_s)$ with $u_1 = 0$. Then since

$$\partial P = (\#_{i=1}^{r_1} S_i^{p_1} \times S_i^{q_1}) \# \cdots \# (\#_{i=1}^{r_s} S_i^{p_s} \times S_i^{q_s}),$$

it follows from 1.1 and 1.2 that there are embeddings

$$f', g' : X(u_1', \cdots, u_s', v_1', \cdots, v_s') \to S^{n+1}$$

such that $S^{n+1} - P^\circ (S^{n+1} - Q^\circ)$ is a regular neighborhood of $f'(X)(g'(X))$ where $u_j' = r_j - u_j$ and $v_j' = r_j - v_j$, $j = 1, \cdots, s$. Therefore since $u_1' \geq 1$ there is, by Case I, a homeomorphism $h : S^{n+1} \to S^{n+1}$ of degree 1 such that $h(S^{n+1} - P^\circ) = S^{n+1} - Q^\circ$. But then $h(P) = Q$ completing the proof of Case II and the theorem.

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